(partial) Notes on BBD(G)

Time : Spring 2017.

BBD(G)

6 functors & Verdier duality

$$\mathrm{Hom}^n(M^\cdot, N^\cdot) = \prod_i \mathrm{Hom}(M^i, N^{i+n})$$

$$d = df + (-1)^{n+1} f\circ d$$

Ker d = chain maps. Im d = null-homotopic maps.

$$H^i(\mathrm{Hom}^\cdot(M^\cdot, N^\cdot)) = \mathrm{Hom}_{K(A)}(M^\cdot, N^\cdot[i]) = \mathrm{Ext}^i_{D(A)}(M^\cdot, N^\cdot)$$

$$(M^\cdot \otimes^L N^\cdot)^i = \mathrm{Tot}^i(M^\cdot \otimes N^\cdot)$$

$$Rf_! = Rf_* \circ j_!$$

Thm & Defn. let X, Y finite type sch. $/S$, S Noetherian. $\exists f^!$ s.t. $\mathrm{Hom}(f_! M^\cdot, N^\cdot) = \mathrm{Hom}(M^\cdot, f^! N^\cdot)$.

E.g. $f: X \to Y$ sm. rel dim d. $f^!(L^\cdot) = f^* L^\cdot(d)[2d]$.

E.g. $f: X \to pt$, X sm. n-dim'l real oriented mfld. $f^! \mathbb{Q} = \mathbb{Q}_X[n]$.

Defn. $\omega_X = f^! \mathbb{Q}_{pt}$ is called dualizing cplx on X. ω_X

$D: K \longmapsto R\mathcal{H}om(K, \omega_X)$

E.g. $\mathcal{H}^i(f^! \mathbb{Q})(U) = \mathrm{Hom}(\mathbb{Q}_U, (f^! \mathbb{Q})[i]) = \mathrm{Hom}(f_! \mathbb{Q}_U[-i], \mathbb{Q})$
$= (H_c^{-i}(U, \mathbb{Q}))^\vee$.

E.g. $f: X \to Y$ closed immersion normally nonsing. of pure codim d
$f^! = f^*[-2d](-d)$

Prop.
① $Rf_* R\mathcal{H}om(f^* K, C) = R\mathcal{H}om(K, Rf_* C)$
② $Rf_* R\mathcal{H}om(C, f^! K) = R\mathcal{H}om(f_! C, K)$
③ $R\mathcal{H}om(K_1 \otimes^L K_2, K_3) = R\mathcal{H}om(K_1, R\mathcal{H}om(K_2, K_3))$.
④ $f^! R\mathcal{H}om(K, K') = R\mathcal{H}om(f^* K, f^! K')$.
⑤ $K \otimes^L f_! C = f_!(f^* K \otimes^L C)$.
⑥ $D \circ D = \mathrm{Id}$ on constructible bdd derived cat.

Defn. A t-structure on a triangulated cat. D consists of 2 strictly full subcat $D^{\le 0}$ & $D^{\ge 0}$ s.t. ($D^{\le n} = D^{\le 0}[-n]$)
- $\mathrm{Hom}(D^{\le 0}, D^{\ge 1}) = 0$
- $D^{\le 0} \subseteq D^{\le 1}$, $D^{\ge 1} \subset D^{\ge 0}$
- $\forall E \in D$ $\exists$ a distinguished triangle $A \to E \to B \to A[1]$ w/ $A \in D^{\le 0}$, $B \in D^{\ge 1}$.

Prop. A & B are canonically determined by E, $\tau^{\le 0} E$, $\tau^{\ge 0} E$.
let $\iota_{\le 0}: D^{\le 0} \to D$. $R\mathcal{H}om(-, \tau_{\le 0}) = R\mathcal{H}om(\iota_{\le 0}, -)$.

Defn. $\tau_{\ge n} = \tau_{\ge 0} \circ [n]$.

Thm. $\tau_{\le 0} \circ \tau_{\ge 0} = \tau_{\ge 0} \circ \tau_{\le 0}$

Lemma. $E \in D^{\ge 1} \Leftrightarrow \mathrm{Hom}(D^{\le 0}, E) = 0$.

Defn. Core of D is $D^{\le 0} \cap D^{\ge 0}$.

Thm. $\mathrm{Core}_t(D)$ is abelian
~~$f: X$~~ $E^\cdot \xrightarrow{f} F^\cdot$, $\ker(f) = \tau_{\le 0}(\mathrm{Cone}(f)[-1])$
$\mathrm{Coker}(f) = \tau_{\ge 0}$

Hodge II (1.4)

Lemma: If we have $\mathrm{Fil}^{\cdot}$ on $K^{\cdot}$, $T: A \to B$ left exact, then:

$$E_1^{p,q} = R\Gamma^{p+q}(\mathrm{Gr}^p K^{\cdot}) \Rightarrow R\Gamma^{p+q}(K^{\cdot})$$

Lemma: $\forall$ sequence $\cdots \to K_n \to K_{n+1} \to$ in $D(A)$ s.t. $K_i = 0$ $i \ll 0$, $K_i \xrightarrow{\sim} K_{i+1}$ for $i \gg 0$, then $\exists$ $(K^{\cdot}, \mathrm{Fil}^{\cdot})$ s.t.

$\cdots \to K_n \to K_{n+1} \to \cdots$ is q-isom. to

$\cdots \to \mathrm{Fil}^n K^{\cdot} \to \mathrm{Fil}^{n+1} K^{\cdot} \to \cdots$

Apply the above 2 lemmata:

$$E_2^{p,q} = H^p(X, {}^p\mathcal{H}^q(K^{\cdot})) \Rightarrow \mathbb{H}^{p+q}(X, K^{\cdot}).$$

Thm: $D =$ disc. $\Lambda = (\{0\}, D - \{0\})$.

$\mathcal{P}_{\Lambda} = \{$perverse + constr. w.r.t. $\Lambda\}$ is equivalent to the cat. of diagrams: $E \underset{v}{\overset{c}{\rightleftarrows}} F$ s.t.

① E, F f.dim'l $\mathbb{Q}$-v.s.

② $c \circ v + \mathrm{Id}_F$, $v \circ c + \mathrm{Id}_E$ are isom.

On $D \setminus \{0\}$, it's the local system w/ stalk E and monodromy $v \circ c + \mathrm{Id}_E$.

Defn & Prop: If $F \in \mathrm{Core}_U$, $\exists !$ $(G, \alpha: j^* G \xrightarrow{\sim} F)$ w/ $G \in \mathrm{Core}_X$ and ① $i^* G \in D_Z^{<0}$

② $i^! G \in D_Z^{>0}$.

Lefschetz hyperplane section Thm of perverse cohomology

$X/\mathbb{C}$ $\quad D_X = D^b_c(X)$ $\quad P_X = \{\text{perverse sheaves}\}$.

Thm $f: Y \longrightarrow X$ affine between alg. vties $/\mathbb{C}$. Then Rf_* (resp $Rf_!$) $D_Y \longrightarrow D_X$ is right (resp. left) t-exact w.r.t. perverse t-str.

Thm (Artin) $f: Y \longrightarrow X$ affine of f.t. k-schemes. Then:
$$d(R^q f_* F) \le d(F) - q.$$
for F any tors. ab. sheaf on Y_{et}.

pf of choose $C \in {}^pD_Y^{\le 0}$. $\quad R^i f_*(\mathcal{H}^j(C')) \Rightarrow \mathcal{H}^{i+j}(Rf_* C')$
$$d(LHS) \underset{\text{(Artin's Thm)}}{\le} d(\mathcal{H}^j(C')) - i \le -j-i.$$
as $D_X(Rf_*(D_Y(\cdot))) = Rf_!(\cdot)$, we get the other one.

Thm
$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "h"'] & Y \arrow[d, "f"] & Z \arrow[l, "i"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$
w/ f proper, h affine. Then for any $C \in {}^pD_Y^{\ge 0}$, the natural map
$${}^p\mathcal{H}^i(Rf_*C') \longrightarrow {}^p\mathcal{H}^i(Rg_* i^* C')$$
is an isom for $i \le -2$ and a mono for $i = -1$.

pf. $j_!j^*C' \longrightarrow C' \longrightarrow i_*i^*C' \xrightarrow{[1]}$
applying f_* ~~gives~~, use ${}^p\mathcal{H}^i(Rh_! j^* C') = 0$ if $i \le -1$

Thm (perverse Lefschetz hyperplane) Let X be a proj. vty, $Z \subseteq X$ a hyperplane section. Then $IH^i(X) \longrightarrow IH^i(Z)$ is inj. for $i = \dim X - 1$, isom. for $i \le \dim X - 2$.

pf.
$$\begin{tikzcd} U \arrow[r] \arrow[dr] & X \arrow[d] & Z \arrow[l] \arrow[dl] \\ & pt & \end{tikzcd}$$
use $Z \subseteq X$ ample to get $i^*IC_X \cong IC_Z[1]$.
$${}^p\mathcal{H}^i(Rf_*C) = IH^{i+\dim X}(X)$$
$$\downarrow$$
$${}^p\mathcal{H}^i(IC_Z[1]) = IH^{i+\dim Z+1}(Z)$$

Intermediate extns

Fix X, and let $Y \xrightarrow{j} X$ be the inclusion of a loc. ~~local~~ closed subvty. Then, we might hope for some "nice" functor $j_?: P_Y \longrightarrow P_X$ s.t. (a) $j_? C$ "extends C to X", $j^* j_? C = C$

(b) $C \longmapsto j_? C$ commutes w/ Verdier duality.

when j = closed immersion, $j_? = j_*$ satisfies (a) and (b). reduce to the case j is a (dense) open immersion.

2 natural candidates: ${}^p\mathcal{H}^0(j_! C)$ and ${}^p\mathcal{H}^0(Rj_* C)$ both satisfies (a), neither satisfies (b).

Defn. $j_{!*}(C) = \underset{\text{in } P_X}{\mathrm{Im}}\left({}^p\mathcal{H}^0(j_!C) \longrightarrow {}^p\mathcal{H}^0(j_*C)\right).$

Prop. Given $C \in P_Y$, $j_{!*}C$ is the unique perverse sheaf $\tilde{C}$ on X, extending C and s.t.
1. $\tilde{C}$ has no subs nor qts in P_X supported on $Z = X - Y$. (but one can have subqts supported there...)

or 2. $i^*\tilde{C} \in {}^pD_Z^{\le -1}$ and $i^!\tilde{C} \in {}^pD_Z^{\ge 1}$.

Thm $D_X(j_{!*}C) = j_{!*}(D_Y(C))$

pf. show RHS satisfies condition 2 above and extending $D_Y(C)$!

Semismall Maps

Defn f is semismall if the following eq. conditions hold:
(1) $f_*\mathbb{Q}_X[n]$ is a perverse sheaf on Y.
(2) $\dim X\times_Y X \le n$
(3) $\dim Y_k + 2d_k \le n$, $\forall k$.
where $n = \dim X$

$$\begin{array}{ccc} X = \sqcup X_k & & \\ f\downarrow & \downarrow & d_k = \dim X_k - \dim Y_k \\ Y = \sqcup Y_k & & \end{array}$$

Defn We say Y_k is relevant if $\dim Y_k + 2d_k = n$.

Thm If $f: X \to Y$ is semismall, then:
(1) $f_*\mathbb{Q}_X[n] = \bigoplus_{k \text{ relevant}} IC_{\overline{Y_k}}(\mathcal{L}_k)$ where $\mathcal{L}_k = f_*\mathbb{Q}_{X_k}$ (not derived)
(2) $\mathrm{End}_{D_Y}(f_*\mathbb{Q}_X[n]) = \bigoplus_{k \text{ relevant}} \mathrm{End}_{\pi_1(Y_k)} V_k \leftarrow \pi_1(Y_k)$-repn corresp to $\mathcal{L}_k$.
(3) $\mathrm{End}_{D_Y}(f_*\mathbb{Q}_X[n]) \cong H^{BM}_{2n}(X\times_Y X)$.

pf of (3): $H^{BM}_{2n}(X\times_Y X) \cong H^0(X\times_Y X; \mathbb{D}_{X\times_Y X})$
$\cong H^0(X\times_Y X, i^!\mathbb{D}_{X\times X}) \cong H^0(Y, (f\times_{id} f)_* i^! \mathbb{D}_{X\times X})$
$\cong H^0(Y, \Delta^!(f\times f)_*\mathbb{D}_{X\times X}) \cong H^0(Y, \Delta^!(f_*\mathbb{Q}_X)\boxtimes(f_*\mathbb{D}_X))$
$\cong H^0(Y, \mathcal{H}om(f_*\mathbb{Q}_X, f_*\mathbb{Q}_X)) \cong \mathrm{Hom}(f_*\mathbb{Q}_X, f_*\mathbb{Q}_X)$.

Geometric Satake

Defn G/F is reductive if every alg. repn of $G_{\overline{F}}$ is semisimple
$\overset{\text{if char}(F)=0}{\Longleftrightarrow}$ G has trivial <u>unipotent radical</u>.

Defn H is unipotent if $\forall \rho: H \to GL_n$ and all $h \in H(\overline{F})$, $\rho(h)$ is unipotent
- torus if $G_{\overline{F}} \simeq \mathbb{G}^r_{m,\overline{F}}$
- solvable if $\exists\ 1 \to U \to G \to T \to 1$

unipotent radical of G is the max'l connected normal unipotent subgp.

"geom. Langlands" $\hat{G}$-local systems on a sm. proj. curve $X \longleftrightarrow$ objects living on Bun_G.

Q: given G, construct $\hat{G}$ directly?
A: Geometric Satake.

"Gr_G": $R \longmapsto \underbrace{G(R((t)))}_{LG} / \underbrace{G(R[[t]])}_{L^+G}$.

$L^+G \curvearrowright \mathrm{Gr}_G$
$\mathcal{P}_{L^+G}(\mathrm{Gr}_G)$ = cat. of L^+G-eq. perverse sheaves on Gr_G.
endow w/ convolution: $\mathcal{A}_1, \mathcal{A}_2 \longmapsto \mathcal{A}_1 * \mathcal{A}_2$ s.t.
$\omega(-): \mathcal{A} \longmapsto \bigoplus H^i(\mathrm{Gr}_G, \mathcal{A})$. is a tensor functor, i.e., $\omega(\mathcal{A}_1 * \mathcal{A}_2) = \omega(\mathcal{A}_1)\otimes\omega(\mathcal{A}_2)$.

punchline. $\exists$ unique ~~up to an~~ isom: $\mathcal{A}_1 * \mathcal{A}_2 \cong \mathcal{A}_2 * \mathcal{A}_1$, endowing $(\mathcal{P}_{L^+G}(\mathrm{Gr}_G), *)$ w/ the structure of a symmetric monoidal category over $\overline{\mathbb{Q}}_\ell$; then ω is a fiber functor and in particular $\mathcal{P}_{L^+G}(\mathrm{Gr}_G)$ is a neutral Tannakian cat. $/\overline{\mathbb{Q}}_\ell$.
Finally, $\hat{G}_{\overline{\mathbb{Q}}_\ell}$ is defined as $\mathrm{Aut}^{\otimes}(\omega(-))$, i.e.,

$$(P_{L^+G}(Gr_G), *) \cong (Rep(\hat{G}), \otimes)$$

The irreducible objects in $P_{L^+G}(Gr_G)$ are the perverse sheaves $IC_\mu = i_{\mu *} j_{\mu,!*} \widetilde{\mathbb{Q}}_{\ell, \mathcal{O}_\mu}[\dim \mathcal{O}_\mu]$, which corresp to irrep of $\hat{G}$ of highest wt μ.

$$\begin{array}{ccccccc} Gr_G \times Gr_G & \xleftarrow{p} & LG \times Gr_G & \xleftrightarrow{q} & LG \overset{L^+G}{\times} Gr_G & \xrightarrow{m} & Gr_G \\ A_1 \boxtimes A_2 & & & & (g, x) & \longmapsto & gx \end{array}$$

Claim $\exists!$ $A_1 \tilde{\boxtimes} A_2$ on $LG \overset{L^+G}{\times} Gr_G$ s.t.

$$p^*(A_1 \boxtimes A_2) \underset{L^+G\text{-eq.}}{\simeq} q^*(A_1 \tilde{\boxtimes} A_2)$$

$$A_1 * A_2 := Rm_* A_1 \tilde{\boxtimes} A_2$$

$P = W$

X - sproj $\exists$ increasing filtration $W^\cdot \subseteq H^*(X, \mathbb{Q})$ (weight)
decreasing filtration $F^\cdot \subseteq H^*(X, \mathbb{C})$ (Hodge)
s.t. $F^\cdot$ induces pure Hodge structure of wt i on $Gr^i_{W^\cdot} H^*(X,\mathbb{Q}) \underset{\mathbb{Q}}{\otimes} \mathbb{C}$

Non-Abelian Hodge Theory: Idea is do Hodge for cohomology w/ coefficient in non-abelian gp, say, GL_n. e.g. $H^1(X, \mathbb{C}) = Hom(\pi_1(X), \mathbb{C})$

Betti side: ~~Hom~~ $Rep(\pi_1(X), GL_n(\mathbb{C}))$ = rk n local systems

de Rham side: Hol. v.b.'s w/ flat connection

$\mathcal{E} \downarrow X \in Vec(X)$, $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \Omega^1_X$, $\mathbb{C}$-linear s.t.

$$\nabla(f . s) = f \nabla s + s \otimes df$$

flat: $\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^2_X$, composite 0

Dolbeault side: Defn: A Higgs bundle on X is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E} \in Vec(X)$, $\theta \in Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \Omega^1_X)$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\theta"] \arrow[dr, "0"'] & \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \Omega^1_X \arrow[d] \\ & \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \Omega^2_X \end{tikzcd}$$

e.g. C is a cpt Riemann surface, $\mathcal{L}^{\otimes 2} \overset{f}{\simeq} \omega_C$

$\mathcal{L} \xrightarrow{f \otimes \mathcal{L}^\vee} \mathcal{L}^\vee \otimes \omega_C$ $\quad \mathcal{E} = \mathcal{L} \oplus \mathcal{L}^\vee \xrightarrow{\begin{pmatrix} 0 & f \otimes \mathcal{L}^\vee \\ 0 & 0 \end{pmatrix}} \mathcal{E} \otimes \omega_C$

Defn $\mu(\mathcal{E}, \theta) = \dfrac{\deg(\mathcal{E})}{rk(\mathcal{E})}$. $(\mathcal{E}, \theta)$ is (semi)stable if $\forall$ θ-stable coh. subsheaves $\mathcal{F} \subsetneq \mathcal{E}$, we have $\mu(\mathcal{F}) \underset{(-)}{<} \mu(\mathcal{E})$.

Thm (1) $\exists$ natural equivalence of cats: (X sm. proj. conn'd)
{semi~~simple~~ stable flat v.b.'s} $\simeq$ {semi~~stable~~ simple reps of $\pi_1(X)$}
$\simeq$ {polystable Higgs bdles w/ $c_i = 0$}.

(2) $\{\text{all reps of } \pi_1\} = \{\text{all flat v.b.'s}\} = \{\text{semistable Higgs bundles w/ } c_1=0\}$

E.g. (rk 1) Betti: $\mathrm{Reps}(\pi_1(X) \to \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$

de Rham: line bundles w/ flat connection:

$$\mathbb{H}^1(\mathcal{O}_X^* \xrightarrow{d\log} \Omega^1_X) = \mathbb{H}^1(\mathcal{O}_X^* \xrightarrow{d\log} \Omega^1_X \to \Omega^2_X \to \cdots)$$

$$= \mathbb{H}^1(\mathbb{Z} \to \mathcal{O}_X \to \Omega^1_X \to \cdots) = \mathbb{H}^1(\mathbb{Z} \to \mathbb{C}) = H^1(\mathbb{C}^*)$$

Dolbeault: $\mathcal{L} \xrightarrow{\theta} \mathcal{L} \otimes \Omega^1_X$, $c_1(\mathcal{L}) = 0$.

$\mathcal{L} \in \mathrm{Pic}^0_X$. $\mathrm{Hom}(\mathcal{L}, \mathcal{L} \otimes \Omega^1_X) = H^0(\Omega^1)$

deg 0 Higgs bundle $:= \mathrm{Pic}^0_X \oplus H^0(X, \Omega^1_X) = T^*\mathrm{Pic}^0_X$.

(unipotent rk 2) $\cdot\ \mathrm{Ext}^1(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = H^1(X, \mathbb{C})$

$\cdot\ \mathbb{H}^1(X, \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots)$

$\cdot\ \mathbb{H}^1(X, \mathcal{O} \xrightarrow{0} \Omega^1_X \xrightarrow{0} \cdots)$

E.g. $\mathbb{V} \in \mathrm{VHS}(X)$ $F^\cdot \subseteq \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}, \nabla \cdots \longrightarrow \mathrm{gr}_F(\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}, \mathrm{gr}\nabla)$.

P=W X = cpt Riemann surface, genus ≥ 2.

(i) $\mathcal{M}_B$ "$=\{$moduli of ~~rk~~ n dim'l repn of $\pi_1(X)\}$"

$= \{(A_1, \dots, A_g, B_1, \dots, B_g) \mid \prod [A_i, B_i] = -1\} / PGL_n$.

(ii) $\mathcal{M}_{Dol} = \{$moduli of stable Higgs bundles of rk $n\}$.

Prop. $\mathcal{M}_B$ is diffeomorphic to $\mathcal{M}_{Dol}$ for $n=2$. (non-biholomorphic).

e.g. rk 1: $\mathcal{M}_B = (\mathbb{C}^*)^{h_1(X)}$. $\mathcal{M}_{Dol} = T^*\mathrm{Pic}^0_X$.

Thm $\dim(\mathrm{gr}^W_{\dim \mathcal{M}_B - 2k} H^*(\mathcal{M}_B)) = \dim(\mathrm{gr}^W_{\dim \mathcal{M}_B + 2k} H^{*+2k}(\mathcal{M}_B))$

Moreover, $\exists$ class $\alpha \in H^*(\mathcal{M}_B)$ s.t. $\cup \alpha^k$ induces an isom.

P=W: attempt to explain this by identifying wt filtration on $H^*(\mathcal{M}_B)$ w/ "perverse filtration" on $H^*(\mathcal{M}_{Dol}, -)$.

Hitchin fibration $\theta \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)$, $\mathrm{rk}\,\mathcal{E} = 2$

$\mathrm{tr}\,\theta \in H^0(X, \Omega^1_X)$ $\det\theta \in H^0(X, (\Omega^1_X)^{\otimes 2})$

gives $\mathcal{M}_{Dol} \xrightarrow{h} H^0(X, \Omega^1_X) \oplus H^0(X, (\Omega^1_X)^{\otimes 2}) = \mathcal{A}$

Defn (Perverse Leray filtration)

$$P_r H^*(\mathcal{M}_{Dol}, \mathbb{Q}) = \mathrm{Im}(H^*(\mathcal{A}, {}^p\tau^{\leq r} Rh_* \underline{\mathbb{Q}}) \longrightarrow H^*(\mathcal{A}, Rh_* \underline{\mathbb{Q}}) = H^*(\mathcal{M}_{Dol}, \underline{\mathbb{Q}}))$$

Defn The spectral curve $C_\theta \in \mathrm{Tot}(\omega_X) \xrightarrow{\pi} X$ is defined by

$$C_\theta = \{y \in \mathrm{Tot}(\omega_X) \mid y^2 - \pi^*\mathrm{tr}(\theta) \cdot y + \pi^*\det(\theta) = 0\}$$

Fact: fiber of $(\mathcal{E}, \theta)$ under h is cptified Picard of C_θ.

Prop. X sm. proper curve, $\mathcal{L} \in \mathrm{Pic}\,X$, $s_i \in \Gamma(X, \mathcal{L}^{\otimes i})_{i=1,\dots,n}$.

$X_s \subseteq \mathrm{Tot}(\mathcal{L})$, $X_s := \{x \in \mathrm{Tot}(\mathcal{L}) \mid x^n + p^*s_1 x^{n-1} + \cdots + p^*s_n = 0\}$

If X_s is integral, $\exists$ bij.

① torsion-free sheaves on X_s

② v.b. $\mathcal{E}$ of rk n on X w/ a map $\mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{L}$ whose char. polynomial is $(\sum s_i x^{n-i})$.