

COMPARISON OF PRISMATIC COHOMOLOGY AND DERIVED DE RHAM COHOMOLOGY

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ABSTRACT. We establish a comparison isomorphism between prismatic cohomology and derived de Rham cohomology respecting various structures, such as their Frobenius actions and filtrations. As an application, when X is a proper smooth formal scheme over \mathcal{O}_K with K being a p -adic field, we improve Breuil–Caruso’s theory on comparison between torsion crystalline cohomology and torsion étale cohomology.

CONTENTS

1. Introduction	2
Acknowledgement	6
2. Preliminaries	6
2.1. Transversal prisms	6
2.2. Envelopes and derived de Rham cohomology	8
2.3. Frobenii	12
2.4. Naïve comparison	16
3. Comparing prismatic and derived de Rham cohomology	16
3.1. The comparison	16
3.2. Functorial endomorphisms of derived de Rham complex	21
4. Filtrations	25
4.1. Hodge filtration on $dR_{R/A}^\wedge$	25
4.2. Nygaard filtration	31
4.3. Promote to filtered map	33
4.4. Comparing Hodge and Nygaard filtrations	36
5. Connection on $dR_{-/ \mathfrak{E}}^\wedge$ and structure of torsion crystalline cohomology	39
5.1. Connection on $dR_{-/ \mathfrak{E}}^\wedge$	39
5.2. Structures of torsion crystalline cohomology	41
5.3. Galois action on torsion crystalline cohomology	42
6. Torsion Kisin module, Breuil module and associated Galois representations	42
6.1. (Generalized) Kisin modules	42
6.2. Galois representation attached to étale Kisin modules	44
6.3. Torsion Breuil modules	45
7. Torsion cohomology and comparison with étale cohomology	49
7.1. Prismatic cohomology and (generalized) Kisin modules	49
7.2. Nygaard filtration and Breuil–Kisin filtration	50
7.3. Torsion crystalline cohomology	54
8. Some calculations on T_S	60
8.1. Identification on (6.20) and (6.19)	60
8.2. T_S and $T_{st, \star}$	60
References	61

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1. INTRODUCTION

Let k be a perfect field of characteristic $p > 0$ and K a totally ramified degree e field extension of $W(k)[1/p]$. Fix an algebraic closure \bar{K} of K , denote its p -adic completion by \mathbf{C} , and use $\mathcal{O}_{\mathbf{C}}$ to denote its ring of integers. Let X be a smooth proper formal scheme over \mathcal{O}_K with (rigid analytic) geometric generic fiber $X_{\bar{\eta}}$. Write $X_n := X \times_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Starting from [FM87] and [Kat87], lots of efforts have been made in investigating the relationship between the crystalline cohomology (and other variants) and the étale cohomology attached to X .

When $e = 1$, it is proved by Fontaine–Messing ([FM87]) and Kato ([Kat87]) that if X is a proper¹ smooth scheme over $\mathcal{O}_K = W(k)$, then $H_{\text{crys}}^i(X_n/W_n(k))$ admits a Fontaine–Laffaille module structure when $i \leq p - 1$ and the functor T_{crys} on the category of Fontaine–Laffaille modules (from Fontaine–Laffaille theory) satisfies $T_{\text{crys}}(H_{\text{crys}}^i(X_n/W_n(k))) \simeq H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ as G_K -modules when $i \leq p - 2$.

When $e > 1$, more complicated base ring has to be introduced. Fix a uniformizer π of K and $E = E(u) \in W(k)[u]$ the Eisenstein polynomial of π . Let S be the p -adic completion of the PD envelope of $W(k)[u]$ for the ideal (E) . Note that S admits:

- a Frobenius action $\varphi : S \rightarrow S$ which extends the Frobenius φ on $W(k)$ and satisfies $\varphi(u) = u^p$;
- a filtration $\text{Fil}^i S$ which is the p -complete i -th PD ideal; and
- a monodromy operator $N : S \rightarrow S$ via $N(f(u)) = \frac{df}{du}(-u)$.

In [Bre98], Breuil introduced the notion of a *Breuil module* to describe the structure of $H_{\text{crys}}^i(X_n/S_n)$, and constructed a functor $T_{\text{st},*}$ from the category of Breuil modules to the category of \mathbb{Z}_p -representations of G_K . Here, a Breuil module is a datum consisting of a finite S -module \mathcal{M} together with a one-step filtration $\text{Fil}^h \mathcal{M} \subset \mathcal{M}$, a “divided Frobenius” $\varphi_h : \text{Fil}^h \mathcal{M} \rightarrow \mathcal{M}$, and a monodromy operator $N : \mathcal{M} \rightarrow \mathcal{M}$ which satisfies some conditions given in §6.3.

Following ideas of Breuil, Caruso proved the following.

Theorem 1.1 ([Car08]). *Let X be a proper semi-stable scheme over \mathcal{O}_K . Then its log-crystalline cohomology $H_{\text{log-crys}}^i(X_n/S_n)$ has a Breuil module structure and $T_{\text{st},*}(H_{\text{log-crys}}^i(X_n/S_n)) \simeq H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i)$ as G_K -modules for $e(i+1) < p - 1$ if $n > 1$ and $ei < p - 1$ for $n = 1$.*

As new cohomology theories have been introduced in [BMS18], [BMS19] and [BS19], it is natural to ask whether these new cohomology theories can recover the aforementioned results due to Fontaine–Messing, Breuil, and Caruso, and hopefully even improve these results. In this paper, we use these new cohomology theories, in particular, prismatic cohomology and derived de Rham cohomology, to study torsion crystalline cohomology, torsion étale cohomology, and their relationship. We obtain the following result:

Theorem 1.2. *Let X be a smooth proper formal scheme over \mathcal{O}_K with geometric generic fiber $X_{\bar{\eta}}$, and let i be an integer satisfying $ei < p - 1$. Then $H_{\text{crys}}^i(X_n/S_n)$ has a structure of Breuil modules and $T_{\text{st},*}(H_{\text{crys}}^i(X_n/S_n)) \simeq H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i)$ as $\mathbb{Z}_p[G_K]$ -modules.*

Here the additional data of the Breuil module structure is roughly given by the following:

- the filtration is given by the cohomology of the PD powers of a natural PD ideal sheaf $\mathcal{I}_{\text{crys}}$ on the crystalline site $H_{\text{crys}}^i(X_n/S_n, \mathcal{I}_{\text{crys}}^{[h]})$;
- the N is a disguise of the connection given by the crystal nature of crystalline cohomology; and
- the divided Frobenius is induced by a natural map of (quasi-)syntomic sheaves.

From now on, when we talk about $H_{\text{crys}}^i(X_n/S_n)$, we always implicitly think of it carrying these additional data.

Remark 1.3.

- (1) Let us highlight the difference between Caruso’s results and our theorem above.
 - (a) The X in our theorem is a *smooth proper formal* scheme over \mathcal{O}_K , whereas the X in [Car08] is a semi-stable \mathcal{O}_K -model of a smooth proper K -variety.
 - (b) Our restriction on e and i is $ei < p - 1$ for any n while the restriction in [Car08] is $ei < p - 1$ for $n = 1$ and $e(i+1) < p - 1$ for $n > 1$.

¹Projective in Kato’s paper.

- (2) We actually use another functor T_S relating torsion crystalline and étale cohomology in the above theorem. But T_S and $T_{\text{st},*}$ are essentially the same. See §8.2.

Now let us discuss the strategy of this paper to see how prismatic cohomology and (derived) de Rham cohomology come into the picture. Let $\mathfrak{S} = W(k)[[u]]$ equipped with the Frobenius morphism φ extending (arithmetic) Frobenius φ on $W(k)$ and $\varphi(u) = u^p$. Then $(\mathfrak{S}, (E))$ is the so-called Breuil–Kisin prism. Classically, an (étale) Kisin module of height h is a finite u -torsion free \mathfrak{S} -module \mathfrak{M} , together with a semi-linear map $\varphi_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathfrak{M}$ so that the cokernel of $1 \otimes \varphi_{\mathfrak{M}}: \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}$ is killed by E^h . By definition, $\varphi^* \mathfrak{M} := \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ admits a Breuil–Kisin (BK) filtration $\text{Fil}^h \varphi^* \mathfrak{M} := (1 \otimes \varphi_{\mathfrak{M}})^{-1}(E^h \mathfrak{M})$, which plays an important technical role later. It is well-known that Kisin module theory is a powerful tool in *abstract* integral p -adic Hodge theory: the study of \mathbb{Z}_p -lattices in crystalline (semi-stable) representations and their modulo p^n -representations, which can be seen as the arithmetic counterpart of $H_{\text{ét}}^n(X_{\overline{\eta}}, \mathbb{Z}_p)$ and $H_{\text{ét}}^i(X_{\overline{\eta}}, \mathbb{Z}/p^n \mathbb{Z})$. Also the relationship between Kisin modules, Galois representations and Breuil modules are known in the abstract theory. In particular, the functor $\underline{\mathcal{M}}: \mathfrak{M} \mapsto \underline{\mathcal{M}}(\mathfrak{M}) := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ sends a Kisin module \mathfrak{M} of height $h \leq p-1$ to a Breuil module (without N -structures) where

$$\text{Fil}^h \mathcal{M}(\mathfrak{M}) := \{x \in \mathcal{M}(\mathfrak{M}) \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in \text{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M}\} \subset \underline{\mathcal{M}}(\mathfrak{M})$$

and $\varphi_h: \text{Fil}^h \mathcal{M}(\mathfrak{M}) \xrightarrow{1 \otimes \varphi_{\mathfrak{M}}} \text{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_h \otimes 1} S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \underline{\mathcal{M}}(\mathfrak{M})$ where $\varphi_h: \text{Fil}^h S \rightarrow S$ is defined by $\varphi_h(x) = \frac{\varphi(x)}{p^h}$. See §6.3 for more details.

It turns out that prismatic cohomology $H_{\Delta}^i(X/\mathfrak{S})$ gives geometric realizations of Kisin modules, in the sense that $H_{\Delta}^i(X/\mathfrak{S})$ modulo its u^∞ -torsion submodule is an étale Kisin module of height i (see §6.2, §7.1 and the discussion below for more details). Suggested by the functor $\underline{\mathcal{M}}$ in the abstract theory, one naturally expects the following comparison between Breuil–Kisin prismatic cohomology and crystalline cohomology:

$$(1.4) \quad \text{R}\Gamma_{\Delta}(X/\mathfrak{S}) \otimes_{\mathfrak{S}, \varphi}^{\mathbb{L}} S \simeq \text{R}\Gamma_{\text{crys}}(X/S).$$

This comparison follows from [BS19, Theorem 5.2] and base change of prismatic cohomology. This is pointed out to us by Koshikawa.

Inspired by the above discussion, we show in this paper the following comparison result:

Theorem 1.5 (see Theorem 3.5 and Theorem 3.11). *Let (A, I) be a bounded prism, and let X be a smooth proper (p -adic) formal scheme over $\text{Spf}(A/I)$. Then we have a functorial isomorphism*

$$\text{R}\Gamma_{\Delta}(X/A) \otimes_{A, \varphi_A}^{\mathbb{L}} A \otimes_A^{\mathbb{L}} \text{dR}_{(A/I)/A}^{\wedge} \cong \text{R}\Gamma(X, \text{dR}_{-/A}^{\wedge}),$$

which is compatible with base change in the prism (A, I) .

Here $\text{dR}_{-/A}^{\wedge}$ denotes the (relative to A) p -adic derived de Rham complex introduced by Illusie in [Ill72, Chapter VIII] and studied extensively by Bhatt in [Bha12]. In fact, when A/I is p -torsion free, this is known due to [BS19, Theorem 5.2]. Our proof follows closely the proof of crystalline comparison in [BS19].

As a consequence, the above gives several comparison results, all of which were known due to work of Bhatt, Morrow, and Scholze [BMS18], [BMS19], [BS19].

Example 1.6. By [Bha12, Theorem 3.27], we know that when A/I is p -torsion free, the derived de Rham complex appearing above is given by certain crystalline cohomology. With this being said, we can explain what the above comparison gives in concrete situations.

- (1) BMS2/Breuil–Kisin prism: when $(A, I) = (\mathfrak{S}, (E))$, then the above comparison becomes Equation (1.4) which, as mentioned above, was obtained in [BS19]. As a consequence, we see that Breuil’s crystalline cohomology groups $H_{\text{crys}}^i(X/S)$ are finitely presented S -module, see Proposition 7.19. To the best of our knowledge, coherence of S is unknown, and we are unaware of any other means showing that these cohomology groups are finitely presented. We thank Bhatt for pointing out this application to us.
- (2) BMS1: when $(A, I) = (A_{\text{inf}}, \ker(\theta))$ is the perfect prism associated with $\mathcal{O}_{\mathbb{C}}$, then the above comparison says

$$\text{R}\Gamma_{\Delta}(X/A_{\text{inf}}) \otimes_{A_{\text{inf}}, \varphi}^{\mathbb{L}} A_{\text{inf}} \otimes_{A_{\text{inf}}}^{\mathbb{L}} A_{\text{crys}} \cong \text{R}\Gamma_{\text{crys}}(X/A_{\text{inf}}).$$

Recall [BS19, Theorem 17.2] states that the first base change of the left hand side gives the A_{inf} -cohomology theory constructed in [BMS18]. Then our comparison here becomes the one established by [BMS18, Theorem 1.8.(iii)] (see also [Yao19]).

- (3) PD prism: suppose $I \subset A$ admits a PD structure γ . Then our comparison implies

$$\mathrm{R}\Gamma_{\Delta}(X/A) \otimes_{A, \varphi_A}^{\mathbb{L}} A \cong \mathrm{R}\Gamma_{\text{crys}}(X/(A, I, \gamma)).$$

When $I = (p)$, then the above is nothing but the crystalline comparison established in [BS19, Theorem 1.8.(1)]. Notice here the left hand side does not depend on the choice of γ , consequently neither does the right hand side. Another class of potentially interesting PD prisms consists of $(W(S), V(1))$ for any bounded p -complete ring S .

- (4) de Rham comparison: there is a natural map $\mathrm{gr}^0: \mathrm{dR}_{R/A}^{\wedge} \rightarrow R^{\wedge}$ given by “quotient out” first Hodge filtration. Our comparison result above, after composing with this further base change, gives

$$\mathrm{R}\Gamma_{\Delta}(X/A) \otimes_{A, \varphi_A}^{\mathbb{L}} A \otimes_A^{\mathbb{L}} A/I \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/(A/I))^{\wedge};$$

here, we have used [GL21, Proposition 3.11] to identify the result of right hand side under this base change. This is the de Rham comparison given by [BS19, Theorem 1.8.(3)].

In (1)-(3) above, the crystalline comparison [BS19, Theorem 5.2] also yields comparison isomorphisms. Note that there are at least two comparison isomorphisms in above discussion, and we just claimed that they give rise to commutative diagrams, which might worry some readers. To assure these readers, we establish the following rigidity of p -adic derived de Rham cohomology theory.

Theorem 1.7 (see Theorem 3.14 and Remark 3.15). *Let (A, I) be a prism such that A/I is p -torsion free. Then the functor $R \mapsto \mathrm{dR}_{R/A}^{\wedge}$ from the category of smooth (A/I) -algebras to $\mathrm{CAlg}(D(\mathrm{dR}_{(A/I)/A}^{\wedge}))$ has no automorphism. Similar statement holds for the functor $R \mapsto \mathrm{dR}_{R/(A/I)}^{\wedge}$.*

Therefore, whenever one has a diagram of functorial comparisons between various cohomology theories and the p -adic derived de Rham cohomology, the diagram is always forced to be commutative. Our method of proving such rigidity is largely inspired by [BLM18, Sections 10.3 and 10.4] and [BS19, Section 18]. In view of rigidity aspects of p -adic derived de Rham complexes, we would like to mention a recent result of Mondal [Mon20]: roughly speaking, there is a *unique* deformation of de Rham cohomology from characteristic p to Artinian local rings given by crystalline cohomology (c.f. [BLM18, Theorem 10.1.2] for the case of deformation over \mathbb{Z}_p). Let us mention that a recent collaboration between Mondal and the first named author [LM21] computes endomorphisms of p -adic derived de Rham cohomology in various p -adic settings.

Next, we discuss compatibility of additional structures on both sides being compared in Theorem 1.5, most notably the Frobenius action and filtration. In §2.3 we define a natural Frobenius action on p -adic derived de Rham complex assuming the base ring A is a p -torsion free δ -ring. Therefore the right hand side is equipped with a Frobenius action. The left hand side admits a Frobenius action as well, by extending the Frobenius action on prismatic cohomology, as $A \rightarrow \mathrm{dR}_{(A/I)/A}^{\wedge}$ is compatible with Frobenii on them. The two Frobenii on two sides in Theorem 1.5 agree when A is p -torsion free, see Remark 3.6. Let us remark that these p -torsion free conditions most likely can be relaxed, with extra work in developing the theory of “derived δ -rings”. We expect the above theoretical results to hold verbatim.

The story of comparing filtrations is our main new contribution to this theory and it is quite involved. Let us rewrite the comparison:

$$\varphi^* \mathrm{R}\Gamma_{\Delta}(X/A) \otimes_A^{\mathbb{L}} \mathrm{dR}_{(A/I)/A}^{\wedge} \cong \mathrm{R}\Gamma(X, \mathrm{dR}_{-/A}^{\wedge}).$$

There are 3 natural filtrations here:

- the Nygaard filtration $\mathrm{Fil}_{\mathbb{N}}^{\bullet}(\Delta_{-/A}^{(1)})$ on $\varphi^* \mathrm{R}\Gamma_{\Delta}(X/A)$, see [BS19, Section 15];
- the I -adic filtration on A ; and
- the Hodge filtration $\mathrm{Fil}_{\mathbb{H}}^{\bullet}(\mathrm{dR}_{-/A}^{\wedge})$ on $\mathrm{dR}_{(A/I)/A}^{\wedge}$ and $\mathrm{R}\Gamma(X, \mathrm{dR}_{-/A}^{\wedge})$.

They are related in the following fashion.

Theorem 1.8 (see Corollary 4.23). *Let (A, I) be a prism such that A/I is p -torsion free, and let X be a smooth proper (p -adic) formal scheme over $\mathrm{Spf}(A/I)$.*

(1) The isomorphism in Theorem 1.5 refines to a filtered isomorphism:

$$\left(\mathrm{R}\Gamma(X, \mathrm{Fil}_N^\bullet(\Delta_{-/A}^{(1)})) \right) \widehat{\otimes}_{(\mathcal{A}, \mathcal{I}^\bullet)}^{\mathbb{L}} (\mathcal{A}, \mathcal{I}^\bullet) \cong \mathrm{R}\Gamma(X, \mathrm{Fil}_H^\bullet(\mathrm{dR}_{-/A}^\wedge)),$$

where the left hand side denotes the p -complete derived tensor of filtered objects over the filtered ring $(\mathcal{A}, \mathcal{I}^\bullet)$ provided by the lax symmetric monoidal structure on the filtered derived infinity category.

In particular, we obtain a graded isomorphism between graded algebras:

$$\left(\mathrm{gr}_N^* \mathrm{R}\Gamma(X, \Delta_{-/A}^{(1)}) \widehat{\otimes}_{\mathrm{Sym}_{A/I}^*(I/I^2)}^{\mathbb{L}} \Gamma_{A/I}^*(I/I^2) \right) \cong \left(\mathrm{gr}_H^* \mathrm{R}\Gamma(X, \mathrm{dR}_{-/A}^\wedge) \right).$$

(2) The isomorphism in Theorem 1.5 induces natural isomorphisms:

$$\mathrm{R}\Gamma(X, \Delta_{-/A}^{(1)} / \mathrm{Fil}_N^i) \cong \mathrm{R}\Gamma(X, \mathrm{dR}_{-/A}^\wedge / \mathrm{Fil}_H^i)$$

for all $i \leq p$.

Moreover these isomorphisms are functorial in X and A .

Here \mathcal{A} denotes the p -adic PD envelope of $A \twoheadrightarrow A/I$, and \mathcal{I}^\bullet denotes the filtration of PD powers of the ideal $\ker(\mathcal{A} \twoheadrightarrow A/I)$. For a (somewhat) concrete description of the filtration on $\varphi^* \mathrm{R}\Gamma_\Delta(X/A) \otimes_A^{\mathbb{L}} \mathrm{dR}_{(A/I)/A}^\wedge$ appearing in the above Theorem, we refer readers to [GL21, Construction 3.9]. Note that by combining aforesaid result of Bhatt [Bha12, Theorem 3.27] and a classical result of Illusie [Ill72, Corollaire VIII.2.2.8], there is a natural filtered isomorphism $(\mathrm{dR}_{(A/I)/A}^\wedge, \mathrm{Fil}_H^\bullet) \cong (\mathcal{A}, \mathcal{I}^\bullet)$.

In [Ill20, Section 2], a comparison of Nygaard and Hodge filtration is established for crystalline base prism $(A, I) = (W(k), (p))$, in particular his A/I is entirely p -torsion. It seems reasonable to expect the comparison of filtration holds for general base prisms. Both Bhatt and Illusie have sketched to us an approach of resolving base prism by prisms (A, I) such that A/I is p -torsion free, to reduce the general comparison to our theorem above. We choose to not pursue that direction further in this paper, as our final application only uses the comparison when the base is the Breuil–Kisin prism.

With the above general preparation, we are ready to show $\underline{\mathcal{M}}(\mathrm{H}_\Delta^i(X/\mathfrak{S})) \simeq \mathrm{H}_{\mathrm{crys}}^i(X/S)$ (when $\mathrm{H}_\Delta^i(X/\mathfrak{S})$ is u -torsion free). In order to treat p^n -torsion cohomologies in Theorem 1.2, we consider the derived mod p^n variants of the aforementioned cohomology theories. For example, we denote the p^n -torsion prismatic cohomology as $\mathrm{R}\Gamma_\Delta(X_n/A_n) := \mathrm{R}\Gamma_\Delta(X/A) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$. As pointed out by Warning 7.1, such p^n -torsion prismatic cohomology *does not only* depend on $X_n = X \times_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}$. But it is enough for our purpose to understand the p^n -torsion crystalline cohomology $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ and its relation with étale cohomology $\mathrm{H}_{\mathrm{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$.

Note that the cohomology groups of $\mathrm{R}\Gamma_\Delta(X_n/\mathfrak{S}_n)$ do fit in our setting of *generalized* Kisin module \mathfrak{M} of height h (discussed in §6.1), i.e. a finitely generated \mathfrak{S} -module \mathfrak{M} together with a $\varphi_{\mathfrak{S}}$ -semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ and an \mathfrak{S} -linear map $\psi : \mathfrak{M} \rightarrow \varphi^*\mathfrak{M}$ so that $\psi \circ (1 \otimes \varphi_{\mathfrak{M}}) = E^h \mathrm{id}_{\varphi^*\mathfrak{M}}$ and $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi = E^h \mathrm{id}_{\mathfrak{M}}$. The generalized Kisin module is a natural extension of classical (étale) Kisin module discussed above allowing u -torsions. In particular, an étale Kisin module \mathfrak{M} of height h is a generalized Kisin module of height h without u -torsion, where ψ is just defined by $\mathfrak{M} \simeq E^h \mathfrak{M} \xrightarrow{(1 \otimes \varphi_{\mathfrak{M}})^{-1}} \mathrm{Fil}_{\mathrm{BK}}^h \varphi^*\mathfrak{M} \subset \varphi^*\mathfrak{M}$, and similarly the BK filtration can be extended to generalized Kisin module by defining $\mathrm{Fil}_{\mathrm{BK}}^h \varphi^*\mathfrak{M} := \mathrm{Im}(\psi : \mathfrak{M} \rightarrow \varphi^*\mathfrak{M})$. Most importantly, $\mathrm{H}_\Delta^i(X_n/\mathfrak{S}_n)$ is a generalized Kisin module of height i , and the BK filtration on $\varphi^*\mathrm{H}_\Delta^i(X_n/\mathfrak{S}_n)$ exactly matches with the image of the Nygaard filtration $\mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^i \Delta_n^{(1)}) \rightarrow \mathrm{H}_{\mathrm{qSyn}}^i(X, \Delta_n^{(1)})$ where $\mathrm{Fil}_N^i \Delta_n^{(1)} = \mathrm{Fil}_N^i \Delta_{-/ \mathfrak{S}}^{(1)} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$ and $\Delta_n^{(1)} = \Delta_{-/ \mathfrak{S}}^{(1)} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$, see Proposition 7.2 and Corollary 7.11. One can apply many methods in the study of étale Kisin modules to treat $\mathrm{H}_\Delta^i(X_n/\mathfrak{S}_n)$ as well. As a consequence, we prove the following:

Theorem 1.9. *Let $A = (\mathfrak{S}, E)$ be the Breuil–Kisin prism and write $\mathfrak{M}_n^i := \mathrm{H}_\Delta^i(X_n/A_n)$. Let $i \leq p - 2$ be an integer. Then $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ has a Breuil module structure if and only if \mathfrak{M}_n^i has no u -torsion for $j = i, i + 1$. In this scenario, we have $\underline{\mathcal{M}}(\mathfrak{M}_n^i) \simeq \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ and $T_S(\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)) \simeq \mathrm{H}_{\mathrm{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i)$ as G_K -modules.*

Finally, by using Caruso's Theorem 1.1 for $n = 1$, we can show that \mathfrak{M}_n^{i+1} has no u -torsion if $ei < p - 1$, hence deducing Theorem 1.2.

In the end of this introduction, let us report what we now, a year since writing this paper, know slightly beyond the case of $e \cdot i < p - 1$. Recall Breuil asked [Bre02, Question 4.1] whether, assuming $i < p - 1$, it is true that $H_{\text{crys}}^i(X_n/S_n)$ always supports a Breuil module structure with associated Galois representation given by $H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$. In view of our Theorem 1.9, what Breuil asked for is really some module theoretic structure of prismatic cohomology of X : whether the prismatic cohomology always has no u -torsion when $i < p$. In the upcoming sequel to this paper, among other things, we study u^∞ -torsions in prismatic cohomology. We obtain some results in the boundary case: that is $e \cdot i = p - 1$. Let us restrict ourselves further to the two extrema of the boundary case, and our relevant findings is summarized below:

- When $i = 1$, Breuil's question amounts to vanishing of u -torsion in the first and second prismatic cohomology. The first prismatic cohomology is always u -torsion free. The second prismatic cohomology having u -torsion is showed to be equivalent to the failure of having Albanese abelian (formal) scheme of X , by which we mean a map $X \rightarrow A$ with both central and generic fibres being the Albanese map. This was studied by Raynaud [Ray79] and we extend some of his results using this prismatic perspective. We generalize a construction in [BMS18, Subsection 2.1] to produce counterexamples to Breuil's question with $e = p - 1 > 1$. In fact the module structure of $H_{\text{crys}}^1(X_n/S_n)$ of this example is too pathological that it cannot possibly support a Breuil module structure. In particular, in hindsight, our Theorem 1.2 is sharp.
- When $e = 1$, this is what Fontaine–Messing [FM87] and Kato [Kat87] studied. In the boundary case $i = p - 1$, we show that the Galois representation ρ^{p-1} attached to the Fontaine–Laffaille structure on the $(p - 1)$ -st crystalline cohomology is not far from the $(p - 1)$ -st étale cohomology of the geometric generic fibre: the latter embeds in the former with cokernel being a p -torsion unramified Galois representation. Moreover the cokernel can be understood as a piece of the p -adic vanishing cycle of X in cohomological degree p .

The case of $e \cdot i > p - 1$ remains mysterious to us. We believe the first step of investigation would be a better understanding of u^∞ -torsions in prismatic cohomology, extending our results so far.

We arrange our paper as follows: after collecting rudiments on prismatic cohomology and derived de Rham cohomology in §2, we establish our comparison isomorphism between the two cohomologies in §3 together with Frobenius structures. We devote efforts to discuss various filtrations in §4 and establish a filtered comparison. We remark that the theory in §2–§4 accommodates quite general classes of prisms, which opens the possibilities to develop, for example, Breuil–Caruso theory, for more general base rings. We hope to report the generalization in this direction in future work. Starting from §5, we restrict ourselves to the Breuil–Kisin prism $(\mathfrak{S}, (E))$ and focus on structures of torsion prismatic cohomology and torsion crystalline cohomology for proper smooth formal scheme X over \mathcal{O}_K . In §5 we construct a connection ∇ on derived de Rham cohomology and hence on crystalline cohomology. The §6 recalls classical theory of Kisin modules, Breuil modules, functors to Galois representations and the functor \mathcal{M} connecting Kisin modules and Breuil modules. Finally §7 assembles all previous preparations to prove Theorem 1.9.

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2. PRELIMINARIES

Starting with this section through Section 4, unless stated otherwise, all completions and (completed) tensor products are derived.

2.1. Transversal prisms.

Lemma 2.1. *Let (A, I) be an oriented prism with $I = (d)$. The following are equivalent:*

- (1) the sequence (p, d) is Koszul regular;
- (2) the sequence (p, d) is regular;
- (3) the morphism $\mathbb{Z}_p[[T]] \rightarrow A$ sending T to d is flat.

Proof. (3) implies (1): as (3) implies that $A \otimes_{\mathbb{Z}_p[[T]]} \mathbb{Z}_p[[T]]/(p, T)$ is discrete.

(1) implies (2): (1) implies that the p -torsions in A is uniquely d -divisible, and A/p has no d -torsion. On the other hand, we know the p -torsions in A is derived d -complete, hence must vanish. Therefore (p, d) is a regular sequence.

(2) implies (3): it suffices to show that for any prime ideal $\mathfrak{p} \subset \mathbb{Z}_p[[T]]$ the derived tensor $A \otimes_{\mathbb{Z}_p[[T]]} \mathbb{Z}_p[[T]]/\mathfrak{p}$ is discrete. When \mathfrak{p} is the unique maximal ideal, this follows immediately from (2). So we only have to deal with height 1 primes which are always generated by a polynomial of the form

$$f = T^n + p \cdot (\text{lower order terms}),$$

and we need to show that A is f -torsionfree. Suppose $a \in A$ is an f -torsion, modulo p we see that $\bar{a} \in A/p$ is a d^n -torsion, now (2) implies that $\bar{a} = 0 \in A/p$. Therefore we see that f -torsions in A is divisible by p . As (2) also implies that A is p -torsionfree, we see that f -torsions in A is uniquely p -divisible. Since A is derived p -complete, we see that A must in fact be f -torsionfree. \square

We can globalize to non-oriented prisms (A, I) . The following easily follows from Lemma 2.1.

Lemma 2.2. *Let (A, I) be a prism. The following are equivalent:*

- (1) there is a (p, I) -completely faithfully flat cover by an oriented prism (A', IA') , which satisfies the equivalent conditions in Lemma 2.1;
- (2) the ideal I is p -completely regular;
- (3) Zariski locally (p, I) is a regular sequence;
- (4) the natural morphism $\mathrm{Spf}(A) \rightarrow [\mathrm{Spf}(\mathbb{Z}_p[[T]])/(\mathbb{G}_m)_{\mathbb{Z}_p}]$ classified by I is flat.

Let us explain the morphism in (4) above: Zariski locally I is generated by a nonzerodivisor d , hence Zariski locally we get a map $\mathrm{Spf}(A) \rightarrow \mathrm{Spf}(\mathbb{Z}_p[[T]])$, and on overlap these generators differ by a unit in A , hence globally we have a morphism to the quotient stack. Alternatively, we can understand this map as the composition of the universal map $\mathrm{Spf}(A) \rightarrow \Sigma$ introduced by Drinfeld [Dri20, Section 1.2], and $\Sigma \rightarrow [\mathrm{Spf}(\mathbb{Z}_p[[T]])/(\mathbb{G}_m)_{\mathbb{Z}_p}]$ induced by $W_{\mathrm{prim}} \rightarrow \mathrm{Spf}(\mathbb{Z}_p[[T]])$ sending a Witt vector (x_0, x_1, \dots) to x_0 .

Definition 2.3. A prism (A, I) is said to be *transversal* if it satisfies the equivalent conditions in Lemma 2.2.

For the remaining of this subsection, let us assume (A, I) to be a transversal prism. Denote the p -completed PD envelope of $A \twoheadrightarrow A/I$ by \mathcal{A} , and denote the kernel of $\mathcal{A} \twoheadrightarrow A/I$ by \mathcal{I} .

Example 2.4. Let us list some examples of transversal prisms.

- (1) The universal oriented prism is transversal.
- (2) The Breuil–Kisin prism [BS19, Example 1.3.(3)] is transversal. We have $A = \mathfrak{S}$ and \mathcal{A} is classically denoted by S in classical literature concerning Breuil modules.
- (3) Let \mathbf{C} be an algebraically closed complete non-Archimedean field extension of \mathbb{Q}_p . Then the perfect prism associated with $\mathcal{O}_{\mathbf{C}}$ is transversal. We have $A = A_{\mathrm{inf}}$ and $\mathcal{A} = A_{\mathrm{crys}}$.

Although \mathcal{A} is usually not flat over A , it has p -completely finite Tor dimension. In the next subsection we shall see that this is a general phenomenon about derived de Rham complex and regularity of I .

Lemma 2.5. *Let (A, I) be a transversal prism. Then $A \rightarrow \mathcal{A}$ has p -complete amplitude in $[-1, 0]$, in particular p -completely base changing along $A \rightarrow \mathcal{A}$ commutes with taking totalizations in $D^{\geq 0}(A)$.*

Proof. It suffices to check the statement Zariski locally on $\mathrm{Spf}(A)$, hence we may assume the prism to be oriented, say $I = (d)$. Then we may base change to A/p . So we need to check that given an \mathbb{F}_p -algebra R , and a nonzerodivisor $d \in R$, the divided power algebra $S = D_R(d)$ has Tor amplitude in $[-1, 0]$ over R . This follows from the fact that $d^p = 0$ in S and S is a free $R/(d^p)$ -module. The commutation of tensor and totalization now follows from [BS19, Lemma 4.20]. \square

2.2. Envelopes and derived de Rham cohomology. Let (A, I) be a bounded prism. In this subsection we review derived de Rham complex of simplicial A -algebras relative to A .

First we want to spell out explicitly the process of freely adjoining divided powers or delta powers of elements mentioned in [BS19, subsections 2.5-2.6 and section 3].

Construction 2.6. (0) Recall I is locally generated by a nonzerodivisor in A . Let A_i be an affine open cover of $\mathrm{Spf}(A)$ such that $I \cdot A_i = (d_i)$ where $d_i \in I$. There is an A -algebra $A[I \cdot x]$ by glueing $A_i[x_i]$'s via $x_i = \frac{d_i}{d_j} x_j$, it has a surjection $A[I \cdot x] \twoheadrightarrow A$ by glueing maps $x_i \mapsto d_i$. Alternatively one may directly define

$$A[I \cdot x] := \bigoplus_{n \geq 0} I^n$$

with the evident surjection being the natural inclusion on each factor. It can also be seen as the ring of functions on the total space of the line bundle I^{-1} on $\mathrm{Spec}(A)$.

Similarly there is a δ - A -algebra $A\{I \cdot x\}$ by glueing $A_i\{x_i\}$'s via $A_i\{x_i\} \otimes_{A_i} A_{ij} \xrightarrow{x_i \mapsto \frac{d_i}{d_j} x_j} A_j\{x_j\} \otimes_{A_j} A_{ij}$ with a surjection $A\{I \cdot x\} \twoheadrightarrow A$ by glueing maps $x_i \mapsto d_i$. Alternatively one may directly define

$$A\{I \cdot x\} := \bigotimes_A \left(\bigoplus_{n \geq 0} (\delta^n(I))^n \right)$$

with the evident surjection being the natural map on each tensor factor. This can also be seen as the ring of functions on the total space of an infinite rank vector bundle on $\mathrm{Spec}(A)$.

Note that the above construction can be generalized to the case where I is replaced by a line bundle \mathcal{L} on $\mathrm{Spec}(A)$. In particular one can make sense of $A\{I^{-1} \cdot x\}$ and $A\{\varphi(I) \cdot x\}$. We remark that there is a natural map $A\{x\} \rightarrow A\{I^{-1} \cdot y\}$ by glueing the maps $x \mapsto d_i y_i$ which we short hand as $x \mapsto y$.

(1) Let B be an A -algebra, let f_1, \dots, f_r be a finite set of elements in B . The *simplicial B -algebra obtained by freely adjoining divided powers of f_i* is denoted by $B\langle\langle f_i \rangle\rangle$ and defined to be the derived tensor of the following:

$$\begin{array}{ccc} \mathbb{Z}[x_1, \dots, x_r] & \xrightarrow{x_i \mapsto f_i} & B \\ \downarrow & & \\ D_{\mathbb{Z}[x_1, \dots, x_r]}(x_1, \dots, x_r). & & \end{array}$$

The *simplicial A -algebra obtained by freely adjoining divided powers of I* is denoted by $A\langle\langle I \rangle\rangle$ and defined to be the derived tensor of the following:

$$\begin{array}{ccc} A[I \cdot x] & \xrightarrow{x_i \mapsto d_i} & A \\ \downarrow & & \\ D_{A[I \cdot x]}(\ker(A[I \cdot x] \twoheadrightarrow A)), & & \end{array}$$

alternatively one may define it as the glueing of the simplicial A -algebras $A_i \otimes_{x \mapsto d_i, A[x]} D_{A[x]}(x)$.

The *simplicial B -algebra obtained by freely adjoining divided powers of I, f_i* , denoted by $B\langle\langle I, f_i \rangle\rangle$, is defined as the derived tensor of the above two algebras over A .

(2) Let B be a δ - A -algebra, let f_1, \dots, f_r be a finite set of elements in B . We define $B\{\frac{f_i}{p}\}$ as derived pushout of the following diagram of simplicial algebras:

$$\begin{array}{ccc} A\{x_1, \dots, x_r\} & \xrightarrow{x_i \mapsto f_i} & B \\ \downarrow x_i \mapsto p \cdot y_i & & \\ A\{y_1, \dots, y_r\}. & & \end{array}$$

We define $A\{\varphi(I)/p\}$ as derived pushout of the following:

$$\begin{array}{ccc} A\{I \cdot x\} & \xrightarrow{\varphi} & A \\ \downarrow x \mapsto p \cdot y & & \\ A\{I \cdot y\}, & & \end{array}$$

alternatively one may define it as the glueing of the simplicial δ - A -algebras $A_i\{\frac{\varphi_A(d_i)}{p}\}$.

Analogously $B\{\frac{\varphi(I)}{p}, \frac{f_i}{p}\}$ is defined as derived tensoring the above two algebras over A .

(3) Given a sequence (f_1, \dots, f_r) of elements inside a ring B , we use notation

$$\mathrm{dR}_B(f_1, \dots, f_r)^\wedge := \mathrm{dR}_{\mathrm{Kos}(B; f_1, \dots, f_r)/B}^\wedge$$

to denote the derived p -completed derived de Rham complex of $\mathrm{Kos}(B; f_1, \dots, f_r)$, viewed as a simplicial B -algebra, over B .

Similarly when B is an A -algebra, we denote

$$\mathrm{dR}_B(I)^\wedge := \mathrm{dR}_{(B \otimes_A (A/I))/B}^\wedge,$$

and

$$\mathrm{dR}_B(I, f_i)^\wedge := \mathrm{dR}_{(\mathrm{Kos}(B; f_1, \dots, f_r) \otimes_A (A/I))/B}^\wedge.$$

Let J be an ideal inside B , then we denote

$$\mathrm{dR}_B(J)^\wedge := \mathrm{dR}_{(B/J)/B}^\wedge.$$

Here all the completion are derived p -completion.

Remark 2.7. (1) Let $B = A\{x\}^\wedge$, note that x is (p, I) -completely regular relative to A . Using [BS19, Proposition 3.13], we can get a B -algebra $C := B\{\frac{x}{p}\}^\wedge$ which is locally (on $\mathrm{Spf}(A)$ as one needs to trivialize the line bundle I) given by $C = A\{y\}^\wedge$ together with B -algebra structure $x \mapsto d \cdot y$ where d is the local generator of I . One checks immediately that, in our notation here, we have $C \cong A\{I^{-1} \cdot y\}^\wedge$ with B -structure given by (p, I) -completion of $x \mapsto y$.

In fact, after examining the proof of [BS19, Proposition 3.13], one finds that in the situation described in loc. cit. , the algebra $B\{\frac{J}{p}\}^\wedge$ is the derived (p, I) -complete pushout of the following diagram:

$$\begin{array}{ccc} A\{x_1, \dots, x_r\} & \longrightarrow & B \\ \downarrow x_i \mapsto y_i & & \\ A\{I^{-1} \cdot y_i\}, & & \end{array}$$

(2) We warn readers that when $J = (f_1, \dots, f_r)$ is an ideal inside B , the two simplicial B -algebras $\mathrm{dR}_B(J)^\wedge$ and $\mathrm{dR}_B(f_1, \dots, f_r)^\wedge$ are usually different. These two agree when (f_i) is a p -completely Koszul regular sequence.

Below we shall see the relation between derived de Rham complex, divided power envelopes, and prismatic envelopes which directly follows from [BS19, Subsection 2.5].

Lemma 2.8. (1) Let B be an A -algebra, let $\{f_1, \dots, f_r\}$ be a finite set of elements of B . Then we have the following identification of derived p -complete simplicial B -algebras:

$$\mathrm{dR}_B(f_1, \dots, f_r)^\wedge \cong B\langle\langle f_i \rangle\rangle^\wedge.$$

Similarly we have an identification:

$$\mathrm{dR}_B(I)^\wedge \cong B\langle\langle I \rangle\rangle^\wedge.$$

(2) Let B be a δ - A -algebra, let $\{f_1, \dots, f_r\}$ be a finite set of elements of B . Then we have the following identification of derived p -complete simplicial B -algebras:

$$B\langle\langle f_i \rangle\rangle^\wedge \cong B\left\{\frac{\varphi(f_i)}{p}\right\}^\wedge.$$

Similarly we have an identification:

$$B\langle\langle I \rangle\rangle^\wedge \cong B\left\{\frac{\varphi(I)}{p}\right\}^\wedge.$$

Proof. By base change property of their constructions, we reduce ourselves to the case where $B = A\{x_1, \dots, x_r\}$ with $f_i = x_i$. Again by base change we may assume A is the initial oriented prism, in particular it is flat over \mathbb{Z}_p and $I = (d)$ is generated by a nonzerodivisor. So we can focus on the case concerning finite set of elements of B , we may further reduce to the case where the set is a singleton.

Now the identification in (1) follows from (the limit version of) [Bha12, Theorem 3.27] and [Ber74, Théorème V.2.3.2]. The identification in (2) follows from [BS19, Lemma 2.36]. \square

We deduce a consequence concerning the Tor amplitude of $dR_A(I)^\wedge$ over A , generalizing Lemma 2.5.

Lemma 2.9. *Let (A, I) be a prism. Then $A \rightarrow dR_A(I)^\wedge$ has p -complete amplitude in $[-1, 0]$, in particular p -completely base changing along $A \rightarrow dR_A(I)^\wedge$ commutes with taking totalizations in $D^{\geq 0}(A)$.*

Proof. We may check this statement locally on $\mathrm{Spf}(A)$, hence we may assume $I = (d)$. Next, by base change, we may assume A to be the initial oriented prism, in particular we may assume it to be transversal. Using Lemma 2.8 (1), we see now $dR_A(I)^\wedge$ is the p -completion of the divided power envelope $D_A(I)^\wedge$. This reduces the Lemma to Lemma 2.5. \square

We also have a prototype base change formula which will be used in the next section to establish a general comparison.

Lemma 2.10. *Let (A, I) be a prism, denote the composition $A\{x\} \xrightarrow{\varphi_A, x \mapsto \varphi(z)} A\{z\} \rightarrow dR_{A\{z\}}(I)^\wedge$ by f . Then we have a base change formula:*

$$A\{I^{-1} \cdot x\} \widehat{\otimes}_{A\{x\}, f} dR_{A\{z\}}(I)^\wedge \cong dR_{A\{z\}}(I, z)^\wedge,$$

Here the completion on the left hand side is derived p -completion. As $\varphi(I) = (p)$ inside $\pi_0(dR_{A\{z\}}(I)^\wedge)$, it is the same as derived (p, I) -completion when viewed as an A -complex via $\varphi_A: A \rightarrow dR_A(I)^\wedge$.

Proof. Note that by Lemma 2.8 we have identifications $dR_{A\{z\}}(I)^\wedge \cong A\{z\}\{\frac{\varphi(I)}{p}\}^\wedge$ as p -complete simplicial $A\{z\}$ -algebras. Similarly we can identify $dR_{A\{z\}}(I, z)^\wedge$ with $A\{z\}\{\frac{\varphi(z)}{p}, \frac{\varphi(I)}{p}\}^\wedge$.

Now we look at the following diagram

$$\begin{array}{ccccc} A\{x\} & \longrightarrow & A\{z\} & \longrightarrow & A\{z\}\{\frac{\varphi(I)}{p}\} \\ \downarrow & & \downarrow & & \downarrow \\ A\{I^{-1} \cdot x\} & \longrightarrow & A\{z\}\{\frac{\varphi(z)}{\varphi(I)}\} & \longrightarrow & A\{z\}\{\frac{\varphi(z)}{p}, \frac{\varphi(I)}{p}\}. \end{array}$$

The left square is a pushout diagram by definition. Hence it suffices to show the right square, after derived p -completion, is also a pushout diagram of p -complete simplicial $A\{z\}$ -algebras.

To that end, we may work Zariski locally on A , so we can assume $I = (d)$ is generated by one element. This square is the base change of the same diagram when A is the initial oriented prism, so we have reduced ourselves to that case. Now every ring in sight is discrete, and the p -completed square is a pushout diagram because $\varphi(d)$ and p differ by a unit inside $A\{\frac{\varphi(d)}{p}\}^\wedge \cong D_A(d)^\wedge$. \square

In [Bha12, Proposition 3.25], for any p -complete A -algebra B , Bhatt constructed a natural map

$$\mathrm{Comp}_{B/A}: dR_{B/A}^\wedge \rightarrow \mathrm{R}\Gamma_{\mathrm{crys}}(B/A).$$

Here the right hand side denotes the p -complete crystalline cohomology defined using PD thickenings of B relative to $(A, (p), \gamma)$ with $\gamma_i(p) = p^i/i!$. This natural map is functorial in $A \rightarrow B$ and agrees with Berthelot's de Rham–crystalline comparison [Ber74, Théorème IV.2.3.2] when it is formally smooth (viewed as p -adic algebras). It is shown that when both A and B are flat over \mathbb{Z}_p and $A \rightarrow B$ is p -completely locally complete intersection, then the natural map above is an isomorphism [Bha12, Theorem 3.27].

For our purpose we shall be interested in the situation where B is formally smooth over A/I , we cannot summon the above Theorem in loc. cit. to say that the natural map in this situation is an isomorphism. In fact, when $B = A/I$ the left hand side is $dR_A(I)^\wedge$ and the right hand side is the classical p -adic completion of the PD envelope of A along I (compatible with the natural PD structure on $(A, (p))$), denoted as \mathcal{A} ². These two need not be the same, e.g. take $A = \mathbb{Z}_p$ and $I = (p)$, then $dR_A(I)^\wedge = \mathbb{Z}_p[[T^i/i!]]^\wedge/(T-p)$ but $\mathcal{A} = \mathbb{Z}_p$. However this turns out to be the only problem.

Proposition 2.11. *Let B be an A/I -algebra.*

(1) *If B is formally smooth over A/I , then we have a natural identification*

$$R\Gamma_{\text{crys}}(B/A) \cong R\Gamma_{\text{crys}}(B/\mathcal{A}),$$

where the right hand side is the usual crystalline cohomology of $\text{Spf}(B)$ over the PD base \mathcal{A} .

(2) *There is a natural map*

$$\text{Comp}_{B/A}: dR_{B/A}^\wedge \widehat{\otimes}_{dR_A(I)^\wedge} \mathcal{A} \rightarrow R\Gamma_{\text{crys}}(B/\mathcal{A}),$$

which is functorial in $A/I \rightarrow B$.

(3) *If B is formally smooth over A/I , then the above is an isomorphism.*

Proof. (1) is an easy consequence of the fact that B is an A/I -algebra. In fact we only need $A/I \rightarrow B$ to be a local complete intersection. Indeed we use Čech–Alexander complex to compute both crystalline cohomology, and one reduces to the following: Let P be a polynomial A -algebra with a surjection $P \twoheadrightarrow B$ of A -algebras, then there is a naturally induced surjection $P \otimes_A \mathcal{A} \twoheadrightarrow B$ of \mathcal{A} -algebras, and we have an identification of PD envelopes

$$D_{(A,(p),\gamma)}(P \twoheadrightarrow B) = D_{(A,\mathcal{I},\gamma)}(P \otimes_A \mathcal{A} \twoheadrightarrow B).$$

(2) The functoriality of Bhatt’s $\text{Comp}_{B/A}$ asserts that the map is compatible with the natural map $dR_A(I)^\wedge \rightarrow \mathcal{A}$, hence we get our natural map $\text{Comp}_{B/A}$.

(3) Choose a formal lift \tilde{B} over A (note that A is (p, I) -complete). By the functoriality of Bhatt’s $\text{Comp}_{B/A}$, we get the following commutative diagram:

$$\begin{array}{ccc} dR_{\tilde{B}/A}^\wedge \widehat{\otimes}_A \mathcal{A} & \longrightarrow & R\Gamma_{\text{crys}}(\tilde{B}/A) \widehat{\otimes}_A \mathcal{A} \\ \downarrow & & \downarrow \\ dR_{B/A}^\wedge \widehat{\otimes}_{dR_A(I)^\wedge} \mathcal{A} & \longrightarrow & R\Gamma_{\text{crys}}(B/\mathcal{A}). \end{array}$$

The top horizontal arrow is an isomorphism by Berthelot’s de Rham–crystalline comparison. The left vertical arrow is an isomorphism by the Künneth formula of derived de Rham complex: $dR_{\tilde{B}/A}^\wedge \widehat{\otimes}_A dR_A(I)^\wedge \cong dR_{B/A}^\wedge$. The right vertical arrow is an isomorphism by base change formula of crystalline cohomology. Therefore we conclude that the bottom horizontal arrow, which is our $\text{Comp}_{B/A}$, must also be an isomorphism. \square

The above proposition and Bhatt’s results discussed before suggest that derived de Rham complex is a substitute of crystalline cohomology. Inspired by this philosophy, below let us show that derived de Rham complex only “depends on the reduction mod p of the input algebra”. We need to introduce some notations first: denote the p -adic derived de Rham complex $dR_{\mathbb{F}_p/\mathbb{Z}_p}^\wedge$ by D . Bhatt’s result implies that the natural map $\mathbb{Z}_p \rightarrow D$ admits a retraction $D \rightarrow \mathbb{Z}_p$. In Example 2.16 (1) below, one finds a detailed description of D .

Remark 2.12. In fact, one can show that D is the p -complete PD envelope of \mathbb{Z}_p along the ideal (p) by D . Moreover under this identification one can easily see that the retraction above is unique, and is given by the fact that there is a unique PD structure on $(\mathbb{Z}_p, (p))$ (as \mathbb{Z}_p has no p -torsion). Notice that when taking PD envelope, one has to fix a PD base ring, and we always take it to be the trivial PD ring $(\mathbb{Z}_p, (0), \gamma_{\text{triv}})$ when we say PD envelope without mentioning a PD base ring.

²This notation agrees with the previous subsection as we assumed (A, I) to be a transversal prism there.

Proposition 2.13. *Let R be a ring with its derived p -completion R^\wedge , let B be a simplicial R -algebra. Then there is a natural isomorphism:*

$$dR_{\text{Kos}(B;p)/R}^\wedge \widehat{\otimes}_D \mathbb{Z}_p \cong dR_{B/R}^\wedge$$

which is functorial in $R \rightarrow B$.

Here the map $D \rightarrow dR_{\text{Kos}(B;p)/R}^\wedge$ is induced by the following natural diagram

$$\begin{array}{ccc} \mathbb{F}_p & \longrightarrow & \text{Kos}(B;p) = B \otimes_{\mathbb{Z}} \mathbb{F}_p \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & R. \end{array}$$

Proof. This follows from the Künneth formula of derived de Rham complex:

$$dR_{\text{Kos}(B;p)/R}^\wedge \cong dR_{B/R}^\wedge \widehat{\otimes}_R dR_R(p)^\wedge,$$

and the base change formula $dR_R(p)^\wedge \cong D \widehat{\otimes}_{\mathbb{Z}_p} R^\wedge$ as $\text{Kos}(R;p) = R \otimes_{\mathbb{Z}} \mathbb{F}_p$. \square

2.3. Frobenii. Let A be a p -torsionfree δ -ring. Using Proposition 2.13 we can define a Frobenius action on $dR_{B/A}^\wedge$ which is functorial in (A, φ_A) and the A -algebra B .

Construction 2.14. Let A be a p -torsionfree δ -ring and B a simplicial A -algebra. Recall there is a functorial endomorphism on simplicial \mathbb{F}_p -algebras given by left Kan extending the usual Frobenius on polynomial \mathbb{F}_p -algebras, see [Lur11, Construction 2.2.6]. For discrete \mathbb{F}_p -algebras, it is just the usual Frobenius. We may view $B/p = B \otimes_A A/p$, using the fact that φ_A on A is a lift of Frobenius on A/p we get the following commutative diagram:

$$\begin{array}{ccc} B/p & \xrightarrow{\varphi_{B/p}} & B/p \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi_A} & A \end{array}$$

it induces a Frobenius map $\tilde{\varphi}: dR_{\text{Kos}(B;p)/A}^\wedge \rightarrow dR_{\text{Kos}(B;p)/A}^\wedge$ which is functorial in $(A \rightarrow B, \varphi_A)$.

Similar diagram for $\mathbb{Z} \rightarrow \mathbb{F}_p$ (where $A = B = \mathbb{Z}_p$) induces identity on D , hence we have a commutative diagram:

$$\begin{array}{ccc} dR_{\text{Kos}(B;p)/A}^\wedge & \xrightarrow{\tilde{\varphi}} & dR_{\text{Kos}(B;p)/A}^\wedge \\ & \searrow & \nearrow \\ & D & \end{array}$$

Finally we define a Frobenius map $\varphi_{B/A}: dR_{\text{Kos}(B;p)/A}^\wedge \widehat{\otimes}_D \mathbb{Z}_p \cong dR_{B/A}^\wedge \xrightarrow{\tilde{\varphi} \widehat{\otimes}_{\text{id}_D} \text{id}_{\mathbb{Z}_p}} dR_{B/A}^\wedge$ which is functorial in $(A \rightarrow B, \varphi_A)$.

Remark 2.15. (1) It is conceivable that the above works for general δ -rings. In private communication we learned from Bhatt that a δ -structure on a ring A is equivalent to specifying a commutative diagram as follows:

$$\begin{array}{ccc} A/p & \xrightarrow{\varphi_{A/p}} & A/p \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi_A} & A \end{array}$$

note that here A/p is a simplicial \mathbb{F}_p -algebra that has nontrivial π_1 when A is not p -torsionfree. Hence for any simplicial A -algebra B , one can also define a Frobenius on $dR_{B/A}^\wedge$ as above. However we do not work out the full story here as we do not need this great generality for our intended applications later.

(2) By letting $n \rightarrow \infty$ in [Bha12, Proposition 3.47], one gets another construction of Frobenius on $dR_{A/\mathbb{Z}_p}^\wedge$ for any \mathbb{Z}_p -algebra A . However later on we shall see in Remark 3.15 that there is only one Frobenius that is

functorial enough (in a suitable sense) on p -completed derived de Rham complexes when the base algebra is a p -torsionfree δ -algebra. In particular, our construction above agrees with Bhatt's whenever both are defined (i.e. when the base is \mathbb{Z}_p).

Let us work out some examples.

Example 2.16. (1) As an illustrative example, let us contemplate with $A = \mathbb{Z}_p$ and $B = \mathbb{F}_p$. We have a derived pushout square of rings:

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & B \\ T \mapsto p \uparrow & & \uparrow \\ \mathbb{Z}_p[T] & \xrightarrow{T \mapsto 0} & A \end{array}$$

moreover the bottom map is a map of δ -rings if we give $\mathbb{Z}_p[T]$ a δ -structure with $\varphi(T) = T^p$. Then we get a pushout diagram of derived de Rham complex which says $D \cong dR_{\mathbb{Z}_p/\mathbb{Z}_p[T]}^{\wedge} \widehat{\otimes}_{\mathbb{Z}_p[T]} \mathbb{Z}_p$. The latter is the same as $\mathbb{Z}_p \langle\langle T \rangle\rangle^{\wedge} / (T)$ where we have used the fact that p has divided powers in \mathbb{Z}_p (hence adjoining divided powers of $T - p$ is the same as adjoining divided powers of T). It is easy to see that the Frobenius defined on $dR_{\mathbb{Z}_p/\mathbb{Z}_p[T]}^{\wedge} \cong \mathbb{Z}_p \langle\langle T \rangle\rangle$ is induced by $T \mapsto T^p$ because it has to be compatible with the Frobenius on $\mathbb{Z}_p[T]$. Therefore the induced Frobenius on $dR_{B/A}^{\wedge}$ is *not* the identity. This might be surprising as one would naïvely think that the Frobenius on the pair $(\mathbb{Z}_p, \mathbb{F}_p)$ is identity, hence must induce identity on the derived de Rham complex. However the Frobenius on $\mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is *not* the identity (as Frobenius always kills cohomology classes in negative degrees, see [Lur11, Remark 2.2.7]), and it is this Frobenius that induces a map on the derived de Rham complex. On a related note, Bhatt has pointed out to us that the identity map is also *not* a lift of Frobenius on $D \cong \mathbb{Z}_p \langle\langle T \rangle\rangle^{\wedge} / (T)$.

(2) Let $J \subset A$ be an ideal which is Zariski locally on $\text{Spec}(A)$ a colimit of ideals generated by a p -completely regular sequence. Then by Lemma 2.8 (1), we have an identification: $dR_A(J)^{\wedge} \cong D_A(J)^{\wedge}$. Since the Frobenius map obtained is compatible with φ_A and $D_A(J)^{\wedge}$ is p -torsionfree, we see that this pins down the Frobenius on $dR_A(J)^{\wedge}$: any $\gamma_n(f)$ with $f \in J$ must be sent to $\frac{\varphi_A(f)^n}{n!}$. Note that f^p is divisible by p in $D_A(J)^{\wedge}$, hence $\varphi_A(f)$ is divisible by p in $D_A(J)^{\wedge}$.

(3) Let A be p -complete, and let $B = A\langle X^{1/p^\infty} \rangle$. Since $A \rightarrow B$ is relatively perfect modulo p , there is a unique lift of Frobenius φ_B on B covering the Frobenius on A and it is given by $\varphi_B(X^i) = X^{i \cdot p}$. By [GL21, Proposition 3.4.(1)], we see the natural map to 0-th graded piece of Hodge filtration induces an isomorphism $dR_{B/A}^{\wedge} \cong B$. Applying the functoriality of the Construction 2.14 to the map of triples: $(A \rightarrow B, \varphi_A) \rightarrow (B \rightarrow B, \varphi_B)$, we see that the Frobenius on $dR_{B/A}^{\wedge} \cong B$ must be φ_B .

When the map $A \rightarrow B$ is a surjection with good regularity properties, we see in Lemma 2.8 that one can express $dR_{B/A}^{\wedge}$ in terms of prismatic envelopes. Since prismatic envelopes are δ -rings, they possess a Frobenius map by design. We can use this to give an alternative construction of the Frobenius for derived de Rham cohomology of certain regular A -algebras relative to A . To that end, we need to first establish a sheaf property for derived de Rham cohomology.

Proposition 2.17. *Let S be an R -algebra. Assume:*

- *the cotangent complex $\mathbb{L}_{S/R} \in D(S)$ has p -completely Tor amplitude in $[-1, 0]$;*
- *the (relative to R/p) Frobenius twist of S/p is in $D^{\geq -m}(\mathbb{F}_p)$.*

Consider the category \mathcal{C} consisting of triangles $R \rightarrow P \rightarrow S$ with P being an ind-polynomial R -algebra, equipped with indiscrete topology. Let $dR_{S/-}^{\wedge}$ be the sheaf that associates any triangle $R \rightarrow P \rightarrow S$ with $dR_{S/P}^{\wedge}$. Then we have:

- (1) *For any $R \rightarrow P \rightarrow S$, the $dR_{S/P}^{\wedge}$ is in $D^{\geq -m}(R)$.*
- (2) *The natural map $dR_{S/R}^{\wedge} \rightarrow \lim_{\mathcal{C}} dR_{S/P}^{\wedge}$ is an isomorphism.*
- (3) *For any $R \rightarrow P \rightarrow S$ with $P \twoheadrightarrow S$ surjective, the natural map $dR_{S/R}^{\wedge} \rightarrow \lim_{\Delta} dR_{S/P_{\bullet}}^{\wedge}$ is an isomorphism. Here $P_n := P^{\otimes_R(n+1)}$ for any $[n] \in \Delta$, with induced maps $P_n \twoheadrightarrow S$.*

Proof. We shall prove this by reduction modulo p . Hence we may assume R and S are simplicial \mathbb{F}_p -algebras.

For (1) we use the conjugate filtrations on the derived de Rham complex. Since $\mathbb{L}_{S/R}$ has Tor amplitude in $[-1, 0]$, so is $\mathbb{L}_{S^{(1,P)}/P}$ where $S^{(1,P)}$ is the (relative to P) Frobenius twist of S . The above estimate shows that the graded pieces of the conjugate filtration has Tor amplitude at least 0 over $S^{(1,P)}$. Since $S^{(1)}$ is assumed to be in $D^{\geq -m}(\mathbb{F}_p)$ and the relative Frobenius for P is flat, we see that all the graded pieces of the conjugate filtration lives in $D^{\geq -m}(R)$.

Note that $P \twoheadrightarrow S$ is surjective if and only if $R \rightarrow P \rightarrow S$ is weakly final in \mathcal{C} . Since these $dR_{S/P}^\wedge$ are cohomologically uniformly bounded below, [Bha18a, Lecture V, Lemma 4.3] (see also [Sta20, Tag 07JM]) reduces (2) to (3).

Lastly to show (3) we appeal to the conjugate filtration again. Since the graded pieces of the conjugate filtration is cohomologically uniformly bounded below by our proof of (1) above, it suffices to show $\mathbb{L}_{S^{(1,n)}/R} \rightarrow \lim_\Delta \mathbb{L}_{S^{(1,\bullet)}/P_\bullet}$ is an isomorphism, where $S^{(1,n)}$ is the (relative to P_n) Frobenius twist of S . This follows easily from the fact that $\lim_\Delta \mathbb{L}_{P_\bullet/R} \cong 0$. \square

The above Proposition gives us a way to describe the Frobenius action of the p -completed derived de Rham complex in more cases than those listed in Example 2.16.

Proposition 2.18. *Let A be a p -torsionfree p -complete δ -algebra, and let $I \subset A$ be an ideal which is Zariski locally on $\text{Spec}(A)$ generated by a p -completely regular element. Let B be a p -completely smooth A/I -algebra. Then we have:*

- (1) *For any (p, I) -completely ind-polynomial A -algebra P with a surjection $P \twoheadrightarrow B$, the kernel J is Zariski locally on $\text{Spf}(P)$ colimit of ideals generated by a p -completely regular sequence.*
- (2) *For any such $A \rightarrow P \rightarrow B$ as in (1), the $dR_{B/P}^\wedge$ is an ordinary algebra.*
- (3) *For any (p, I) -completely free δ - A -algebra F with a surjection $F \twoheadrightarrow B$, there is a unique δ -algebra structure on $dR_{B/F}^\wedge$ compatible with that on F . With this δ -structure, we have an identification:*

$$dR_{B/F}^\wedge \cong F\left\{\frac{\varphi_F(J)}{p}\right\}^\wedge.$$

- (4) *Consider the category \mathcal{C} of all triples $A \rightarrow F \twoheadrightarrow B$ as in (3), we have*

$$dR_{B/A}^\wedge \cong \lim_{\mathcal{C}} dR_{B/F}^\wedge.$$

In fact, it suffices to take limit over the Cech nerve of one such $F \twoheadrightarrow B$. Together with (3) we get a natural Frobenius action on $dR_{B/A}^\wedge$.

- (5) *The Frobenius on $dR_{B/A}^\wedge$ obtained in (4) agrees with the one in Construction 2.14.*

The notation $F\left\{\frac{\varphi_F(J)}{p}\right\}^\wedge$ is defined analogously as in [BS19, Corollary 3.14]. Using J is Zariski locally given by an ind- p -completely regular ideal, we may define $F\left\{\frac{\varphi_F(J)}{p}\right\}^\wedge$ as the glueing of the colimit of $F\left\{\frac{\varphi_F(f_i)}{p}\right\}^\wedge$, where (f_i) is the ind-regular sequence generating J on a Zariski open.

Proof. (1) follows easily from the fact that B is formally smooth over A/I and I is Zariski locally generated by a p -completely regular element.

(2) follows from the argument of Proposition 2.17 (1). Indeed we set $R = A$ and $S = B$. The Frobenius twist of B/p is smooth over $A/(\varphi_A(I), p) = A/(I^p, p)$, and the latter is an ordinary algebra. Hence in our situation, we have $m = 0$ in the condition of Proposition 2.17. This shows that the $dR_{B/P}^\wedge$ is in $D^{\geq 0}$. Using conjugate filtration again, it is easy to see that the p -completed derived de Rham complex of any surjection must be in $D^{\leq 0}$. Hence our $dR_{B/P}^\wedge$ must in fact be an ordinary algebra.

(3) essentially follows from (1) and Lemma 2.8. Indeed by description of J , we see that $dR_{B/F}^\wedge \cong D_F(J)^\wedge$. Since J is Zariski locally an ind- p -completely regular ideal, we see that $D_F(J)^\wedge$ is p -torsionfree, hence having a δ -structure is equivalent to having a lift of Frobenius. The argument in Example 2.16 (2) tells us that there is at most one Frobenius structure on it compatible with that on F . Lastly Lemma 2.8 shows that we can put a δ -structure on it by identifying

$$dR_{B/F}^\wedge \cong D_F(J)^\wedge \cong F\left\{\frac{\varphi_F(J)}{p}\right\}.$$

(4) follows from Proposition 2.17 (2)-(3).

As for (5), it suffices to notice that for any of these $A \rightarrow F \rightarrow B$ the two Frobenii defined on $dR_{B/F}^\wedge$ agree and they are both functorial in $A \rightarrow F \rightarrow B$. \square

The following is similar to Proposition 2.17, and will be used later in the next section.

Proposition 2.19. *Let (A, I) be a bounded prism. Let R be a formally smooth A/I -algebra. Consider \mathcal{C} the category of all triples $A \rightarrow P \rightarrow R$ where P is a p -completed polynomial algebra over A . Associated with such a triple is the following diagram:*

$$\begin{array}{ccccc} A & \longrightarrow & P & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ A/I & \longrightarrow & R & \longrightarrow & S, \end{array}$$

where F is the p -completed free δ - A -algebra associated with P , and S is the p -completed tensor product $S := R \widehat{\otimes}_P F$. Then we have:

- (1) Choose an object $A \rightarrow P \rightarrow R$, consider the n -th self-fiber product $A \rightarrow P^n := P^{\widehat{\otimes}_{A^n}} \rightarrow R$ for any positive integer n . Then the associated p -completed free δ - A -algebra is $F^n := F^{\widehat{\otimes}_{A^n}}$, and we have

$$R \widehat{\otimes}_{P^n} F^n \cong S^{\widehat{\otimes}_{R^n}},$$

which we shall denote by S^n below.

- (2) Choose an object $A \rightarrow P \rightarrow R$, then the natural map

$$dR_{R/A}^\wedge \rightarrow \lim_{[n] \in \Delta} dR_{S^n/F^n}^\wedge$$

is an isomorphism.

- (3) The natural map

$$dR_{R/A}^\wedge \rightarrow \lim_{\mathcal{C}} dR_{S/F}^\wedge$$

is an isomorphism.

Notice that we do not need to assume A to be p -torsionfree here.

Proof. For (1): if P is p -completely adjoin a set T of variables, then F is p -completely adjoin the set $\coprod_{\mathbb{N}} T$ of variables, where t in the i -th component represents $\delta^i(x_t)$. The statement on fiber product and the associated F^n is clear. As for the statement about S^n , just notice that we have the following pushout diagrams:

$$\begin{array}{ccccc} P^n & \longrightarrow & R^n := R^{\widehat{\otimes}_{A^n}} & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ F^n & \longrightarrow & S^{\widehat{\otimes}_{A^n}} & \longrightarrow & S^n := S^{\widehat{\otimes}_{R^n}}. \end{array}$$

To prove (2), we may reduce modulo p . Note that $A \rightarrow F$ and $R \rightarrow S$ are p -completely faithfully flat. In a similar manner to the proof of Proposition 2.17 (3), using conjugate filtration, plus the distinguished triangle of cotangent complex, and fpqc descent of cotangent complex (see [BMS19, Theorem 3.1]), one can show this natural map is an isomorphism.

- (3) follows from (2) in the same way as how Proposition 2.17 (2) follows from Proposition 2.17 (3). \square

Remark 2.20. Similar to Proposition 2.18, assume A to be p -torsionfree, then these $dR_{S/F}^\wedge$ appeared above are discrete rings, and we can equip them a natural δ -structure. By the same proof of Proposition 2.18 the induced Frobenius on $dR_{R/A}^\wedge$ agrees with the one provided by Construction 2.14.

Later on we shall see in Remark 3.15 (1) that if (A, I) is a transversal prism, then there is only one Frobenius in a strong sense. So all these different constructions must give rise to the same map.

2.4. Naïve comparison. Consider the composition $f: A \xrightarrow{\varphi_A} A \rightarrow \mathcal{A}$, it induces a morphism of prisms which we still denote by $f: (A, I) \rightarrow (\mathcal{A}, (p))$. Let \mathcal{X} be a p -completely smooth affine formal scheme over $\mathrm{Spf}(A/I)$. Now by base change formula of prismatic cohomology [BS19, Theorem 1.8.(5)], we have

$$\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/A)\widehat{\otimes}_{A,f}\mathcal{A} \cong \mathrm{R}\Gamma_{\Delta}(\mathcal{Y}/\mathcal{A}),$$

where $\mathcal{Y} = \mathcal{X} \times_{\mathrm{Spf}(A/I), f} \mathrm{Spec}(\mathcal{A}/p)$.

Then the crystalline comparison of prismatic cohomology [BS19, Theorem 1.8.(1)] gives us

$$(\square) \quad \varphi_{\mathcal{A}}^*(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/A)\widehat{\otimes}_{A,f}\mathcal{A}) \cong \varphi_{\mathcal{A}}^*(\mathrm{R}\Gamma_{\Delta}(\mathcal{Y}/\mathcal{A})) \cong \mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{Y}/\mathcal{A}) \cong \varphi_{\mathcal{A}}^*(\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{X}/\mathcal{A})).$$

Here the last isomorphism comes from the following commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \twoheadrightarrow & A/(I, p) \\ \varphi_{\mathcal{A}} \downarrow & & \downarrow f \\ \mathcal{A} & \twoheadrightarrow & \mathcal{A}/p. \end{array}$$

In the following, we aim at getting a Frobenius descent of the isomorphism obtained in \square , see Remark 3.8.

3. COMPARING PRISMATIC AND DERIVED DE RHAM COHOMOLOGY

Let (A, I) be a bounded prism. Let X be a p -adic formal scheme which is formally smooth over $\mathrm{Spf}(A/I)$. In this section we shall establish a functorial comparison between the prismatic cohomology $\mathrm{R}\Gamma_{\Delta}(X/A)$ with the derived de Rham cohomology $\mathrm{dR}_{X/A}^{\wedge}$.

3.1. The comparison. In the beginning of this subsection we need to comment on an error in the construction of Čech–Alexander complex in [BS19, Construction 4.16]. We learned this subtlety from Bhatt who was informed by Koshikawa. The issue is as follows, with notation as in loc. cit.: suppose $D \rightarrow D/ID \leftarrow R$ is an object in $(R/A)_{\Delta}$, then one needs to exhibit a morphism $(B\{\frac{J}{I}\}^{\wedge} \rightarrow D)$ in $(R/A)_{\Delta}$. The argument was along the following line, by universal property it suffices to exhibit a map $B \rightarrow D$ sending J into ID , which is amount to filling in the following dotted arrow (of δ -rings)

$$\begin{array}{ccc} R & \longrightarrow & D/ID \\ \uparrow & & \uparrow \\ B & \dashrightarrow & D \end{array}$$

that makes the diagram commutative. At first sight this seems easy, as B is a free δ -ring in a set of variables, we just lift images of those variables under $B \rightarrow R \rightarrow D/ID$ to D to get a map of δ -rings. But there is no way a general lift will make the above diagram commutative for the δ 's of those variables.

Below we describe a fix that we learned from Bhatt. Recall that the forgetful functor from δ - A -algebras to A -algebras admits a left adjoint, see [BS19, Remark 2.7]. One checks the following easily:

- given a derived (p, I) -completed polynomial A -algebra P which is freely generated by a set of variables, then apply this left adjoint we will get a derived (p, I) -completed free δ - A -algebra F generated by the same set of variables.
- this left adjoint commutes with completed tensor product.

In particular the natural map $P \rightarrow F$ is (p, I) -completely ind-smooth.

Construction 3.1 (Čech–Alexander complex for prismatic cohomology). Let R be a p -completely smooth A/I -algebra. Let P be a derived (p, I) -completed polynomial A -algebra along with a surjection $P \twoheadrightarrow R$, and let J be the kernel. Associated with the triple $A \rightarrow P \twoheadrightarrow R$ is a δ - A -algebra $F\{\frac{JF}{I}\}^{\wedge}$, obtained by applying [BS19, Corollary 3.14]. We make three claims about this construction.

Claim 3.2.

- (1) The δ - A -algebra $F\{\frac{JF}{I}\}^{\wedge}$ is naturally an object in $(R/A)_{\Delta}$;
- (2) as such, it is weakly initial in $(R/A)_{\Delta}$; and

(3) if there is a set of triples $A \rightarrow P_i \twoheadrightarrow R$, then the coproduct of associated $F_i\{\frac{J_i F_i}{I}\}^\wedge$ in $(R/A)_\Delta$ is given by the δ - A -algebra associated with the triple $A \rightarrow \widehat{\otimes}_A P_i \twoheadrightarrow R$ where the second map is given by the completed tensor of those $P_i \twoheadrightarrow R$ maps.

Let us postpone the verification of these claims and continue with the construction. At this point we may simply follow the rest of [BS19, Construction 4.16]. Form the derived (p, I) -completed Čech nerve P^\bullet of $A \rightarrow P$, and let $J^\bullet \subset P^\bullet$ be the kernel of the augmentation map $P^\bullet \rightarrow P \rightarrow R$. By the first claim above, we get a cosimplicial object $(F^\bullet\{\frac{J^\bullet F^\bullet}{I}\}^\wedge)$ in $(R/A)_\Delta$. The third claim above shows that this is the Čech nerve of $F\{\frac{JF}{I}\}^\wedge$ in $(R/A)_\Delta$, and according to the second claim the object $F\{\frac{JF}{I}\}^\wedge$ covers the final object of the topos $\text{Shv}((R/A)_\Delta)$. Therefore $\Delta_{R/A}$ is computed by $F^\bullet\{\frac{J^\bullet F^\bullet}{I}\}^\wedge$.

This construction commutes with base change of the prism (A, I) . When (A, I) is fixed, this construction can be carried out in a way which is strictly functorial in R , by setting P to be the completed polynomial A -algebra generated by the underlying set of R .

Proof of Claim 3.2. Proof of (1): form the following pushout diagram:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ P & \longrightarrow & F \end{array}$$

Denote $F\{\frac{JF}{I}\}^\wedge$ by C^0 , by its defining property there is a natural map $S \cong F/JF \rightarrow C^0/IC^0$. Hence C^0 gives rise to a diagram $(C^0 \rightarrow C^0/IC^0 \leftarrow S \leftarrow R)$ which is an object in $(R/A)_\Delta$.

Proof of (2) and (3): this follows from chasing through universal properties. Let $(D \rightarrow D/ID \leftarrow R)$ be an object in $(R/A)_\Delta$, we have the following chain of equivalences:

$$\begin{aligned} F\{\frac{JF}{I}\}^\wedge \rightarrow D \text{ in } (R/A)_\Delta &\iff \text{a map of } \delta\text{-}A\text{-algebras } F \rightarrow D \text{ such that } JF \text{ is mapped into } ID \\ &\iff \text{a map of } A\text{-algebras } P \rightarrow D \text{ such that } J \text{ is mapped into } ID. \end{aligned}$$

It is easy to see that the last statement is equivalent to filling in the following dotted arrow below

$$\begin{array}{ccc} R & \longrightarrow & D/ID \\ \uparrow & & \uparrow \\ P & \dashrightarrow & D \end{array}$$

as A -algebras, making the diagram commutative. Note that there is no requirement from δ -ring consideration here. Now one checks the claims (2) and (3) easily. \square

With the above preparatory discussion, we are ready to compare prismatic cohomology and derived de Rham cohomology. The key computation we need is the following.

Lemma 3.3 (Comparing prismatic and PD envelopes for regular sequences). *Let B be a (p, I) -completely flat δ - A -algebra, let $f_1, \dots, f_r \in B$ be a (p, I) -completely regular sequence. Write $J = (I, f_1, \dots, f_r) \subset B$. Then we have a natural identification of p -completely flat $dR_A(I)^\wedge$ -algebras:*

$$B\{\frac{J}{I}\}^\wedge \widehat{\otimes}_{B, \varphi_B} B \widehat{\otimes}_A dR_A(I)^\wedge \cong dR_B(J)^\wedge.$$

Here the $B\{\frac{J}{I}\}^\wedge$ is as in [BS19, Proposition 3.13], which is (p, I) -completely flat over A . Let us clarify the various completions involved on the left hand side. First we perform derived (p, I) -complete tensor, then we perform derived $(p, \varphi(I))$ -complete tensor which is the same as derived p -complete tensor as $\varphi(I) = (p)$ in $\pi_0(dR_A(I)^\wedge)$.

Proof. Recall that in the proof of [BS19, Proposition 3.13] and also explained in Remark 2.7, the $B\{\frac{J}{I}\}^\wedge$ is constructed as p -completely pushout the following diagram:

$$\begin{array}{ccc} A\{x_1, \dots, x_r\} & \xrightarrow{x_i \mapsto f_i} & B \\ \downarrow & & \\ A\{x_1, \dots, x_r\}\{\frac{x_i}{I}\}. & & \end{array}$$

The left hand side in this Lemma is therefore given by pushing-out the above diagram further along $f_B: B \xrightarrow{\varphi_B} B \rightarrow B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge$. The composition $A\{x_1, \dots, x_r\} \xrightarrow{x_i \mapsto f_i} B \xrightarrow{f_B} B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge$ can now be factored as $A\{x_1, \dots, x_r\} \xrightarrow{\varphi_A, x_i \mapsto \varphi(z_i)} A\{z_1, \dots, z_r\} \rightarrow \mathrm{dR}_{A\{z_1, \dots, z_r\}}(I)^\wedge \rightarrow B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge$, where in the last map z_i is sent to $f_i \otimes 1$. Hence the left hand side becomes the p -completely outer pushout of the following diagram with solid arrows:

$$\begin{array}{ccccc} A\{x_1, \dots, x_r\} & \longrightarrow & \mathrm{dR}_{A\{z_1, \dots, z_r\}}(I)^\wedge & \longrightarrow & B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \\ \downarrow & & \downarrow & & \downarrow \\ A\{x_1, \dots, x_r\}\{\frac{x_i}{I}\} & \longrightarrow & \mathrm{dR}_{A\{z_1, \dots, z_r\}}(I, z_1, \dots, z_r)^\wedge & \longrightarrow & \mathrm{dR}_{B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge}(f_i \otimes 1)^\wedge \cong \mathrm{dR}_B(J)^\wedge. \end{array}$$

Using (multi-variable version of) Lemma 2.10 we see the left square of the above is a p -completely pushout. Base change property of derived de Rham complex now shows the right square of the above to be a p -completely pushout. Here the isomorphism of the right bottom corner follows from the fact that (I, f_1, \dots, f_r) is a Koszul regular sequence in B . \square

Just like how [BS19, Proposition 3.13] implies [BS19, Corollary 3.14], our Lemma 3.3 above gives us the following.

Lemma 3.4. *Let R be a p -completely smooth A/I -algebra. Let P be a p -completed polynomial algebra over A , and let $P \rightarrow R$ be a surjection of A -algebras with kernel J . Consider the following diagram:*

$$\begin{array}{ccccc} A/I & \longrightarrow & R & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & P & \longrightarrow & F \end{array}$$

where F is the p -completed free δ - A -algebra associated with P , and S is the p -completed tensor product $S := R \widehat{\otimes}_P F$. Then we have a natural identification of p -completely flat $\mathrm{dR}_A(I)^\wedge$ -algebras:

$$F\left\{\frac{J \cdot F}{I}\right\} \widehat{\otimes}_{F, \varphi_F} F \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}_{S/F}^\wedge.$$

Proof. Zariski locally on $\mathrm{Spf}(P)$ and $\mathrm{Spf}(F)$, the kernel J and $J \cdot F$ is a colimit of the form considered in Lemma 3.3. Also note that $F/J \cdot F \cong S$, by definition we have $\mathrm{dR}_F(J \cdot F)^\wedge \cong \mathrm{dR}_{S/F}^\wedge$.

Since formation of p -complete derived de Rham complex commutes with taking p -complete colimit (of the algebra over A) and descends from p -completely flat covers, we may glue the local isomorphisms obtained in Lemma 3.3 and take colimit to get our identification here. \square

Using this comparison of prismatic envelope and derived de Rham complex, we get a comparison between prismatic and derived de Rham cohomology as follows.

Theorem 3.5. *Let (A, I) be a bounded prism. For any p -completely smooth A/I -algebra R , there is a natural isomorphism in $\mathrm{CAlg}(A)$:*

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A, \varphi_A} A \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}_{R/A}^\wedge,$$

which is functorial in $A/I \rightarrow R$ and satisfies base change in (A, I) .

Let us emphasize again that when (A, I) is transversal, this follows from [BS19, Theorem 5.2].

Proof. Let us first construct the desired natural morphism

$$\mathrm{R}\Gamma_{\Delta}(R/A)\widehat{\otimes}_{A,\varphi_A}A\widehat{\otimes}_A dR_A(I)^\wedge \rightarrow dR_{R/A}^\wedge.$$

Given any triple $A \rightarrow P \rightarrow R$ as in the setting of Lemma 3.4, we have a natural morphism

$$\mathrm{R}\Gamma_{\Delta}(R/A)\widehat{\otimes}_{A,\varphi_A}A\widehat{\otimes}_A dR_A(I)^\wedge \rightarrow F\left\{\frac{J \cdot F}{I}\right\}^\wedge \widehat{\otimes}_{F,\varphi_F} F\widehat{\otimes}_A dR_A(I)^\wedge \cong dR_{S/F}^\wedge,$$

which is functorial in $A \rightarrow P \rightarrow R$. By Proposition 2.19 (3), the limit of right hand side over all the triples $A \rightarrow P \rightarrow R$ is just $dR_{R/A}^\wedge$, hence we get the desired natural morphism. It is functorial in $A/I \rightarrow R$ and satisfies base change in (A, I) .

Now we need to show the natural arrow constructed above is a natural isomorphism. Let us make more reductions. It suffices to check this is an isomorphism after a faithfully flat cover, and since both sides commute with base change in A , we may Zariski localize on A , hence we may first reduce to the case where A is oriented, i.e. $I = (d)$. Observe that both sides are the left Kan extension of their restriction to the category of polynomial A -algebras, so it suffices to show that the above arrow is a natural isomorphism for algebras of the form $R = A/I[X_1, \dots, X_n]^\wedge$, which is the base change of p -complete polynomial algebras over the universal oriented prism. Hence we can reduce further to the case that A is the universal oriented prism. In particular we may assume that (A, I) is transversal and that φ_A is flat.

Lastly, we shall prove the statement under the assumption that (A, I) is transversal and that φ_A is flat. Choose a (p, I) -completely polynomial A algebra P with a surjection of A -algebras $P \twoheadrightarrow R$, form the cosimplicial object $(F^\bullet\{\frac{J \cdot F^\bullet}{I}\}^\wedge)$ in $(R/A)_\Delta$ computing $\Delta_{R/A}$ as in Construction 3.1. Notice that we have an identification of cosimplicial (p, I) -complete algebras $A \xrightarrow{\cong} F^\bullet$.

Since we have reduced ourselves to the case where (A, I) is transversal and that φ_A is flat, using Lemma 2.5, the natural morphism considered above gives rise to the following identification

$$\begin{aligned} (\square) \quad \mathrm{R}\Gamma_{\Delta}(R/A)\widehat{\otimes}_{A,\varphi_A}A\widehat{\otimes}_A dR_A(I)^\wedge &\cong \lim_{\Delta} \left((F^\bullet\{\frac{F^\bullet \cdot J^\bullet}{I}\}^\wedge) \widehat{\otimes}_{A,\varphi_A} A\widehat{\otimes}_A dR_A(I)^\wedge \right) \\ &\cong \lim_{\Delta} \left((F^\bullet\{\frac{F^\bullet \cdot J^\bullet}{I}\}^\wedge) \widehat{\otimes}_{F^\bullet, \varphi_{F^\bullet}} F^\bullet \widehat{\otimes}_A dR_A(I)^\wedge \right) \cong \lim_{[n] \in \Delta} dR_{S^{\otimes R^n}/F^n}^\wedge \cong dR_{R/A}^\wedge \end{aligned}$$

as desired. Let us comment on the identifications above. Here we have used the cosimplicial replacement $(A, \varphi_A) \xrightarrow{\cong} (F^\bullet, \varphi_{F^\bullet})$ in the second identification. The second-to-last identification is provided by Lemma 3.4, and the last identification is because of Proposition 2.19. \square

Remark 3.6. In this paper we have only defined Frobenius action on $dR_{\square/A}^\wedge$ under the assumption of A being a p -torsionfree δ -ring. Now suppose (A, I) is a p -torsionfree prism, by Remark 2.20, we see that the chain of identifications in (\square) is compatible with Frobenius. Consequently, the identification in Theorem 3.5 is compatible with Frobenius in a functorial manner.

We expect however that one can remove the p -torsionfree condition with additional work in developing the framework of “derived δ -rings”. However since the primary interest of this paper is in the case of p -torsionfree prisms, we choose to not pursue that level of generality here.

Below let us conclude two consequences from Theorem 3.5.

Corollary 3.7. *Let (A, I) be a bounded prism. For any p -completely smooth A/I -algebra R , there is a natural isomorphism in $\mathrm{CAlg}(A)$:*

$$\mathrm{R}\Gamma_{\Delta}(R/A)\widehat{\otimes}_{A,\varphi_A}A\widehat{\otimes}_A \mathcal{A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/A),$$

which is functorial in $A/I \rightarrow R$ and satisfies base change in (A, I) .

Proof. This follows from Theorem 3.5: simply base change along the morphism $dR_A(I)^\wedge \rightarrow \mathcal{A}$, and by Proposition 2.11 we have

$$dR_{R/A}^\wedge \widehat{\otimes}_{dR_A(I)^\wedge} \mathcal{A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/A).$$

\square

Remark 3.8. By chasing diagram, one verifies that the following diagram of isomorphisms:

$$\begin{array}{ccc} \varphi_{\mathcal{A}}^* (\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A, \varphi_A} A \widehat{\otimes}_{A, \varphi_A} \mathcal{A}) & \xrightarrow{\alpha} & \varphi_{\mathcal{A}}^* (\mathrm{R}\Gamma_{\mathrm{crys}}(R/A)) \\ \downarrow \beta & & \downarrow \epsilon \\ \varphi_{\mathcal{A}}^* (\mathrm{R}\Gamma_{\Delta}((R \widehat{\otimes}_{A, \varphi_A} \mathcal{A})/A)) & \xrightarrow{\gamma} & \mathrm{R}\Gamma_{\mathrm{crys}}((R/p \otimes_{A/(p, I), \varphi_A} A/(p, I))/A) \end{array}$$

is commutative, since all comparisons here are expressed in terms of various explicit envelopes. Here these arrows are given by:

- (1) α is Frobenius pullback of the arrow in Corollary 3.7;
- (2) β is the base change of prisms $\varphi_A: (A, I) \rightarrow (A, p)$;
- (3) γ is the crystalline comparison for crystalline prisms [BS19, Theorem 5.2];
- (4) ϵ is the base change of crystalline cohomology.

A bounded prism (A, I) is called a *PD prism*, if there is a PD structure γ on I , compatible with the canonical one on (p) .

Corollary 3.9. *Let (A, I, γ) be a bounded PD prism. Then for any p -completely smooth A/I -algebra R , there is a natural isomorphism in $\mathrm{CALg}(A)$:*

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A, \varphi_A} A \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/(A, I, \gamma)),$$

which is functorial in $A/I \rightarrow R$ and satisfies base change in (A, I) .

Here $\mathrm{R}\Gamma_{\mathrm{crys}}(-/(A, I, \gamma))$ denotes the crystalline cohomology with respect to the p -adic PD base (A, I, γ) .

Proof. The additional PD structure gives us a section $\mathcal{A} \rightarrow A$, which makes the composition of

$$A \rightarrow \mathrm{dR}_A(I)^\wedge \rightarrow \mathcal{A} \rightarrow A$$

being identity. Take the functorial isomorphism in Corollary 3.7, and base change further along $\mathcal{A} \rightarrow A$ gives us

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A, \varphi_A} A \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/\mathcal{A}) \widehat{\otimes}_{\mathcal{A}} A,$$

and the latter is naturally isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(R/(A, I, \gamma))$ due to base change in crystalline cohomology theory. \square

Remark 3.10. (1) Any derived p -complete δ -ring A with bounded p -torsion together with the ideal (p) is a PD prism. In this situation, our Corollary 3.9 is simply the crystalline comparison in [BS19, Theorem 1.8.(1)].

(2) The left hand side of this comparison does not depend on the PD structure γ on I , whereas the right hand side *a priori* does. Therefore this comparison tells us that the right hand side also does not depend on the PD structure γ .

We can “globalize” these comparisons to general quasi-compact quasi-separated smooth formal schemes over $\mathrm{Spf}(A/I)$.

Theorem 3.11. *Let (A, I) be a bounded prism. Let $X \rightarrow \mathrm{Spf}(A/I)$ be a quasi-compact quasi-separated smooth morphism of formal schemes. Then we have natural isomorphisms in $\mathrm{CALg}(A)$:*

$$\mathrm{R}\Gamma_{\Delta}(X/A) \widehat{\otimes}_{A, \varphi_A} A \widehat{\otimes}_{A, \varphi_A} \mathrm{dR}_A(I)^\wedge \cong \mathrm{R}\Gamma(X, \mathrm{dR}_{-/A}^\wedge);$$

and

$$\mathrm{R}\Gamma_{\Delta}(X/A) \widehat{\otimes}_{A, \varphi_A} A \widehat{\otimes}_{A, \varphi_A} \mathcal{A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(X/\mathcal{A}).$$

If (A, I, γ) is a PD prism, then we have a natural isomorphism in $\mathrm{CALg}(A)$:

$$\mathrm{R}\Gamma_{\Delta}(X/A) \widehat{\otimes}_{A, \varphi_A} A \cong \mathrm{R}\Gamma_{\mathrm{crys}}(X/(A, I, \gamma)).$$

All the isomorphisms above satisfy base change in (A, I) . Moreover, if X is also proper over $\mathrm{Spf}(A/I)$, then all the completed tensor products above may be replaced by tensor products.

Proof. Since X is assumed to be quasi-compact and quasi-separated. These cohomologies are computed by a finite limit of the corresponding cohomologies of affine opens of X . Because completed tensor commutes with finite limit, the comparisons here follow from Theorem 3.5, Corollary 3.7, and Corollary 3.9.

To justify the replacement of completed tensor with tensor, just note that $\mathrm{R}\Gamma_{\Delta}(X/A)$ is a perfect complex of A -modules for smooth proper $X \rightarrow \mathrm{Spf}(A/I)$, see the last sentence of [BS19, Theorem 1.8]. \square

3.2. Functorial endomorphisms of derived de Rham complex. Throughout this subsection, we assume (A, I) to be a transversal prism, in particular we have $\mathrm{dR}_A(I)^\wedge \cong \mathcal{A}$ and $\mathrm{dR}_{R/A}^\wedge \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/\mathcal{A})$ where R is any p -adic formally smooth A/I -algebra, see Proposition 2.11.

In this subsection, we aim at understanding all functorial endomorphisms of the derived de Rham complex functor, under this transversality assumption. In particular we shall see that the functorial isomorphism

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A, \varphi_A} A \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \rightarrow \mathrm{dR}_{R/A}^\wedge,$$

appearing in Theorem 3.5 is unique if we assume (A, I) to be a transversal prism. In order to show this, we need to first extend the natural isomorphism to a larger class of A/I -algebras.

Construction 3.12 (c.f. [BS19, Construction 7.6] and [BMS19, Example 5.12]). Fix a bounded prism (A, I) , consider the functor $R \mapsto \mathrm{dR}_{R/A}^\wedge$ on p -completely smooth A/I -algebras R valued in the category of commutative algebras in the ∞ -category of p -complete objects in $\mathcal{D}(A)$. Left Kan extend it to all derived p -complete simplicial A/I -algebras, which is nothing but the p -adic derived de Rham complex relative to A , still denoted by $\mathrm{dR}_{R/A}^\wedge$. Let us record some properties of this construction:

- (1) Since R is an A/I -algebra, the $\mathrm{dR}_{R/A}^\wedge$ is naturally a $\mathrm{dR}_{(A/I)/A}^\wedge$ -algebra. Hence we may actually view the functor as taking values in the category $\mathrm{CAlg}(\mathrm{dR}_A(I)^\wedge)$.
- (2) The formation of $\mathrm{dR}_{R/A}^\wedge$ commutes with base change in A .
- (3) Below we shall see that, following the reasoning of [BMS19, Theorem 3.1 and Example 5.12], the association $R \mapsto \mathrm{dR}_{R/A}^\wedge$ defines a sheaf on the relative quasisyntomic site $\mathrm{qSyn}_{A/I}$.
- (4) By left Kan extending the natural isomorphism obtained in Theorem 3.5, we get an isomorphism of sheaves:

$$\Delta_{R/A}^{(1)} \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}_{R/A}^\wedge,$$

which is compatible with base change in A . Here $\Delta_{R/A}^{(1)} := \Delta_{R/A} \widehat{\otimes}_{A, \varphi_A} A$ is the Frobenius pullback of the derived prismatic cohomology.

- (5) Moreover if we assume that (A, I) is a transversal prism, then for any R which is large quasisyntomic over A/I , the value $\mathrm{dR}_{R/A}^\wedge$ is p -completely flat over \mathcal{A} and lives in cohomological degree 0.

Let us justify the claim (3) above.

Proposition 3.13. *The association $R \mapsto \mathrm{dR}_{R/A}^\wedge$ defines a sheaf on the relative quasisyntomic site $\mathrm{qSyn}_{A/I}$.*

Proof. Let $R \rightarrow S$ be a quasisyntomic cover of objects in $\mathrm{qSyn}_{A/I}$, with Čech nerve S^\bullet . Our task is to show $\mathrm{dR}_{R/A}^\wedge = \lim_{\Delta^{\mathrm{op}}} \mathrm{dR}_{S^\bullet/A}^\wedge$. Since both sides are p -complete, we may check this after derived modulo p . Below we shall always use $-/p$ to denote derived modulo p .

Now we follow closely the argument in [BMS19, Example 5.12], correcting a typo thereof. First there is a functorial exhaustive increasing \mathbb{N} -index filtration, i.e. the conjugate filtration, on $\mathrm{dR}_{R/A}/p \cong \mathrm{dR}_{(R/p)/(A/p)}$ with graded pieces given by $(\wedge_{(R/p)^{(1)}}^i \mathbb{L}_{(R/p)^{(1)}/(A/p)})[-i]$ (and similarly for $\mathrm{dR}_{(S^\bullet/p)/(A/p)}$). Here $(-/p)^{(1)}$ denotes the base change along the Frobenius on A/p , and the loc. cit. has a typo of not adding this Frobenius twist. For a discussion of p -complete derived de Rham complex and conjugate filtration in the realm of animated rings, we refer readers to [KP21, p. 33-35].

We stare at the following diagram (with its S^\bullet analogs in mind):

$$\begin{array}{ccccc}
R & \longrightarrow & R/p & \xrightarrow{\varphi_{A/p}} & (R/p)^{(1)} \\
\uparrow & & \uparrow & & \uparrow \\
A/I & \longrightarrow & (A/I)/p & \xrightarrow{\varphi_{A/p}} & ((A/I)/p)^{(1)} \cong (A/I^p)/p \\
\uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & A/p & \xrightarrow{\varphi_{A/p}} & A/p.
\end{array}$$

Note that every square above is Cartesian. Base change property of cotangent complex implies that these graded pieces $(\wedge_{(R/p)^{(1)}}^i \mathbb{L}_{(R/p)^{(1)}/(A/p)})[-i]$ (and their S^\bullet analogs) can be identified with:

- (1) either $\wedge_R^i \mathbb{L}_{R/A}[-i] \otimes_R \varphi_{A/p,*}(R/p)^{(1)}$;
- (2) or $\wedge_R^i \mathbb{L}_{R/A} \otimes_A \varphi_{A/p,*}(A/p)$,

where the A -module (resp. R -module) structure on $\varphi_{A/p,*}(A/p)$ (resp. $\varphi_{A/p,*}(R/p)^{(1)}$) is given by the top and bottom row of the above diagram.

The identification (1) above implies that all these graded pieces live in $D^{\geq -1}$. Indeed, $\wedge_R^i \mathbb{L}_{R/A}[-i]$ as an R -module has Tor-amplitude in $[0, i]$, and $(R/p)^{(1)}$ is flat over $((A/I)/p)^{(1)} \cong (A/I^p)/p$ which lives in $[-1, 0]$, so $(R/p)^{(1)}$ lives in $D^{\geq -1}$. Similar statements for S^\bullet hold as well. Hence we are reduced to checking these graded pieces satisfy the descent property, here we are using the reasoning of [BMS19, last sentence of Example 5.12]. Now using the identification (2) above, we are reduced to flat descent for “tensoring” wedge powers of cotangent complex, see [LM21, Proposition 3.2] (which is itself a generalization of [BMS19, Theorem 3.1]). \square

Recall that an A/I -algebra is called *large quasisyntomic over A/I* (see [BS19, Definition 15.1]) if

- $A/I \rightarrow R$ is quasisyntomic; and
- there is a surjection $A/I \langle X_j^{1/p^\infty} \mid j \in J \rangle \rightarrow R$ where J is a set.

The following is inspired by [BLM18, Sections 10.3 and 10.4], and our proof is a modification of the proof thereof.

Theorem 3.14. *Let (A, I) be a transversal prism, and assume that $\mathrm{Spf}(A/I)$ is connected. Then*

- (1) *The mapping space*

$$\mathrm{End}_{\mathrm{Shv}(\mathrm{qSyn}_{A/I}, \mathrm{CAlg}(\mathcal{A}))} \left(\mathrm{dR}_{-/A}^\wedge, \mathrm{dR}_{-/A}^\wedge \right)$$

has contractible components given by a submonoid in \mathbb{N} . In particular, the automorphism space has only one contractible component given by identity.

- (2) *The automorphism space*

$$\mathrm{Aut}_{\mathrm{Shv}(\mathrm{qSyn}_{A/I}, \mathrm{CAlg}(A/I))} \left(\mathrm{dR}_{-(A/I)}^\wedge, \mathrm{dR}_{-(A/I)}^\wedge \right)$$

has only one contractible component given by identity.

Since $A/p \rightarrow A/(I, p)$ is a locally nilpotent thickening, we get that $\mathrm{Spf}(\mathcal{A})$ is also connected. In particular, the only idempotents in \mathcal{A} are 0 and 1. It is easy to see that the statement concerning automorphism spaces for these functors hold true without the connectedness assumption, as on each connected component the automorphism must be identity.

Proof. The assertion that all components are contractible follows from the fact that on the basis of large quasisyntomic over A/I -algebras, the sheaves $\mathrm{dR}_{-/A}^\wedge$ and $\mathrm{dR}_{-(A/I)}^\wedge$ are discrete.

All we need to check is that there are not many functorial endomorphisms (resp. automorphisms) for these two sheaves. Since (2) follows from the same proof as that of (1), let us only present the proof of (1) here. To simplify notation, let us denote the set of functorial endomorphisms by $\mathrm{End}(\mathrm{dR}_{-/A}^\wedge)$. By restriction, any functorial endomorphism induces a functorial endomorphism of the functor restricted to the subcategory

of A/I -algebras of the form $A/I\langle X_h^{1/p^\infty} \mid h \in H \rangle$ for some set H . We denote the latter monoidal space by $\text{End}(\text{dR}_{-/A}^\wedge |_{\text{perf}})$, all of whose components are also contractible by the same reasoning. By definition there is a natural map

$$\text{res}: \text{End}(\text{dR}_{-/A}^\wedge) \rightarrow \text{End}(\text{dR}_{-/A}^\wedge |_{\text{perf}})$$

of monoids.

Now we make following three claims:

- the natural map res is injective;
- the monoid $\text{End}(\text{dR}_{-/A}^\wedge |_{\text{perf}})$ is a submonoid of \mathbb{Z} ; and
- the image of res is contained in \mathbb{N} .

Below let us show the map res is injective. In other words, we need to show that any functorial endomorphism of $\text{dR}_{-/A}^\wedge$ is determined by its restriction to the algebras of the form $A/I\langle X_h^{1/p^\infty} \mid h \in H \rangle$ for some set H . To see this, notice that $\text{qSyn}_{A/I}$ has a basis given by large quasi-syntomic over A/I -algebras. Any large quasi-syntomic over A/I -algebra S , by definition, admits a surjection from an algebra of the form $A/I\langle X_l^{1/p^\infty} \mid l \in L \rangle$ for some set L . By choosing a set of generators $\{f_j \mid j \in J\}$ of the kernel, we may form a surjection (c.f. [BS19, proof of Proposition 7.10])

$$S' := A/I\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J \rangle / (Y_j - f_j \mid j \in J)^\wedge \rightarrow S: Y_j^m \mapsto 0.$$

This induces a surjection of shifted cotangent complexes: $\mathbb{L}_{S'/A}[-1] \rightarrow \mathbb{L}_{S/A}[-1]$, therefore it induces a surjection of p -adic derived de Rham complexes: $\text{dR}_{S'/A}^\wedge \rightarrow \text{dR}_{S/A}^\wedge$. For any such S' , we have

$$\text{dR}_{S'/A}^\wedge \cong D_{\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J \rangle} (Y_j - f_j \mid j \in J)^\wedge,$$

i.e. p -completely adjoining divided powers of $Y_j - f_j$ for all $j \in J$ to $\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J \rangle$. Since \mathcal{A} is p -torsionfree, any endomorphism of $\text{dR}_{S'/A}^\wedge$ is determined by its restriction to $\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J \rangle$. Lastly, we know that applying $\text{dR}_{-/A}^\wedge$ functor to the map

$$A/I\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J \rangle \rightarrow S'$$

exactly induces the natural map

$$\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J \rangle \rightarrow \text{dR}_{S'/A}.$$

Therefore we know that any functorial endomorphism of $\text{dR}_{-/A}^\wedge$ must be determined by its restriction to algebras of the form $A/I\langle X_h^{1/p^\infty} \mid h \in H \rangle$.

Next, let us try to understand $\text{End}(\text{dR}_{-/A}^\wedge |_{\text{perf}})$ and show it is a submonoid of integers. Consider a functorial endomorphism f . It is determined by its restriction to the one-variable ‘‘perfect’’ A/I -algebra $R = A/I\langle X^{1/p^\infty} \rangle$. We know $\text{dR}_{R/A}^\wedge \cong \mathcal{A}\langle X^{1/p^\infty} \rangle$. Suppose $f(x) = \sum_{i \in \mathbb{N}[1/p]} a_i X^i \in \mathcal{A}\langle X^{1/p^\infty} \rangle$. Consider the map $R \rightarrow S := A/I\langle Y^{1/p^\infty}, Z^{1/p^\infty} \rangle$ sending $X^i \mapsto Y^i Z^i$. This map induces the corresponding map $\mathcal{A}\langle X^{1/p^\infty} \rangle \rightarrow \mathcal{A}\langle Y^{1/p^\infty}, Z^{1/p^\infty} \rangle$ which also sends $X^i \mapsto Y^i Z^i$. Now the functoriality of the endomorphism tells us that $f(YZ) = f(Y) \cdot f(Z)$. We immediately get $a_i^2 = a_i$ and $a_i \cdot a_j = 0$ for any pair of distinct indices $i, j \in \mathbb{N}[1/p]$. By connectedness assumption of $\text{Spf}(A/I)$, we see there is at most one index $i \in \mathbb{N}[1/p]$, with nonzero $a_i = 1$. To see there is at least one nonzero a_i , we use the map $R \rightarrow A/I$ given by $X^i \mapsto 1$ for all $i \in \mathbb{N}[1/p]$.

We want to show the $i \in \mathbb{N}[1/p]$ got in the previous paragraph defining the functorial endomorphism f must in fact lie in $p^\mathbb{Z}$. Assume $i = \frac{\ell}{p^N}$ where ℓ is an integer coprime to p . Now we contemplate the map $R \rightarrow S$ given by $X \mapsto \lim_n (Y^{1/p^n} + Z^{1/p^n})^{p^n}$, it induces a map of $\text{dR}_{-/A}^\wedge$ with the image of X given by the same formula. Functoriality of f implies that we have

$$\left(\lim_n (Y^{1/p^n} + Z^{1/p^n})^{p^{n-N}} \right)^\ell = \lim_n (Y^{\ell/p^{n-N}} + Z^{\ell/p^{n-N}})^{p^n}.$$

Reduction modulo p tells us that

$$(Y^{1/p^N} + Z^{1/p^N})^\ell = Y^{\ell/p^N} + Z^{\ell/p^N} \in \mathbb{F}_p[Y^{1/p^\infty}, Z^{1/p^\infty}],$$

forcing $\ell = 1$. Therefore we see that $\text{End}(\text{dR}_{-/A}^\wedge |_{\text{perf}}) \subset p^\mathbb{Z}$, i.e. it is a submonoid inside \mathbb{Z} .

Finally let us prove the image of res lands in $p^\mathbb{N}$. We want to rule out negative powers of p . To that end consider $R \rightarrow R/(X)$, which induces the map of p -adic derived de Rham complex:

$$\tilde{R} := \mathcal{A}\langle X^{1/p^\infty} \rangle \rightarrow \tilde{S} := D_{\mathcal{A}\langle X^{1/p^\infty} \rangle}(X)^\wedge.$$

Here the latter denotes the p -complete PD envelope of the former along the ideal X , and this is the natural map. Take a positive integer j , and we need to argue that $X \mapsto X^{1/p^j}$ on \tilde{R} does not extend to an endomorphism of \tilde{S} . Suppose otherwise, then the extended endomorphism of \tilde{S} must send X^p to $X^{p^{1-j}}$, but X^p is divisible by p in \tilde{S} whereas $X^{p^{1-j}}$ is not (here we use the fact that $j > 0$), hence we get a contradiction.

The only invertible element in the additive monoid \mathbb{N} is 0, corresponding to $X \mapsto X^{(p^0)} = X$, hence the only functorial automorphism of $\text{dR}_{-/A}^\wedge$ is identity. \square

Remark 3.15. Let (A, I) be a transversal prism. Then

- (1) By the same argument, there are not many functorial homomorphisms from $\varphi_A^* \text{dR}_{-/A}^\wedge$ to $\text{dR}_{-/A}^\wedge$. Similarly, these are determined by its restriction to $R = A/I\langle X^{1/p^\infty} \rangle$. If we require the restriction sends X to X^p , then there is a unique one given by Frobenius constructed in Section 2.3. Therefore in a strong sense, there is a unique Frobenius.
- (2) Due to previous remark, we see the comparison in Theorem 3.5 must be compatible with Frobenius.
- (3) It is unclear which positive integer i , corresponding to $X \mapsto X^{p^i}$, can occur as a functorial endomorphism. When $A/(p, I)$ has transcendental (relative to \mathbb{F}_p) elements, then none of these can occur. This can be seen by considering the map $R \rightarrow R/(X - a)$ for some lift a of the transcendental element $\bar{a} \in A/(p, I)$.

Consequently we get the following uniqueness of the functorial comparison established in Theorem 3.5, readers shall compare with [BS19, Section 18].

Corollary 3.16. *Fix a transversal prism (A, I) . There is a unique natural isomorphism of p -complete commutative algebra objects in $\mathcal{D}(\mathcal{A})$:*

$$\text{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A, \varphi_A} A \widehat{\otimes}_A \mathcal{A} \rightarrow \text{R}\Gamma_{\text{crys}}(R/A),$$

which is functorial in the p -completely smooth A/I -algebra R .

Proof. The existence part is given by Theorem 3.5, we need to show uniqueness. Suppose there are two such functorial isomorphisms. Then consider the composition of one with the inverse of the other, we get a natural automorphism of the functor $\text{dR}_{-/A} \cong \text{R}\Gamma_{\text{crys}}(-/A)$ on smooth A/I -algebras. By left Kan extension, this will induce a natural automorphism of the functor $\text{dR}_{-/A}$ on quasisyntomic A/I -algebras. We conclude by Theorem 3.14 that this automorphism must be identity. \square

Corollary 3.17. *Let \mathbf{C} be an algebraically closed complete non-Archimedean field extension of \mathbb{Q}_p , and let (A, I) be the associated perfect prism (denoted as $(A_{\text{inf}}, \ker(\theta))$ in literature). Then the comparison in Theorem 3.5 is compatible with the crystalline comparison over $\mathcal{A} = A_{\text{crys}}$ of the $A\Omega$ -theory obtained in [BMS18]. Concretely, the following diagram of isomorphisms is commutative:*

$$\begin{array}{ccc} \text{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A, \varphi_A} A \widehat{\otimes}_A \mathcal{A} & \longrightarrow & \text{R}\Gamma_{\text{crys}}(R/A) \\ \downarrow & & \downarrow \\ A\Omega(R) \widehat{\otimes}_A \mathcal{A} & \longrightarrow & \text{R}\Gamma_{\text{crys}}((R/p)/A), \end{array}$$

where the left vertical arrow is given by [BS19, Theorem 17.2] and the bottom horizontal arrow is given by [BMS18, Theorem 12.1] or [Yao19].

Proof. This follows from the uniqueness statement in Corollary 3.16. \square

Both sides of the isomorphism obtained in Theorem 3.5 after completely tensoring $\mathcal{A}/\mathcal{I} \cong A/I$ over \mathcal{A} , are naturally isomorphic to $dR_{R/(A/I)}^\wedge$. For the left hand side this follows from the de Rham comparison of (Frobenius pullback of) the prismatic cohomology:

$$\Delta_{R/A}^{(1)} \widehat{\otimes}_{\mathcal{A}} \mathcal{A} \widehat{\otimes}_{\mathcal{A}} A/I \cong \Delta_{R/A}^{(1)} \widehat{\otimes}_{\mathcal{A}} A/I \cong dR_{R/(A/I)}^\wedge,$$

where the last equality follows from [BS19, Theorem 6.4 or Corollary 15.4]. For the right hand side this is just the base change of the derived de Rham complex (or base change of crystalline cohomology and the comparison of de Rham and crystalline cohomology for smooth morphism). We observe that similar argument as above forces these natural isomorphisms to be compatible with each other.

Corollary 3.18. *Let (A, I) be a transversal prism. The following triangle of natural isomorphisms*

$$\begin{array}{ccc} & dR_{R/(A/I)}^\wedge & \\ \cong \nearrow & & \nwarrow \cong \\ \Delta_{R/A}^{(1)} \widehat{\otimes}_{\mathcal{A}} \mathcal{A} \widehat{\otimes}_{\mathcal{A}} A/I & \xrightarrow{\cong} & dR_{R/A}^\wedge \widehat{\otimes}_{\mathcal{A}} A/I \end{array}$$

is a commutative diagram.

Proof. Observe that all three natural isomorphisms are functorial in R , hence going around the circle produces a functorial automorphism of $dR_{R/(A/I)}^\wedge$.

Now we argue as in the proof of Corollary 3.16: using Theorem 3.14 we may conclude that this functorial automorphism must be identity. Hence the above diagram must commute functorially. \square

4. FILTRATIONS

Throughout this section, we assume (A, I) to be a transversal prism and let $(\mathcal{A}, \mathcal{I})$ be the p -adic PD envelope of A along I . By Theorem 3.5, for any p -completely smooth A/I -algebra R we have a functorial isomorphism:

$$\varphi^*(R\Gamma_{\Delta}(R/A)) \widehat{\otimes}_{\mathcal{A}} \mathcal{A} \cong dR_{R/A}^\wedge.$$

All objects involved here have interesting filtrations, they are: Nygaard filtration on $\varphi^*(R\Gamma_{\Delta}(R/A))$, I -adic filtration on A , PD ideal filtration $\mathcal{I}^{[\bullet]}$ on \mathcal{A} , and Hodge filtration on $dR_{R/A}^\wedge$. In this section, we discuss how these filtrations are related.

Unless otherwise specified, we shall use R to denote a general A/I -algebra, and S will be used to denote a large quasi-syntomic over A/I -algebra (see the discussion right after Construction 3.12).

Let us briefly remind readers how these filtrations are defined and their properties.

4.1. Hodge filtration on $dR_{R/A}^\wedge$. Recall that $R\Gamma_{\text{crys}}(R/A)$ is the cohomology of the structure sheaf $\mathcal{O}_{\text{crys}}$ on the (absolute) crystalline site $(R/A)_{\text{crys}}$. The crystalline structure sheaf admits a natural surjection to the Zariski structure sheaf, whose kernel is an ideal sheaf $\mathcal{I}_{\text{crys}}$ admitting divided powers. Concretely, given a PD thickening (U, T) , with U a p -adic formal $\text{Spf}(A)$ -scheme with an $\text{Spf}(A)$ -map $U \rightarrow \text{Spf}(R)$ and $U \hookrightarrow T$ a p -completely nilpotent PD thickening, then we have $\mathcal{O}_{\text{crys}}|_{(U, T)} = \mathcal{O}_T$ and $\mathcal{I}_{\text{crys}}|_{(U, T)} = \ker(\mathcal{O}_T \rightarrow \mathcal{O}_U)$ which is a PD ideal sheaf inside $\mathcal{O}_{\text{crys}}$. For any integer $r \geq 0$, we get a natural filtration on $R\Gamma_{\text{crys}}(R/A)$ given by $R\Gamma_{\text{crys}}(R/A, \mathcal{I}_{\text{crys}}^{[r]})$. Results of Bhatt [Bha12, Section 3.3] and Illusie [Ill72, Section VIII.2] help us to understand this natural filtration in terms of p -adic derived de Rham complex and its Hodge filtrations.

Theorem 4.1 (see [Bha12, Proposition 3.25 and Theorem 3.27] and [Ill72, Corollaire VIII.2.2.8], see also [GL21, Theorem 3.4.(4)]). *Let R be a p -completely locally complete intersection A/I -algebra. Then there is a natural identification of filtered \mathbb{E}_∞ - \mathcal{A} -algebras:*

$$(dR_{R/A}^\wedge, \text{Fil}_{\text{H}}^r) \xrightarrow{\cong} (R\Gamma_{\text{crys}}(R/A), R\Gamma_{\text{crys}}(R/A, \mathcal{I}_{\text{crys}}^{[r]})).$$

Here $\text{Fil}_{\text{H}}^\bullet$ denotes the (derived p -completed) Hodge filtration on $dR_{R/A}^\wedge$, whose graded pieces are given by

$$\text{gr}_{\text{H}}^*(dR_{R/A}^\wedge) \cong \Gamma_R^*(\mathbb{L}_{R/A}^\wedge[-1]),$$

where Γ^* denotes the derived divided power algebra construction and $\mathbb{L}_{R/A}^\wedge$ denotes the derived p -completed cotangent complex of R over A . The triangle $A \rightarrow A/I \rightarrow R$ now gives us a triangle relating various p -completed cotangent complexes:

$$R\widehat{\otimes}_{A/I} I/I^2[1] \cong R\widehat{\otimes}_{A/I} \mathbb{L}_{(A/I)/A}^\wedge \rightarrow \mathbb{L}_{R/A}^\wedge \rightarrow \mathbb{L}_{R/(A/I)}^\wedge,$$

where the (shifted) map $R\widehat{\otimes}_{A/I} I/I^2 \rightarrow \mathbb{L}_{R/A}^\wedge[-1]$ comes from the \mathcal{A} -algebra structure on $dR_{R/A}^\wedge$. Indeed the multiplicativity of Hodge filtrations and the fact that $I/I^2 \cong \mathcal{I}/\mathcal{I}^{[2]} \cong \mathrm{gr}_\mathbb{H}^1(dR_{(A/I)/A}^\wedge)$ naturally sits inside $\mathrm{gr}_\mathbb{H}^1(dR_{R/A}^\wedge)$ giving rise to

$$\mathrm{gr}_\mathbb{H}^0(dR_{R/A}^\wedge) \widehat{\otimes}_{\mathrm{gr}_\mathbb{H}^0(dR_{(A/I)/A}^\wedge)} \mathrm{gr}_\mathbb{H}^1(dR_{(A/I)/A}^\wedge) \rightarrow \mathrm{gr}_\mathbb{H}^1(dR_{R/A}^\wedge),$$

which is identified with the shifted map $R\widehat{\otimes}_{A/I} I/I^2 \rightarrow \mathbb{L}_{R/A}^\wedge[-1]$.

The above discussion naturally extends to all A/I -algebras via left Kan extension. We restrict ourselves to those algebras that are quasisyntomic over A/I so that everything in sight is a sheaf with respect to the quasisyntomic topology. Recall that a basis of the quasisyntomic site is given by algebras that are large quasisyntomic over A/I (see [BS19, Definition 15.1]). Below we shall show that, on this basis, all these sheaves have values living in cohomological degree zero. The proof is inspired by [BS19, Subsection 12.5].

Lemma 4.2. *Let B be an \mathbb{F}_p -algebra and let S be a B -algebra which is relatively semiperfect with $\mathbb{L}_{S/B}[-1]$ given by a flat S -module. Then $dR_{S/B}$ and its Hodge filtrations all live in cohomological degree 0.*

Proof. Using the conjugate filtration and Cartier isomorphism, we see that $dR_{S/B}$ (being its 0-th Hodge filtration) lives in degree 0. On the other hand, we also know that the graded pieces of the Hodge filtrations are given by divided powers $\Gamma_S^*(\mathbb{L}_{S/B}[-1])$, hence all the graded pieces live in degree 0 as well. In order to prove the statement about Hodge filtrations, we need to show the natural map $dR_{S/B} \rightarrow dR_{S/B} / \mathrm{Fil}_\mathbb{H}^r$ is surjective (note that both sides live in degree 0 by last sentence).

To this end, we proceed by mimicking [BS19, proof of Theorem 12.2]. First we may replace B by the relative perfection of S , as the relevant cotangent complexes $\mathbb{L}_{S/B}$ and $\mathbb{L}_{S^{(1)}/B}$ are unchanged. Hence we may assume $B \rightarrow S$ is a surjection, as S/B is assumed to be relatively semiperfect. Next, by choosing the surjection $\mathbb{F}_p[X_b \mid b \in B] \rightarrow B$ and base change along the fully faithful map $\mathbb{F}_p[X_b \mid b \in B] \rightarrow \mathbb{F}_p[X_b^{1/p^\infty} \mid b \in B]$, we may further assume that B is semiperfect (as surjectiveness of a map can be tested after fully faithful base change). In particular, any element in the kernel of $B \rightarrow S$ admits compatible p -power roots in B .

Now if the kernel is generated by a regular sequence, then the map $dR_{S/B} \rightarrow dR_{S/B} / \mathrm{Fil}_\mathbb{H}^r$ is identified as $D_B(S) \rightarrow D_B(S)/J^{[r]}$ where $D_B(S)$ denotes the PD envelope and $J^{[r]}$ is the r -th divided power ideal of $J = \ker(D_B(S) \rightarrow S)$. Therefore $dR_{S/B} \rightarrow dR_{S/B} / \mathrm{Fil}_\mathbb{H}^r$ is surjective by this concrete description.

Lastly given any such surjection $B \rightarrow S$, call the underlying set of its kernel by I . Then we look at the surjection of B -algebras

$$\tilde{S} := B[X_i^{1/p^\infty} \mid i \in I]/(X_i \mid i \in I) \rightarrow S,$$

where X_i^{1/p^∞} is sent to (the image of) a compatible p -power roots of the corresponding element $f_i \in I$ in S . We have that the induced map $\mathbb{L}_{\tilde{S}/B}[-1] \rightarrow \mathbb{L}_{S/B}[-1]$ sends X_i to f_i , hence is a surjection. Therefore we get that the map $\mathrm{gr}_\mathbb{H}^*(dR_{\tilde{S}/B}) \rightarrow \mathrm{gr}_\mathbb{H}^*(dR_{S/B})$ is a surjection. Since \tilde{S} is a quotient of a relatively perfect algebra over B by an ind-regular sequence. Applying (filtered colimit of) what we proved in the previous paragraph, we get that $dR_{\tilde{S}/B} \rightarrow dR_{\tilde{S}/B} / \mathrm{Fil}_\mathbb{H}^r$ is also a surjection. Looking at the following commutative diagram

$$\begin{array}{ccc} dR_{\tilde{S}/B} & \longrightarrow & dR_{S/B} \\ \downarrow & & \downarrow \\ dR_{\tilde{S}/B} / \mathrm{Fil}_\mathbb{H}^r & \twoheadrightarrow & dR_{S/B} / \mathrm{Fil}_\mathbb{H}^r, \end{array}$$

we conclude that the right arrow must be surjective, which is what we need to show. \square

Lemma 4.3. *Let S be a large quasisyntomic over A/I algebra. Then all of the Hodge filtrations on $dR_{S/A}^\wedge$ and $dR_{S/(A/I)}^\wedge$ are given by submodules, equivalently all the filtrations and their graded pieces are cohomologically supported in degree 0. Moreover the Hodge filtrations of $dR_{S/(A/I)}^\wedge$ are p -completely flat over A/I .*

Proof. Derived modulo p , we see that the first claim follows from Lemma 4.2. Also we see that $dR_{S/(A/I)}^\wedge \rightarrow dR_{S/(A/I)}^\wedge / \text{Fil}_H^r$ is surjective. So the statement of p -completely flatness of $dR_{S/(A/I)}^\wedge$ and its Hodge filtrations now follows from p -completeness of $dR_{S/(A/I)}^\wedge$ and the graded pieces of its Hodge filtrations. Using conjugate filtration and Cartier isomorphism, both p -completely flatness follow from the fact that $\mathbb{L}_{S/(A/I)}^\wedge[-1]$ is p -completely flat over S and S is p -completely flat over A/I (as S is large quasisyntomic over A/I). \square

Since $dR_{R/A}^\wedge$ is naturally an \mathcal{A} -algebra for any A/I -algebra R , the filtration on \mathcal{A} by the divided powers of \mathcal{I} gives rise to another functorial decreasing filtration on $dR_{R/A}^\wedge$:

$$\text{Fil}_{\mathcal{I}}^r(dR_{R/A}^\wedge) := dR_{R/A}^\wedge \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]}.$$

We caution readers that this is *not* the \mathcal{I} -adic filtration, as we are using divided powers of \mathcal{I} instead of symmetric powers. A basic understanding of these filtrations are given by the following:

Lemma 4.4. *All of these $\text{Fil}_{\mathcal{I}}^r(dR_{R/A}^\wedge)$ are quasisyntomic sheaves, whose values on large quasisyntomic over A/I algebras are supported in degree 0. The graded pieces are given by*

$$\text{gr}_{\mathcal{I}}^r \cong dR_{R/(A/I)}^\wedge \widehat{\otimes}_{A/I} \mathcal{I}^{[r]} / \mathcal{I}^{[r+1]}.$$

Proof. The statement about graded pieces follows from the following chain of identifications

$$\text{gr}_{\mathcal{I}}^r \cong dR_{R/A}^\wedge \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]} / \mathcal{I}^{[r+1]} \cong dR_{R/A}^\wedge \widehat{\otimes}_{\mathcal{A}} \mathcal{A} / \mathcal{I} \widehat{\otimes}_{\mathcal{A}/\mathcal{I}} \mathcal{I}^{[r]} / \mathcal{I}^{[r+1]} \cong dR_{R/(A/I)}^\wedge \widehat{\otimes}_{A/I} \mathcal{I}^{[r]} / \mathcal{I}^{[r+1]},$$

where the last identification comes from $dR_{R/A}^\wedge \widehat{\otimes}_{\mathcal{A}} \mathcal{A} / \mathcal{I} \cong dR_{R/(A/I)}^\wedge$ (c.f. [GL21, Proposition 3.11]) and $\mathcal{A} / \mathcal{I} \cong A/I$. In particular, these graded pieces are given by $dR_{R/(A/I)}^\wedge$ twisted by a rank 1 locally free sheaf on $\text{Spf}(A/I)$, hence are quasisyntomic sheaves themselves.

Since $dR_{R/A}^\wedge$ and all these graded pieces are quasisyntomic sheaves, each $\text{Fil}_{\mathcal{I}}^r$ is also a quasisyntomic sheaf.

If S is large quasisyntomic over A/I , then $dR_{S/A}^\wedge$ and all these graded pieces are supported in cohomological degree 0 by Lemma 4.3. By induction, in order to show the filtrations are in degree 0, it suffices to show $dR_{S/A}^\wedge \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]} \rightarrow dR_{S/A}^\wedge \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]} / \mathcal{I}^{[r+1]}$ is surjective for any r , which follows from the right exactness of p -complete tensor. \square

The filtration $\text{Fil}_{\mathcal{I}}^\bullet(dR_{R/A}^\wedge)$ is a disguise of the Katz–Oda filtration $\text{Fil}_{\text{KO}}^\bullet(dR_{C/A})$ discussed in [GL21], applied to the triple $(A \rightarrow B \rightarrow C) = (A \rightarrow A/I \rightarrow R)$. More precisely, we have

$$\text{Fil}_{\mathcal{I}}^i dR_{R/A}^\wedge \cong \text{Fil}_{\text{KO}}^\bullet(dR_{R/A})^\wedge.$$

We refer readers to the Subsection 3.2 of loc. cit. for a general discussion of additional structures on the derived de Rham complex of $A \rightarrow C$ when it factorizes through $A \rightarrow B \rightarrow C$.

Let R be an A/I -algebra. By p -completing the double filtrations obtained in [GL21, Construction 3.12], we see that $dR_{R/A}^\wedge$ can be naturally equipped a decreasing filtration indexed by $\mathbb{N} \times \mathbb{N}$:

$$\text{Fil}^{i,j}(dR_{R/A}^\wedge) := \left(\text{Fil}_{\text{KO}}^i \text{Fil}_H^j(dR_{R/A}) \right)^\wedge$$

The following proposition will describe $\text{Fil}^{i,j}(dR_{R/A}^\wedge)$ and declare its relation with the two systems of filtrations $\text{Fil}_H^\bullet dR_{R/A}^\wedge$ and $\text{Fil}_{\mathcal{I}}^\bullet dR_{R/A}^\wedge$.

Proposition 4.5. *Let R be an A/I -algebra. Then:*

- (1) *For any j , we have an identification $\text{Fil}^{0,j}(dR_{R/A}^\wedge) \cong \text{Fil}_H^j(dR_{R/A}^\wedge)$.*
- (2) *For each pair $0 \leq j \leq i$, we have an identification*

$$\text{Fil}^{i,j}(dR_{R/A}^\wedge) \cong \text{Fil}_{\mathcal{I}}^i(dR_{R/A}^\wedge).$$

(3) For each pair $0 \leq i \leq j$, we have a natural identification

$$\text{Cone}\left(\text{Fil}^{i+1,j}(\text{dR}_{R/A}^\wedge) \rightarrow \text{Fil}^{i,j}(\text{dR}_{R/A}^\wedge)\right) \cong \text{Fil}_H^{j-i} \text{dR}_{R/(A/I)}^\wedge \widehat{\otimes}_{A/I} \Gamma_{A/I}^i(I/I^2).$$

Moreover this identification is compatible with

$$\begin{array}{ccc} \text{Cone}\left(\text{Fil}^{i+1,j}(\text{dR}_{R/A}^\wedge) \rightarrow \text{Fil}^{i,j}(\text{dR}_{R/A}^\wedge)\right) & \xrightarrow{\cong} & \text{Fil}_H^{j-i} \text{dR}_{R/(A/I)}^\wedge \widehat{\otimes}_{A/I} \Gamma_{A/I}^i(I/I^2) \\ \downarrow & & \downarrow \\ \text{Cone}\left(\text{Fil}^{i+1,0}(\text{dR}_{R/A}^\wedge) \rightarrow \text{Fil}^{i,0}(\text{dR}_{R/A}^\wedge)\right) & \xrightarrow{\cong} & \text{dR}_{R/(A/I)}^\wedge \widehat{\otimes}_{A/I} \Gamma_{A/I}^i(I/I^2). \end{array}$$

(4) The association $R \mapsto \text{Fil}^{i,j}(\text{dR}_{R/A}^\wedge)$ defines a sheaf on the quasi-syntomic site of A/I for any (i, j) .

Proof. For (1): this follows from the [GL21, Construction 3.12], $\text{Fil}^{0,j}$ is the p -completed j -th filtration on $\text{dR}_{R/A} \otimes_{\text{dR}_A(I)} \text{Fil}_H^0(\text{dR}_A(I)) \cong \text{dR}_{R/A}$. Since this is a filtered isomorphism, we see that this is nothing but p -completed j -th Hodge filtration on $\text{dR}_{R/A}$, hence it is $\text{Fil}_H^j(\text{dR}_{R/A}^\wedge)$.

For (2): this follows from the [GL21, Construction 3.9]. Indeed, the inequality $j \leq i$ implies that the Fil^j of each term showing in [GL21, Construction 3.9] is the whole term. Hence the colimit just gives $\text{dR}_{R/A} \otimes_{\text{dR}_A(I)} \text{Fil}_H^i(\text{dR}_A(I))$ back. After p -completing, we see that by definition we have $\text{Fil}^{i,j}(\text{dR}_{R/A}^\wedge) \cong \text{Fil}_T^i(\text{dR}_{R/A}^\wedge)$.

(3) follows from p -completing [GL21, Proposition 3.13.(1)].

For (4): first we claim the associations $R \mapsto \text{Fil}_H^m(\text{dR}_{R/A}^\wedge)$ and $R \mapsto \text{Fil}_H^n(\text{dR}_{R/(A/I)}^\wedge)$ define sheaves for all m and n . For $m = 0$ this is Proposition 3.13, and for $n = 0$ this is [BMS19, Example 5.12]. Induction on m and n reduces us to showing the sheaf property of graded pieces, which are given by $\wedge_R^i \mathbb{L}_{R/A}^\wedge[-i]$ and $\wedge_{R/(A/I)}^i \mathbb{L}_{R/(A/I)}^\wedge[-i]$. p -Completing [BMS19, Theorem 3.1] gives the desired sheaf property of these graded pieces.

Fix a natural number j , then by (1) we see that $\text{Fil}^{0,j}$ is a quasisyntomic sheaf. Each graded piece with respect to i , by (2) and (3), is also a sheaf. Therefore we see that by increasing induction on i , each $\text{Fil}^{i,j}$ defines a sheaf. \square

To understand these sheaves more concretely, we look at their value on the basis of large quasisyntomic over A/I -algebras.

Proposition 4.6. *Let S be a large quasisyntomic over A/I algebra. Then:*

- (1) For any pair $(i, j) \in \mathbb{N} \times \mathbb{N}$, the $\text{Fil}^{i,j}(\text{dR}_{S/A}^\wedge)$ is concentrated in degree 0, and the natural map $\text{Fil}^{i,j}(\text{dR}_{S/A}^\wedge) \rightarrow \text{dR}_{S/A}^\wedge$ is injective.
- (2) For any j , the natural map

$$\text{Fil}_H^j(\text{dR}_{S/A}^\wedge) \rightarrow \text{Fil}_H^j(\text{dR}_{S/(A/I)}^\wedge)$$

is surjective.

- (3) For each pair $0 \leq i \leq j$, we have an equality:

$$\text{Fil}^{i,j}(\text{dR}_{S/A}^\wedge) = \sum_{r=i}^j \left(\text{Fil}_H^{j-r} \text{dR}_{S/A}^\wedge \cdot \mathcal{I}^{[r]} \right),$$

where $\text{Fil}_H^{j-r} \text{dR}_{S/A}^\wedge \cdot \mathcal{I}^{[r]}$ denotes the image of $\text{Fil}_H^{j-r} \text{dR}_{S/A}^\wedge \widehat{\otimes}_A \mathcal{I}^{[r]} \rightarrow \text{dR}_{S/A}^\wedge$, and the sum is inside the algebra $\text{dR}_{S/A}^\wedge$.

- (4) We have another description:

$$\text{Fil}^{i,j}(\text{dR}_{S/A}^\wedge) = \left(\text{Fil}_H^j \text{dR}_{S/A}^\wedge \right) \cap \left(\text{Fil}_T^i \text{dR}_{S/A}^\wedge \right),$$

where the intersection happens inside the algebra $\text{dR}_{S/A}^\wedge$.

Proof. Proof of (1): we shall prove by decreasing induction on i . When $j \leq i$, by Proposition 4.5 (2) we see that $\mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge) \cong \mathrm{Fil}_{\mathcal{I}}^i(\mathrm{dR}_{S/A}^\wedge)$, which is concentrated in degree 0 by Lemma 4.4. By Proposition 4.5 (3), the graded pieces with respect to i are all concentrated in degree 0 by Lemma 4.3. This in turn implies that,

- All of $\mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge)$ are in degree 0 for any (i, j) ; and
- We have short exact sequences:

$$0 \rightarrow \mathrm{Fil}^{i+1,j}(\mathrm{dR}_{S/A}^\wedge) \rightarrow \mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge) \rightarrow \mathrm{Fil}_{\mathbb{H}}^{j-i} \mathrm{dR}_{R/(A/I)}^\wedge \widehat{\otimes}_{A/I} \Gamma_{A/I}^i(I/I^2) \rightarrow 0.$$

In particular $\mathrm{Fil}^{i+1,j}(\mathrm{dR}_{S/A}^\wedge) \rightarrow \mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge)$ is injective. Using Proposition 4.5 (1) and Lemma 4.3, we see that the map $\mathrm{Fil}^{0,j}(\mathrm{dR}_{S/A}^\wedge) \cong \mathrm{Fil}_{\mathbb{H}}^j(\mathrm{dR}_{S/A}^\wedge) \rightarrow \mathrm{dR}_{S/A}^\wedge$ is also injective. Therefore the composition $\mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge) \rightarrow \mathrm{dR}_{S/A}^\wedge$ is injective as well for any (i, j) .

(2) follows from the short exact sequence obtained in the previous paragraph, specializing to $i = 0$.

(3) follows from the combination of (2), Proposition 4.5 (3), and the fact that p -completed tensor is right exact.

For (4): first notice that this is true for $i = 0$, due to Proposition 4.5 (1). Next let us look at the commutative diagram in Proposition 4.5 (3). Since the right hand side is an injection, we see that the map

$$\mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge) / \mathrm{Fil}^{i+1,j}(\mathrm{dR}_{S/A}^\wedge) \rightarrow \mathrm{Fil}^{i,0}(\mathrm{dR}_{S/A}^\wedge) / \mathrm{Fil}^{i+1,0}(\mathrm{dR}_{S/A}^\wedge)$$

is injective. Therefore, by Proposition 4.5 (2), we know that

$$\mathrm{Fil}^{i+1,j}(\mathrm{dR}_{S/A}^\wedge) = \left(\mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge) \right) \cap \left(\mathrm{Fil}_{\mathcal{I}}^{i+1} \mathrm{dR}_{S/A}^\wedge \right).$$

By increasing induction on i , we may assume

$$\mathrm{Fil}^{i,j}(\mathrm{dR}_{S/A}^\wedge) = \left(\mathrm{Fil}_{\mathbb{H}}^j \mathrm{dR}_{S/A}^\wedge \right) \cap \left(\mathrm{Fil}_{\mathcal{I}}^i \mathrm{dR}_{S/A}^\wedge \right).$$

Hence we have

$$\mathrm{Fil}^{i+1,j}(\mathrm{dR}_{S/A}^\wedge) = \left(\mathrm{Fil}_{\mathbb{H}}^j \mathrm{dR}_{S/A}^\wedge \right) \cap \left(\mathrm{Fil}_{\mathcal{I}}^i \mathrm{dR}_{S/A}^\wedge \right) \cap \left(\mathrm{Fil}_{\mathcal{I}}^{i+1}(\mathrm{dR}_{S/A}^\wedge) \right) = \left(\mathrm{Fil}_{\mathbb{H}}^j \mathrm{dR}_{S/A}^\wedge \right) \cap \left(\mathrm{Fil}_{\mathcal{I}}^{i+1} \mathrm{dR}_{S/A}^\wedge \right).$$

□

Let us draw a table to summarize these filtrations on $dR_{R/A}^\wedge$:

$$\begin{array}{ccccccc}
& & & R & & \mathbb{L}_{R/(A/I)}^\wedge[-1] & & (\wedge_R^2 \mathbb{L}_{R/(A/I)})^\wedge[-2] & & \cdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
\text{---} & & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
A/I & & & M_0 \widehat{\otimes}_{A/I} N_0 & & M_0 \widehat{\otimes}_{A/I} N_1 & & M_0 \widehat{\otimes}_{A/I} N_2 & & \cdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
\mathcal{I}/\mathcal{I}^{[2]} & & & M_1 \otimes_{A/I} N_0 & & M_1 \widehat{\otimes}_{A/I} N_1 & & M_1 \widehat{\otimes}_{A/I} N_2 & & \cdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
\mathcal{I}^{[2]}/\mathcal{I}^{[3]} & & & M_2 \widehat{\otimes}_{A/I} N_0 & & M_2 \widehat{\otimes}_{A/I} N_1 & & M_2 \widehat{\otimes}_{A/I} N_2 & & \cdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

In the diagram above, $M_i = \mathcal{I}^{[i]}/\mathcal{I}^{[i+1]}$, and $N_j = (\wedge_R^j \mathbb{L}_{R/(A/I)})^\wedge[-j]$, for $i, j \in \mathbb{N}$. Here rows indicate graded pieces of the filtration $\text{Fil}_{\mathcal{I}}^r$, and each term in i -th row indicates the graded piece of the induced filtration on $dR_{R/(A/I)}^\wedge \widehat{\otimes}_{A/I} \Gamma_{A/I}^i(I/I^2)$. The skewed dotted line indicate the Hodge filtration on $dR_{R/A}^\wedge$ (given by things below the dotted line). See also [GL21, p.10].

As a consequence we get a structural result on the graded algebra associated with the Hodge filtration on $dR_{R/A}^\wedge$.

Lemma 4.7. *There is a functorial increasing exhaustive filtration Fil_i^v on the graded algebra $\text{gr}_H^*(dR_{R/A}^\wedge)$ by graded- $(\text{gr}_{\mathcal{I}}^* \mathcal{A} \cong \Gamma_{A/I}^*(I/I^2))$ -submodules with graded pieces given by*

$$\text{gr}_i^v \left(\text{gr}_H^*(dR_{R/A}^\wedge) \right) \cong (\wedge_R^i \mathbb{L}_{R/(A/I)})^\wedge[-i] \widehat{\otimes}_{A/I} \Gamma_{A/I}^*(I/I^2).$$

Here $(\wedge_R^i \mathbb{L}_{R/(A/I)})^\wedge[-i]$ has degree i and the above is a graded isomorphism.

We refer to this filtration Fil_i^v on $\text{gr}_H^*(dR_{R/A}^\wedge)$ as the *vertical filtration* from now on, c.f. [GL21, Construction 3.14]. This choice of name is because the Fil_i^v is literally the filtration given by vertical columns in the table before this Lemma.

Proof. Use the above table one can see this directly. Equivalently, we may use

$$\text{gr}_H^*(dR_{R/A}^\wedge) \cong \left(\Gamma_R^*(\mathbb{L}_{R/A}^\wedge[-1]) \right)^\wedge,$$

and the triangle

$$R \widehat{\otimes}_{A/I} I/I^2 \rightarrow \mathbb{L}_{R/A}^\wedge[-1] \rightarrow \mathbb{L}_{R/(A/I)}^\wedge[-1].$$

□

Remark 4.8. Let (A, I) be a general bounded prism, and let S be a large quasisyntomic over A/I -algebra. Combining Theorem 3.5, Construction 3.12 (4), and [BS19, Theorem 15.2.(1)], we can see that $dR_{S/A}^\wedge$ is p -completely flat over $dR_A(I)^\wedge$.

Below is suggested to us by Bhatt. Using conjugate filtration and the same argument of Lemma 4.7, we can give an alternative proof of this fact. Indeed we can check this after mod p , hence we shall assume A to be p -torsion. Next we want to appeal to the conjugate filtrations on both algebras: we have the following pushout diagram:

$$\begin{array}{ccccc} & & A & \longrightarrow & A/I^p & \longrightarrow & R^{(1)} \\ & & \uparrow & & \uparrow & & \uparrow \\ \varphi_A & & A & \longrightarrow & A/I & \longrightarrow & R \end{array}$$

There is a similar functorial increasing exhaustive filtration on the graded algebra of the conjugate filtered $dR_{S/A}$, with graded pieces given by $(\wedge_{R^{(1)}}^i \mathbb{L}_{R^{(1)}/(A/I^p)})[-i] \otimes_{A/I^p} \Gamma_{A/I^p}^*(I^p/I^{2p})$. It is flat over $\Gamma_{A/I^p}^*(I^p/I^{2p})$, which is the conjugate graded algebra of $dR_A(I)$. Lastly we conclude by recalling that an increasingly exhaustive filtered module of an increasingly exhaustive filtered algebra is flat if the graded counterpart is flat.

4.2. Nygaard filtration. Recall in [BS19, Section 15], there is a natural decreasing filtration of quasisyntomic subsheaves on $\mathbb{A}_{-/A}^{(1)}$ called the Nygaard filtration with the following properties:

Theorem 4.9 (see [BS19, Theorem 15.2 and 15.3] and proof therein). *Let S be a large quasisyntomic over A/I algebra. Then*

- (1) *The Nygaard filtrations $\text{Fil}_{\mathbb{N}}^\bullet$ on $\mathbb{A}_{S/A}^{(1)}$ are given by p -completely flat A -submodules inside $\mathbb{A}_{S/A}^{(1)}$.*
- (2) *We have an identification of algebras $\mathbb{A}_{S/A}^{(1)}/I \cong dR_{S/(A/I)}^\wedge$, under which the image of Nygaard filtration becomes the Hodge filtration.*
- (3) *For each $i \geq 0$, we have a short exact sequence:*

$$0 \rightarrow \text{Fil}_{\mathbb{N}}^i \mathbb{A}_{S/A}^{(1)} \otimes_A I \rightarrow \text{Fil}_{\mathbb{N}}^{i+1} \mathbb{A}_{S/A}^{(1)} \rightarrow \text{Fil}_{\mathbb{H}}^{i+1} dR_{R/(S/I)}^\wedge \rightarrow 0.$$

Let R be a general quasisyntomic A/I -algebra. On $\mathbb{A}_{R/A}^{(1)}$ there is also an I -adic filtration $\text{Fil}_I^r \mathbb{A}_{R/A}^{(1)} := \mathbb{A}_{R/A}^{(1)} \otimes_A I^r$, by Theorem 4.9 (2), we identify the graded pieces as

$$\text{gr}_I^r \cong \mathbb{A}_{R/A}^{(1)}/I \otimes_{A/I} I^r/I^{r+1} \cong dR_{R/(A/I)}^\wedge \otimes_{A/I} \text{Sym}_{A/I}^r(I/I^2).$$

The I -adic filtration and the Nygaard filtration are related by the following. For any $(i, j) \in \mathbb{N} \times \mathbb{N}$, we define

$$\text{Fil}^{i,j} \mathbb{A}_{R/A}^{(1)} := \text{Fil}_{\mathbb{N}}^{j-i} \mathbb{A}_{R/A}^{(1)} \otimes_A I^i,$$

where we adopt the convention that $\text{Fil}_{\mathbb{N}}^l \mathbb{A}_{R/A}^{(1)} = \mathbb{A}_{R/A}^{(1)}$ if $l \leq 0$. One checks easily that this puts a decreasing filtration on $\mathbb{A}_{R/A}^{(1)}$ indexed by $\mathbb{N} \times \mathbb{N}$. This filtration has very similar behavior as the $\text{Fil}^{i,j}(dR_{R/A}^\wedge)$ studied in previous subsection. The following is the analogue of Proposition 4.5.

Proposition 4.10. *Let R be an A/I -algebra. Then:*

- (1) *For any j , we have $\text{Fil}^{0,j} \mathbb{A}_{R/A}^{(1)} \cong \text{Fil}_{\mathbb{N}}^j \mathbb{A}_{R/A}^{(1)}$.*
- (2) *For each pair $0 \leq j \leq i$, we have*

$$\text{Fil}^{i,j} \mathbb{A}_{R/A}^{(1)} \cong \text{Fil}_I^i \mathbb{A}_{R/A}^{(1)}.$$

(3) For each pair $0 \leq i \leq j$, we have a natural identification

$$\text{Cone}\left(\text{Fil}^{i+1,j} \Delta_{R/A}^{(1)} \rightarrow \text{Fil}^{i,j} \Delta_{R/A}^{(1)}\right) \cong \text{Fil}_{\mathbb{H}}^{j-i}(\text{dR}_{R/(A/I)}^{\wedge}) \otimes_{A/I} \text{Sym}_{A/I}^i(I/I^2).$$

Moreover these identifications fit in the following commutative diagram:

$$\begin{array}{ccc} \text{Cone}\left(\text{Fil}^{i+1,j} \Delta_{R/A}^{(1)} \rightarrow \text{Fil}^{i,j} \Delta_{R/A}^{(1)}\right) & \xrightarrow{\cong} & \text{Fil}_{\mathbb{H}}^{j-i}(\text{dR}_{R/(A/I)}^{\wedge}) \otimes_{A/I} \text{Sym}_{A/I}^i(I/I^2) \\ \downarrow & & \downarrow \\ \text{Cone}\left(\text{Fil}^{i+1,0} \Delta_{R/A}^{(1)} \rightarrow \text{Fil}^{i,0} \Delta_{R/A}^{(1)}\right) & \xrightarrow{\cong} & \text{dR}_{R/(A/I)}^{\wedge} \otimes_{A/I} \text{Sym}_{A/I}^i(I/I^2) \end{array}$$

(4) The association $R \mapsto \text{Fil}^{i,j} \Delta_{R/A}^{(1)}$ defines a sheaf on $\text{qSyn}_{A/I}$ for any (i, j) .

Proof. (1) and (2) follows from definition. (3) follows from Theorem 4.9 (3). (4) follows from (3). \square

Proposition 4.11. *Let S be a large quasisyntomic over A/I algebra. Then:*

(1) *We have an equality:*

$$\text{Fil}^{i,j} \Delta_{S/A}^{(1)} = \sum_{r=i}^j \left(\text{Fil}_{\mathbb{N}}^{j-r}(\Delta_{S/A}^{(1)}) \cdot I^r \right),$$

where the sum is inside the algebra $\Delta_{S/A}^{(1)}$.

(2) *We have another equality:*

$$\text{Fil}^{i,j} \Delta_{S/A}^{(1)} = \left(\text{Fil}_{\mathbb{N}}^j \Delta_{S/A}^{(1)} \right) \cap \left(\text{Fil}_I^i \Delta_{S/A}^{(1)} \right),$$

where the intersection happens inside the algebra $\Delta_{S/A}^{(1)}$.

Proof. The proof is similar to Proposition 4.6 (3) and (4). Notice that $\text{Fil}_{\mathbb{N}}^j \Delta_{S/A}^{(1)} \rightarrow \text{Fil}_{\mathbb{H}}^j(\text{dR}_{R/(A/I)}^{\wedge})$ is surjective by Theorem 4.9 (2). \square

We can express all these structures on $\Delta_{R/A}^{(1)}$ in the following graph similar to what was drawn in the previous subsection. One observes that the distinction is just that divided powers of I/I^2 get replaced by

symmetric powers of I/I^2 .

$$\begin{array}{ccccccc}
 & \vdots & R & \mathbb{L}_{R/(A/I)}^\wedge[-1] & (\wedge_R^2 \mathbb{L}_{R/(A/I)})^\wedge[-2] & \cdots & \\
 & \vdots & & & & & \\
 \hline
 A/I & \vdots & M_0 \widehat{\otimes}_{A/I} N_0 & M_0 \widehat{\otimes}_{A/I} N_1 & M_0 \widehat{\otimes}_{A/I} N_2 & \cdots & \\
 & \vdots & & & & & \\
 I/I^2 & \vdots & M_1 \otimes_{A/I} N_0 & M_1 \widehat{\otimes}_{A/I} N_1 & M_1 \widehat{\otimes}_{A/I} N_2 & \cdots & \\
 & \vdots & & & & & \\
 I^2/I^3 & \vdots & M_2 \widehat{\otimes}_{A/I} N_0 & M_2 \widehat{\otimes}_{A/I} N_1 & M_2 \widehat{\otimes}_{A/I} N_2 & \cdots & \\
 & \vdots & & & & & \\
 & \vdots & & & & &
 \end{array}$$

Here rows indicate graded pieces of the filtration Fil_i^r , and each term in each row indicates the graded piece of the Hodge filtration on $dR_{R/(A/I)}^\wedge$. The skewed dotted line indicate the Nygaard filtration on $\Delta_{R/A}^{(1)}$ (given by things below the dotted line).

Also as a consequence we get a structural result on the graded algebra associated with the Nygaard filtration on $\Delta_{R/A}^{(1)}$.

Lemma 4.12. *There is a functorial increasing exhaustive filtration Fil_i^v on the graded algebra $\text{gr}_N^*(\Delta_{R/A}^{(1)})$ by graded- $(\text{gr}_I^* A \cong \text{Sym}_{A/I}^*(I/I^2))$ -submodules with graded pieces given by*

$$\text{gr}_i^v \left(\text{gr}_N^*(\Delta_{R/A}^{(1)}) \right) \cong (\wedge_R^i \mathbb{L}_{R/(A/I)})^\wedge[-i] \widehat{\otimes}_{A/I} \text{Sym}_{A/I}^*(I/I^2).$$

Here $(\wedge_R^i \mathbb{L}_{R/(A/I)})^\wedge[-i]$ has degree i and the above is a graded isomorphism.

We also call this filtration Fil_i^v on $\text{gr}_N^*(\Delta_{R/A}^{(1)})$ as the *vertical filtration* from now on.

Proof. This follows from Theorem 4.9 (3), see also the proof of Lemma 4.7. \square

4.3. Promote to filtered map. Recall that we use (A, I) to denote a transversal prism. For the rest of this section, we shall use (B, J) to denote a general bounded prism. By Theorem 3.5, we have a map $\Delta_{R/B}^{(1)} \rightarrow dR_{R/B}^\wedge$ functorial in $B/J \rightarrow R$. The goal of this subsection is to show that this map can be promoted to a filtered map where the left hand side is equipped with the Nygaard filtration and the right hand side is equipped with the Hodge filtration. Our plan is:

- show certain rigidity of the map being filtered;
- show the map is filtered when the base prism (A, I) is transversal;
- show the map is filtered when the algebra R is large quasisyntomic over B/J of a particular type; and
- show the map is functorially filtered when R is a p -completely smooth B/J -algebra, and hence finish the argument by left Kan extension.

Fix a natural number i . The main diagram that we shall stare at in this subsection is:

$$\begin{array}{ccccc}
 \text{Fil}_{\mathbb{N}}^i & \longrightarrow & \Delta^{(1)} & \longrightarrow & Q_{1,i} \\
 \vdots \downarrow g_i & & \downarrow & & \vdots \downarrow f_i \\
 \text{Fil}_{\mathbb{H}}^i & \longrightarrow & \text{dR} & \longrightarrow & Q_{2,i}
 \end{array}$$

viewed as a commutative diagram of sheaves on $\text{qSyn}_{B/J}$. Here $Q_{1,i}$ and $Q_{2,i}$ are the cones of the natural maps, so both rows are distinguished triangles of quasisyntomic sheaves. All the solid arrows are defined: for instance, the middle vertical arrow is given by Theorem 3.5. Our main task is to show that one can fill in the dotted arrows f_i and g_i making the diagram commute.

We first need a few lemmas to illustrate that the situation is pretty rigid and there is at most one choice of these dotted arrows.

Lemma 4.13. *Let S be a large quasisyntomic over B/J algebra, then the values of*

$$\text{Fil}_{\mathbb{N}}^i, \Delta^{(1)}, Q_{1,i}, \text{ and } Q_{2,i}$$

at S are concentrated in cohomological degree 0.

Proof. The first three follows from how they are defined, see [BS19, Subsection 15.1]. The claim for $Q_{2,i}$ follows from the fact that $\mathbb{L}_{S/B}^\wedge[-1]$ lives in cohomological degree 0. \square

Lemma 4.14. *Let S be a large quasisyntomic over B/J algebra. Then,*

- (1) *there is at most one choice of f_i making the right square of \square commute;*
- (2) *if $S \rightarrow T$ is a morphism of large quasisyntomic over B/J algebras, and suppose f_i are defined on both of them, then the diagram*

$$\begin{array}{ccc}
 Q_{1,i}(S) & \longrightarrow & Q_{1,i}(T) \\
 \downarrow f_i(S) & & \downarrow f_i(T) \\
 Q_{2,i}(S) & \longrightarrow & Q_{2,i}(T)
 \end{array}$$

is commutative;

- (3) *the existence of $f_i(S)$ is equivalent to the existence of $g_i(S)$ making the left square of \square commute;*
- (4) *the $g_i(S)$, if exists, must be unique.*

Proof. (1): suppose there are two of them, and take their difference. Since precomposing with $\Delta^{(1)} \rightarrow Q_{1,i}$ of this difference is the zero map $\Delta^{(1)} \xrightarrow{0} Q_{2,i}$ due to commutativity, the difference must factor through $\text{Fil}_{\mathbb{N}}^i[1]$. But $\text{Hom}(\text{Fil}_{\mathbb{N}}^i[1], Q_{2,i}) = \{0\}$ by cohomological considerations in Lemma 4.13. Hence the difference must be zero.

(2): the argument is similar to (1). The difference of the two arrows from $Q_{1,i}(S)$ to $Q_{2,i}(T)$ will again factor through $\text{Fil}_{\mathbb{N}}^i(S)[1]$, hence must again be zero.

(3): just apply TR3, notice that the two rows of \square are exact triangles.

(4): similar to (1). The difference of two possible $g_i(S)$'s factors through an arrow $\text{Fil}_{\mathbb{N}}^i(S) \rightarrow Q_{2,i}[-1](S)$ which is again zero by cohomological considerations. \square

After knowing the rigidity of our situation, we shall start proving the existence of f_i following the plan outlined above.

Proposition 4.15. *Let (A, I) be a transversal prism, and let $S = A/I\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle / (f_1, \dots, f_r)$ where (f_i) is a p -completely regular sequence. We have $\mathrm{Fil}_N^i \Delta_{S/A}^{(1)} \subset \mathrm{Fil}_H^i dR_{S/A}^\wedge$.*

Proof. When $i = 0$, there is nothing to prove, when $i = 1$, the triangle in Corollary 3.18 gives us a commutative diagram

$$\begin{array}{ccc} & S & \\ & \nearrow & \nwarrow \\ \Delta_{S/A}^{(1)} & \longrightarrow & dR_{S/A}^\wedge, \end{array}$$

since the kernels of these two surjections define the first Nygaard and Hodge filtrations respectively, we see the containment for $i = 1$. For general i , we prove by induction. Let us look at the induced map $g: \mathrm{Fil}_N^i \Delta_{S/A}^{(1)} \rightarrow dR_{S/A}^\wedge / \mathrm{Fil}_H^i$. We first notice that by induction and $I \subset \mathcal{I}$ the submodule $I \cdot \mathrm{Fil}_N^{i-1} \Delta_{S/A}^{(1)}$ is sent to zero under g . By multiplicativity and the containment for $i = 1$, we have that $\mathrm{Sym}^i(\mathrm{Fil}_N^1 \Delta_{S/A}^{(1)})$ is also sent to zero under g . Now we use Theorem 4.9 (3) to see that $\mathrm{Fil}_N^i / I \cdot \mathrm{Fil}_N^{i-1}$ is identified with $\mathrm{Fil}_H^i dR_{S/(A/I)}^\wedge$ and the image of $\mathrm{Sym}^i(\mathrm{Fil}_N^1 \Delta_{S/A}^{(1)})$ becomes $\mathrm{Sym}^i(\mathrm{Fil}_H^1 dR_{S/(A/I)}^\wedge)$, so we get an induced map $\bar{g}: \left(\mathrm{Fil}_H^i dR_{S/(A/I)}^\wedge / \mathrm{Sym}^i(\mathrm{Fil}_H^1 dR_{S/(A/I)}^\wedge) \right) \rightarrow dR_{S/A}^\wedge / \mathrm{Fil}_H^i$. But the source of this map has its p -power torsions submodule being p -adically dense and the target of this map is p -torsionfree and p -adically complete, so the map \bar{g} must in fact be zero. This proves the containment $\mathrm{Fil}_N^i \subset \mathrm{Fil}_H^i$ as claimed. \square

The following is inspired by [BS19, Subsection 12.4].

Proposition 4.16. *Let (A, I) be a transversal prism, then for any p -completely smooth A/I -algebra R , the map $\Delta_{R/A}^{(1)} \rightarrow dR_{R/A}^\wedge$ can be promoted to a map of filtered algebras. Moreover this lift is functorial in the A/I -algebra R , hence left Kan extends to all animated A/I -algebras.*

Proof. For any surjection $A/I\langle X_1, \dots, X_n \rangle \rightarrow R$, the ring

$$\tilde{R} = A/I\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle \otimes_{A/I\langle X_1, \dots, X_n \rangle} R$$

is large quasisyntomic over A/I and Zariski locally of the form considered in the previous proposition. Therefore the map $\Delta_{\tilde{R}/A}^{(1)} \rightarrow dR_{\tilde{R}/A}^\wedge$ is canonically filtered: (Zariski locally) existence follows from the previous proposition, Zariski glue as well as uniqueness is provided by Lemma 4.14. The same applies to all terms of the Čech nerve \tilde{R}^\bullet of $R \rightarrow \tilde{R}$.

The filtered cosimplicial rings $\Delta_{\tilde{R}^\bullet/A}^{(1)}$ and $dR_{\tilde{R}^\bullet/A}^\wedge$ computes the filtered rings $\Delta_{\tilde{R}/A}^{(1)}$ and $dR_{\tilde{R}/A}^\wedge$ separately. By Lemma 4.14, we get a map of filtered cosimplicial rings.

This construction is independent of the choice of the surjection $A/I\langle X_1, \dots, X_n \rangle \rightarrow R$: adding extra variables to the X_i , one gets a square of maps between filtered cosimplicial algebras, we use Lemma 4.14 to see the maps commute for each term associated with $[m] \in \Delta$. Since the category of such surjections admits pairwise coproducts, it is therefore sifted. The naturality in R follows from exactly the same argument. This way we get the desired functorial map. \square

Now we turn to the general situation where the base prism (B, J) is not necessarily transversal. We bootstrap the previous two propositions.

Proposition 4.17. *Let $S = B/J\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle / (f_1, \dots, f_r)$ where (f_i) is a p -completely regular sequence. Then there exists $f_i(S)$ and $g_i(S)$ making the diagram \square commute.*

Proof. We shall utilize the knowledge when the base prism is transversal. Without loss of generality, let us assume $(B, J) = (B, (d))$ is oriented: Zariski locally it is oriented, and the locally defined $f_i(S)$ and $g_i(S)$ will necessarily glue due to Lemma 4.14.

Following a private communication with Illusie, let us define a transversal prism $(A, (a))$ together with a surjection of prisms $(A, (a)) \twoheadrightarrow (B, (d))$ as follows. Let

$$A := \mathbb{Z}_p\{x_b; b \in B\} \{ \delta(x_d)^{-1} \}_{(x_d, p)}^\wedge$$

be given by first adjoining a free δ -variable corresponding to each element in B to \mathbb{Z}_p together with an inverse of the δ of the variable corresponding to $d \in B$, completed in the end with respect to (p, x_d) . Denote $a := x_d$. Then $(A, (a))$ is a transversal prism, and there is an evident surjection of prisms $(A, (a)) \rightarrow (B, (d))$.

Consider the surjection $A/a\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle \rightarrow B/J\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle$ and lift the elements f_i to \tilde{f}_i . Let

$$\tilde{S} := \text{Kos}(A/a\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle; \tilde{f}_1, \dots, \tilde{f}_r).$$

We have $S = \tilde{S} \otimes_{(A/a)}^{\mathbb{L}} B/d$. We know the analogous map for $\tilde{S}/(A/a)$ exists, thanks to Proposition 4.16. Since both Nygaard and Hodge filtrations satisfy base change, we may base change the maps for $\tilde{S}/(A/a)$ to obtain our desired maps for $S/(B/b)$. \square

Following the same reasoning as in the proof of Proposition 4.16, one obtains the following:

Proposition 4.18. *For any p -completely smooth B/J -algebra R , the map $\Delta_{R/B}^{(1)} \rightarrow \text{dR}_{R/B}^\wedge$ can be promoted to a map of filtered algebras. Moreover this lift is functorial in the B/J -algebra R , hence left Kan extends to all animated B/J -algebras.*

Proof. The argument is exactly the same as Proposition 4.16, note that Lemma 4.14 applies to general bounded base prism (B, J) . \square

Remark 4.19. Fix a bounded base prism (B, J) . Any such functorial filtered map is determined by its effect on p -complete polynomial algebras, by left Kan extension. Then by quasisyntomic descent, such a functorial filtered map is determined by its effect on a basis of $\text{qSyn}_{B/J}$, such as the full subcategory generated by large quasisyntomic over B/J algebras. Therefore Lemma 4.14 implies that there is at most one such functorial filtered map. Combining with the above proposition, we have both existence and uniqueness of it.

Remark 4.20. Following the way these filtered maps are constructed, we have certain compatibility with base change: Let R be an animated B/J -algebra, let $(B, J) \rightarrow (C, JC)$ be a map of bounded prisms and denote $R' := R \otimes_{B/J}^{\mathbb{L}} C/JC$, then the filtered map for R'/C arises as the filtered map for R/B base changed along $B \rightarrow C$. Indeed it suffices to prove this when $R/(B/J)$ is p -completely smooth. Then one simply notices that a surjection $B/J\langle X_1, \dots, X_n \rangle \rightarrow R$ base change along $B \rightarrow C$ will give rise to a surjection $C/JC\langle X_1, \dots, X_n \rangle \rightarrow R'$.

4.4. Comparing Hodge and Nygaard filtrations. We again use (A, I) to denote a transversal prism, and use (B, J) to denote a general bounded prism. All sheaves referred to in this subsection are viewed as objects in $\text{Shv}(\text{qSyn}_{A/I})$ or $\text{Shv}(\text{qSyn}_{B/J})$ depending on the context. Combining Theorem 3.5 and Proposition 4.18, we get a natural map of sheaves of filtered rings:

$$(\Delta_{-/B}^{(1)}, \text{Fil}_{\mathbb{N}}^\bullet) \widehat{\otimes}_{(B, J^\bullet)} (\text{dR}_{(B/J)/B}^\wedge, \text{Fil}_{\mathbb{H}}^\bullet) \longrightarrow (\text{dR}_{-/A}^\wedge, \text{Fil}_{\mathbb{H}}^\bullet),$$

which is an isomorphism on the underlying sheaf of rings. Our objective in this subsection is to show that the above map is an isomorphism of sheaves of filtered rings.

Our plan is again to first understand the case of transversal base prism, then bootstrap to general base prisms. Let us begin by discussing the case of transversal base prism.

Theorem 4.21. *Let S be a large quasisyntomic over A/I algebra.*

- (1) *The map $\Delta_{S/A}^{(1)} \rightarrow \text{dR}_{S/A}^\wedge$ is injective.*
- (2) *We have*

$$\text{Fil}_I^r \Delta_{S/A}^{(1)} = \left(\text{Fil}_I^r \text{dR}_{S/A}^\wedge \right) \cap \left(\Delta_{S/A}^{(1)} \right).$$

- (3) *We have*

$$\text{Fil}_{\mathbb{N}}^i \Delta_{S/A}^{(1)} = \left(\text{Fil}_{\mathbb{H}}^i \text{dR}_{S/A}^\wedge \right) \cap \left(\Delta_{S/A}^{(1)} \right).$$

- (4) For any i , the natural map $\Delta_{S/A}^{(1)}/\mathrm{Fil}_N^i \rightarrow dR_{S/A}^\wedge/\mathrm{Fil}_H^i$ is an injection of p -torsionfree modules, whose cokernel is $(i-1)!$ -torsion. Hence multiplying by $(i-1)!$ gives a natural map backward and compose the two maps in either direction is the same as multiplying by $(i-1)!$. In particular, the natural map

$$\Delta_{S/A}^{(1)}/\mathrm{Fil}_N^i \rightarrow dR_{S/A}^\wedge/\mathrm{Fil}_H^i$$

is an isomorphism for any $i \leq p$.

- (5) The induced map $\mathrm{gr}_N^* \Delta_{S/A}^{(1)} \rightarrow \mathrm{gr}_H^* dR_{S/A}^\wedge$ is compatible with the vertical filtrations on both sides, and the induced map on the graded pieces of the vertical filtrations $(\wedge_S^i \mathbb{L}_{S/(A/I)})^\wedge[-i] \widehat{\otimes}_{A/I} \mathrm{Sym}_{A/I}^*(I/I^2) \rightarrow (\wedge_S^i \mathbb{L}_{S/(A/I)})^\wedge[-i] \widehat{\otimes}_{A/I} \Gamma_{A/I}^*(I/I^2)$ is given by $\mathrm{id} \otimes_{A/I} (\mathrm{gr}_I^* A \rightarrow \mathrm{gr}_{\mathcal{I}}^* \mathcal{A})$.

Contemplating with $R = A/I$ suggests that our estimate in (4) is sharp.

Before the proof, let us remark that p -completely tensor over A with an \mathcal{A} -module is the same as (p, I) -completely tensor. This is because $I^p \mathcal{A} \subset p\mathcal{A}$.

Proof. (1): the map is given by (p, I) -completely tensoring the inclusion $A \hookrightarrow \mathcal{A}$ with $\Delta_{S/A}^{(1)}$ over A . Since $\Delta_{S/A}^{(1)}$ is (p, I) -completely flat over A , see Remark 4.8, we get the injectivity of $\Delta_{S/A}^{(1)} \rightarrow dR_{S/A}^\wedge$.

(2): clearly we have $I^r \Delta_{S/A}^{(1)}$ contained in $\mathcal{I}^{[r]} dR_{S/A}^\wedge$. To check the equality of intersection, it suffices to show the induced map $\Delta_{S/A}^{(1)}/I^r \rightarrow dR_{S/A}^\wedge/\mathcal{I}^{[r]}$ is injective. But this map is given by (p, I) -completely tensoring $\Delta_{S/A}^{(1)}$ with the inclusion $A/I^r \hookrightarrow \mathcal{A}/\mathcal{I}^{[r]}$ over A , so we get the desired injectivity again by (p, I) -completely flatness of $\Delta_{S/A}^{(1)}$ over A .

(3): it suffices to show that the induced map $\tilde{g}: \Delta_{S/A}^{(1)}/\mathrm{Fil}_N^i \rightarrow dR_{S/A}^\wedge/\mathrm{Fil}_H^i$ is injective. The I -adic and $\mathcal{I}^{[\bullet]}$ -filtrations on each side induces maps of graded pieces as

$$dR_{S/(A/I)}^\wedge/\mathrm{Fil}_H^j \widehat{\otimes}_{A/I} I^{i-j}/I^{i-j+1} \rightarrow dR_{S/(A/I)}^\wedge/\mathrm{Fil}_H^j \widehat{\otimes}_{A/I} \mathcal{I}^{[i-j]}/\mathcal{I}^{[i-j+1]}.$$

Here we have used Proposition 4.5 (3) and Proposition 4.10 (3). We conclude that the map \tilde{g} is injective as $dR_{S/(A/I)}^\wedge/\mathrm{Fil}_H^j$ is p -completely flat over A/I for any j and the natural map $I^{i-j}/I^{i-j+1} \rightarrow \mathcal{I}^{[i-j]}/\mathcal{I}^{[i-j+1]}$ is injective.

(4): injectivity follows from the previous paragraph. Let $S = A/I \langle X_l^{1/p^\infty} \mid l \in L \rangle / M$, with each element $m \in M$ corresponding to a series f_m . Below we shall not distinguish m and f_m . Consider

$$S' = A/I \langle X_l^{1/p^\infty}, Y_m^{1/p^\infty} \mid l \in L, m \in M \rangle / (Y_m - f_m; m \in M) =: \tilde{S}/(Y_m - f_m; m \in M).$$

There is a surjection $S' \twoheadrightarrow S$ of A/I -algebras, sending powers of Y_m to 0. This induces a surjection on $\mathbb{L}_{-/A}^\wedge$, hence also a surjection on $dR_{-/A}^\wedge$. Therefore it suffices to prove the statement for S' .

Now we know $dR_{S'/A}^\wedge$ is given by p -completely adjoining divided powers of I and $Y_m - f_m$ to \tilde{S} , and the i -th Hodge filtration is given by the ideal p -completely generated by those degree-at-least- i divided monomials. Since the image of $\Delta_{S'/A}^{(1)}$ contains \tilde{S} already, it suffices to show that $(i-1)!$ times those degree-less-than- i divided monomials lies in \tilde{S} , which follows from definition.

(5): since the generating factor $(\wedge_S^i \mathbb{L}_{S/(A/I)})^\wedge[-i]$ of both vertical filtrations comes from the i -th graded piece of the Hodge filtration on $dR_{S/(A/I)}^\wedge$ (via modulo I and \mathcal{I} respectively), our statement follows from the commutative triangle in Corollary 3.18. \square

The above statements can be immediately extended to our desired statement via several reduction steps.

Corollary 4.22. *Let R be a B/J -algebra. The natural map of filtered algebras:*

$$(\Delta_{R/B}^{(1)}, \mathrm{Fil}_N^\bullet) \widehat{\otimes}_{(B, J^\bullet)} (dR_{(B/J)/B}^\wedge, \mathrm{Fil}_H^\bullet) \longrightarrow (dR_{R/A}^\wedge, \mathrm{Fil}_H^\bullet),$$

is a filtered isomorphism. In particular, filtrations on the left hand side define quasisyntomic sheaves.

We refer readers to [GL21, 3.8-3.10] for a discussion of the filtration on tensor of filtered modules over a filtered algebra. Here we use $\widehat{\otimes}$ to mean that we derived p -complete the [GL21, Construction 3.9].

Proof. We make a few reduction steps. First of all, both sides are left Kan extended from the case of p -complete polynomial algebras, therefore it suffices to show the map is a filtered isomorphism when $R = B/J\langle X_1, \dots, X_n \rangle$. Secondly, it suffices to prove the statement Zariski locally on $\mathrm{Spf}(B/J)$, hence we may assume (B, J) is oriented. Now we look at the universal map from universal oriented prism $(A^{\mathrm{univ}}, I) \rightarrow (B, J)$. Let R^{univ} be the corresponding p -complete polynomial algebras over the reduction of the universal oriented prism. By Remark 4.20, one sees that the filtered map for R is the base change of the analogous map for R^{univ} . Therefore we are finally reduced to the case where the base prism (A, I) is transversal and R is a p -completely smooth A/I -algebra.

Since the underlying algebra is an isomorphism by Theorem 3.5, it suffices to show the induced map of graded algebra is an isomorphism. By derived p -completing [GL21, Lemma 3.10], we see that the graded algebra of left hand side becomes

$$\mathrm{gr}_N^*(\Delta_{R/A}^{(1)}) \widehat{\otimes}_{\mathrm{Sym}_{A/I}^*(I/I^2)} \Gamma_{A/I}^*(I/I^2).$$

Now we invoke the vertical filtrations on graded algebras of both sides, see Lemma 4.7 and Lemma 4.12. The vertical filtration on $\mathrm{gr}_N^*(\Delta_{R/A}^{(1)})$ induces an increasing filtration by $(-)\widehat{\otimes}_{\mathrm{Sym}_{A/I}^*(I/I^2)} \Gamma_{A/I}^*(I/I^2)$, and our morphism induces identifications

$$\mathrm{gr}_i^v \left(\mathrm{gr}_N^*(\Delta_{R/A}^{(1)}) \widehat{\otimes}_{\mathrm{Sym}_{A/I}^*(I/I^2)} \Gamma_{A/I}^*(I/I^2) \right) \cong \mathrm{gr}_i^v \left(\mathrm{gr}_H^*(\mathrm{dR}_{R/A}^\wedge) \right)$$

for all i . Here we have used Theorem 4.21 (5). Since these vertical filtrations are increasing, exhaustive, and uniformly bounded below by 0, we conclude that the natural map

$$\mathrm{gr}_N^*(\Delta_{R/A}^{(1)}) \widehat{\otimes}_{\mathrm{Sym}_{A/I}^*(I/I^2)} \Gamma_{A/I}^*(I/I^2) \longrightarrow \mathrm{gr}_H^*(\mathrm{dR}_{R/A}^\wedge)$$

is also an isomorphism. \square

In particular, we can specialize to the case of quasi-compact quasi-separated smooth formal schemes over $\mathrm{Spf}(B/J)$.

Corollary 4.23 (c.f. [Ill20, Theorem 2.9]). *Let X be a quasi-compact quasi-separated smooth formal scheme over $\mathrm{Spf}(B/J)$. Then we have a natural filtered isomorphism:*

$$\left(\mathrm{R}\Gamma(X, \mathrm{Fil}_N^\bullet(\Delta_{-/B}^{(1)})) \right) \widehat{\otimes}_{(B, J, \bullet)} (\mathrm{dR}_{(B/J)/B}^\wedge, \mathrm{Fil}_H^\bullet) \xrightarrow{\cong} \mathrm{R}\Gamma(X, \mathrm{Fil}_H^\bullet(\mathrm{dR}_{-/B}^\wedge));$$

they are furthermore naturally filtered isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(X, \mathcal{I}_{\mathrm{crys}}^\bullet)$ if (B, J) is transversal. Similarly whenever $i \leq p$, we also have natural isomorphisms

$$\mathrm{R}\Gamma(X, \Delta_{-/A}^{(1)} / \mathrm{Fil}_N^i) \xrightarrow{\cong} \mathrm{R}\Gamma(X, \mathrm{dR}_{-/A}^\wedge / \mathrm{Fil}_H^i);$$

and are furthermore naturally isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(X, \mathcal{O}_{\mathrm{crys}} / \mathcal{I}_{\mathrm{crys}}^i)$ if (B, J) is transversal. These isomorphisms are functorial in X , and satisfies the base change property as in Remark 4.20.

Proof. These functorial isomorphisms are provided by Corollary 4.22 and Theorem 4.21 (4) respectively. The furthermore equality, when the base prism is transversal, follows from Theorem 4.1. \square

Remark 4.24. Back to the transversal base prism case. A posteriori the filtration on the left hand side of Corollary 4.22 is a quasisyntomic sheaf, hence we may define it as the unfolding of its restriction to the basis of large quasisyntomic over A/I -algebras. Also a posteriori, we know the value on such an algebra S must be concentrated in cohomological degree 0, therefore they have to be the image of the augmentation map

$$\mathrm{Fil}^i(\Delta_{S/A}^{(1)} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{A}) \rightarrow \mathrm{dR}_{S/A}^\wedge,$$

where the filtration on the left hand side is given by the usual Day convolution. This implies an equality

$$\mathrm{Fil}_H^n(\mathrm{dR}_{S/A}^\wedge) = \sum_{i=0}^n \left(\mathrm{Fil}_N^i(\Delta_{S/A}^{(1)}) \cdot \mathcal{I}^{[n-i]} \right),$$

which also follows from combining Proposition 4.5 (1), Proposition 4.6 (3), and Theorem 4.9 (2).

Therefore, for any $0 \leq r \leq p-1$, we see that the Frobenius on derived de Rham complex when restricted to the r -th Hodge filtration

$$\mathrm{Fil}_H^r(\mathrm{dR}_{S/A}^\wedge) \xrightarrow{\varphi} \mathrm{dR}_{S/A}^\wedge$$

factors through multiplication by p^r . Since for large quasisyntomic over A/I -algebras S , the $\mathrm{dR}_{S/A}^\wedge$ is p -completely flat over \mathcal{A} (see Remark 4.8) which is p -torsionfree, we may uniquely divide the restriction φ by p^r . By unfolding, this gives rise to divided Frobenii as maps of sheaves on $\mathrm{qSyn}_{A/I}$:

$$\varphi_r : \mathrm{Fil}_H^r(\mathrm{dR}_{-/A}^\wedge) \rightarrow \mathrm{dR}_{-/A}^\wedge.$$

By definition, they also satisfy $\varphi_r \mid_{\mathrm{Fil}_H^{r+1}} = p\varphi_{r+1}$ when $r \leq p-2$. Following the same argument of Theorem 3.14, see also Remark 3.15, such a functorial divided Frobenius is unique for each $0 \leq r \leq p-1$.

When (A, I) is the Breuil–Kisin prism, this gives rise to an alternative definition of the divided Frobenii appeared in [Bre98, p. 10].

5. CONNECTION ON $\mathrm{dR}_{-/S}^\wedge$ AND STRUCTURE OF TORSION CRYSTALLINE COHOMOLOGY

From this section onward, we focus on the Breuil–Kisin prism $A = (\mathfrak{S}, E)$ and crystalline cohomology over $S = \mathrm{dR}_{\mathcal{O}_K/\mathfrak{S}}^\wedge$. Let k be a perfect field with characteristic p , and let K be a finite totally ramified extension over $K_0 = W(k)[\frac{1}{p}]$ with a fixed uniformizer $\pi \in \mathcal{O}_K$. Fix an algebraic closure \overline{K} of K and let \mathbf{C} be p -adic completion of \overline{K} . Write $G_K := \mathrm{Gal}(\overline{K}/K)$ and $e = [K : K_0]$. Let $E = E(u) \in W(k)[u]$ be the Eisenstein polynomial of π with constant term a_0p , recall $\mathfrak{S} := W(k)[[u]]$ is equipped with a Frobenius φ naturally extends that on $W(k)$ by $\varphi(u) = u^p$. Pick $\pi_n \in \mathcal{O}_{\overline{K}}$ so that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Then $\underline{\pi} := (\pi_n)_{n \geq 0} \in \mathcal{O}_{\mathbf{C}}^\flat$. We embed $\mathfrak{S} \hookrightarrow A_{\mathrm{inf}}$ via $u \mapsto [\underline{\pi}]$ which is a map of prisms. Let $K_\infty := \bigcup_{n=0}^\infty K(\pi_n)$ and $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$. It is clear that the embedding $\mathfrak{S} \subset A_{\mathrm{inf}}$ is compatible with G_∞ -actions. We extend φ from \mathfrak{S} to S and let $\mathrm{Fil}^m S$ be the p -adic closure of the ideal generated by $\gamma_i(E) := \frac{E^i}{i!}, i \geq m$. We embed $S \hookrightarrow A_{\mathrm{crys}}$ also via $u \mapsto [\underline{\pi}]$. For $m \leq p-1$, $\varphi(\mathrm{Fil}^m S) \subset p^m S$. We set $\varphi_m := \frac{\varphi}{p^m} : \mathrm{Fil}^m S \rightarrow S$. Similar notation also applies to A_{crys} . Write $c_1 := \frac{\varphi(E)}{a_0p} \in S^\times$. Finally, there exists a $W(k)$ -linear derivation $\nabla_S : S \rightarrow S$ by $\nabla_S(f(u)) = f'(u)$.

For $n \geq 1$, if M is an \mathbb{Z}_p -module then we always use M_n to denote $M/p^n M$. Similar notation applies to (p -adic formal) schemes: i.e., $X_n := X \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}/p^n\mathbb{Z})$. Write $W = W(k)$ and reserve $\gamma_i(\cdot)$ for the i -th divided power.

5.1. Connection on $\mathrm{dR}_{-/S}^\wedge$. Under the philosophy that derived de Rham cohomology behaves a lot like crystalline cohomology, one expects there to be a connection on $\mathrm{dR}_{-/S}^\wedge$. We explain it in this section.

Lemma 5.1. *Let R be an \mathcal{O}_K -algebra. Then the natural morphism $\mathrm{dR}_{R/W[u]}^\wedge \rightarrow \mathrm{dR}_{R/\mathfrak{S}}^\wedge$ is an isomorphism, where R is regarded as an \mathfrak{S} and $W[u]$ algebra via $W[u] \rightarrow \mathfrak{S} \rightarrow \mathcal{O}_K \rightarrow R$.*

Proof. Just notice the following p -completely pushout diagram:

$$\begin{array}{ccccc} \mathfrak{S} & \longrightarrow & \mathcal{O}_K & \longrightarrow & R \\ \uparrow & & \uparrow & & \uparrow \\ W[u] & \longrightarrow & \mathcal{O}_K & \longrightarrow & R \end{array}$$

and appeal to the p -completely base change formula of derived de Rham complexes to get

$$\mathrm{dR}_{R/W[u]}^\wedge \widehat{\otimes}_{W[u]} \mathfrak{S} \xrightarrow{\cong} \mathrm{dR}_{R/\mathfrak{S}}^\wedge.$$

Next we observe that $\mathrm{dR}_{R/W[u]}^\wedge$ is an $S = \mathrm{dR}_{\mathcal{O}_K/W[u]}^\wedge$ -complex and $S \widehat{\otimes}_{W[u]} \mathfrak{S} = S$, hence the base change on the left hand side gives $\mathrm{dR}_{R/W[u]}^\wedge$ back. \square

Construction 5.2 (see also [KO68]). For any $W[u]$ -algebra R , by (p -completely) applying [GL21, Lemma 3.13.(4)] to the triple $W \rightarrow W[u] \rightarrow R$, we see that there is a functorial triangle in filtered derived ∞ -category of W -modules:

$$\mathrm{dR}_{R/W[u]}^{\wedge} \widehat{\otimes}_{W[u]} \Omega_{W[u]/W}^1[-1] \rightarrow \mathrm{dR}_{R/W}^{\wedge} \rightarrow \mathrm{dR}_{R/W[u]}^{\wedge}.$$

Here $\Omega_{W[u]/W}^1[-1]$ is completely put in the first filtration. By choosing the generator $du \in \Omega_{W[u]/W}^1$, the above becomes

$$\mathrm{dR}_{R/W}^{\wedge} \rightarrow \mathrm{dR}_{R/W[u]}^{\wedge} \xrightarrow{\nabla} \mathrm{dR}_{R/W[u]}^{\wedge}(-1),$$

where (-1) indicates the shift of filtrations: $\mathrm{Fil}^i(\mathrm{dR}_{R/W[u]}^{\wedge}(-1)) = \mathrm{Fil}_{\mathbb{H}}^{i-1}(\mathrm{dR}_{R/W[u]}^{\wedge})$. When R is smooth over $W[u]$, then ∇ is given by Lie derivative with respect to $\frac{\partial}{\partial u}$:

$$\nabla(\omega) = \mathcal{L}_{\frac{\partial}{\partial u}}(\omega).$$

Lemma 5.3. *Let R be an \mathcal{O}_K -algebra. Then we have a functorial triangle in the filtered derived ∞ -category:*

$$\mathrm{dR}_{R/W}^{\wedge} \rightarrow \mathrm{dR}_{R/\mathfrak{S}}^{\wedge} \xrightarrow{\nabla} \mathrm{dR}_{R/\mathfrak{S}}^{\wedge}(-1).$$

Moreover we have

$$pu^{p-1} \cdot \varphi \circ \nabla = \nabla \circ \varphi,$$

where $\varphi: \mathrm{dR}_{R/\mathfrak{S}}^{\wedge} \rightarrow \mathrm{dR}_{R/\mathfrak{S}}^{\wedge}$ is the Frobenius defined in Section 2.3.

Proof. The first statement follows from Construction 5.2 and Lemma 5.1.

To check the equality, by left Kan extension it suffices to check it for the polynomials. Then by quasisyntomic descent, it suffices to check the equality for large quasisyntomic over \mathcal{O}_K algebras. Following the proof of Corollary 3.16, we are reduced to showing the equality for algebras of the form

$$R = \mathcal{O}_K \langle X_i^{1/p^\infty}, Y_j^{1/p^\infty} \mid i \in I, j \in J \rangle / (Y_j - f_j \mid j \in J) =: \tilde{R} / (Y_j - f_j \mid j \in J).$$

Now the map $\tilde{R} \rightarrow R$ induces a map between $\mathrm{dR}_{\tilde{R}/\mathfrak{S}}^{\wedge}$ given by

$$S \langle X_i^{1/p^\infty}, Y_j^{1/p^\infty} \mid i \in I, j \in J \rangle =: T \rightarrow D_T(Y_j - f_j; j \in J)^{\wedge}.$$

Here S is the p -complete PD envelope of \mathfrak{S} along (E) and the latter denotes p -completely adjoining divided powers of $(Y_j - f_j)$ in T . Since $D_T(Y_j - f_j; j \in J)^{\wedge}$ is p -complete and p -torsionfree, it suffices to check the identity on T . On T , the Frobenius φ acts by sending variables X, Y, u to their p -th power, and ∇ acts via $\frac{\partial}{\partial u}$. Finally we are reduced to checking the equality

$$pu^{p-1} \cdot \varphi \left(\frac{\partial}{\partial u}(F(u, \underline{X}, \underline{Y})) \right) = \frac{\partial}{\partial u}(\varphi(F(u, \underline{X}, \underline{Y}))),$$

for any $F(u, \underline{X}, \underline{Y}) \in T$. □

Consequently, for any \mathcal{O}_K -algebra R , we always have a long exact sequence:

$$(E) \quad \dots \rightarrow \mathrm{H}^i(\mathrm{dR}_{R/W}^{\wedge}) \rightarrow \mathrm{H}^i(\mathrm{dR}_{R/\mathfrak{S}}^{\wedge}) \xrightarrow{\nabla} \mathrm{H}^i(\mathrm{dR}_{R/\mathfrak{S}}^{\wedge}(-1)) \xrightarrow{+1} \dots$$

and its r -th filtration analogues for all $r \in \mathbb{N}$. In special situation, these will break into short exact sequences. Let us introduce some more notation. Let L be a perfectoid field extension of K containing all p -power roots of π . For instance L could be p -adic completion of K_∞ or \mathbf{C} . Let $A_{\mathrm{inf}}(L) := W(\mathcal{O}_L^b)$ be Fontaine's A_{inf} ring associated with L , and recall there is a natural map $\theta := A_{\mathrm{inf}}(L) \rightarrow \mathcal{O}_L$. Fix a compatible system of p -power roots of π , we obtain a map $\mathfrak{S} \rightarrow A_{\mathrm{inf}}(L)$ with $u \mapsto [\pi]$ compatible with θ and the inclusion $\mathcal{O}_K \rightarrow \mathcal{O}_L$.

Proposition 5.4. *With notation as above. Let R be a quasisyntomic \mathcal{O}_L -algebra. Then we have*

- (1) *The natural map $\mathrm{dR}_{R/W}^{\wedge} \rightarrow \mathrm{dR}_{R/A_{\mathrm{inf}}(L)}^{\wedge}$ is a filtered isomorphism.*
- (2) *The sequence (E) and its r -th filtration analogues break into short exact sequence:*

$$0 \rightarrow \mathrm{H}^i(\mathrm{Fil}_{\mathbb{H}}^r \mathrm{dR}_{R/W}^{\wedge}) \rightarrow \mathrm{H}^i(\mathrm{Fil}_{\mathbb{H}}^r \mathrm{dR}_{R/\mathfrak{S}}^{\wedge}) \xrightarrow{\nabla} \mathrm{H}^i(\mathrm{Fil}_{\mathbb{H}}^{r-1} \mathrm{dR}_{R/\mathfrak{S}}^{\wedge}) \rightarrow 0,$$

for all i and r . In particular $\mathrm{dR}_{R/\mathfrak{S}}^{\wedge} \xrightarrow{\nabla} \mathrm{dR}_{R/\mathfrak{S}}^{\wedge}(-1)$ is surjective on each H^i , and

$$\mathrm{H}^i(\mathrm{dR}_{R/W}^{\wedge}) = \mathrm{H}^i(\mathrm{dR}_{R/\mathfrak{S}}^{\wedge})^{\nabla=0}.$$

Proof. (1) is [GL21, Theorem 3.4.(2)].

As for (2), it suffices to show that the maps $H^i(\mathrm{Fil}_H^r dR_{R/W}^\wedge) \rightarrow H^i(\mathrm{Fil}_H^r dR_{R/\mathfrak{S}}^\wedge)$ are injective for all i and r . By functoriality, we have maps of filtered algebras

$$dR_{R/W}^\wedge \rightarrow dR_{R/\mathfrak{S}}^\wedge \rightarrow dR_{R/A_{\mathrm{inf}}(L)}^\wedge$$

whose composition is a filtered isomorphism by (1). Therefore the first morphism factorizing isomorphism induces injection at the level of cohomology. This explains why the long exact sequence \mathfrak{E} breaks into short exact sequences. The last statement follows easily by letting $r = 0$. \square

5.2. Structures of torsion crystalline cohomology. Let X be a proper smooth formal scheme over \mathcal{O}_K . Let us summarize the structures on $H_{\mathrm{crys}}^i(X/S) := H_{\mathrm{crys}}^i(X/S, \mathcal{O}_{\mathrm{crys}})$ constructed from previous sections.

By Corollary 4.22 and Theorem 4.1, we obtain the following commutative diagram.

$$(5.5) \quad \begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \Delta_{-\mathfrak{S}}^{(1)}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{crys}}(X/S, \mathcal{O}_{\mathrm{crys}}) \simeq S \otimes_{\varphi, \mathfrak{S}}^{\mathbb{L}} \mathrm{R}\Gamma_{\Delta}(X/\mathfrak{S}) \\ \uparrow & & \uparrow \\ \mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \mathrm{Fil}_N^m \Delta_{-\mathfrak{S}}^1) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{crys}}(X/S, \mathcal{I}_{\mathrm{crys}}^{[m]}) \end{array}$$

Here the second isomorphism of the first row follows the canonical isomorphism $\mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \Delta_{-\mathfrak{S}}) \simeq \mathrm{R}\Gamma_{\Delta}(X/\mathfrak{S})$ and the fact that $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is flat.

For $m \leq p-1$, Remark 4.24 allows us to define φ -semi-linear map $\varphi_m : H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m]}) \rightarrow H_{\mathrm{crys}}^i(X/S)$ so that the following diagram commutes for $m+1 \leq p-1$

$$\begin{array}{ccc} H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m]}) & \xrightarrow{\varphi_m} & H_{\mathrm{crys}}^i(X/S) \\ \uparrow & \nearrow p\varphi_{m+1} & \\ H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m+1]}) & & \end{array}$$

We simply denote the above diagram by $\varphi_m|_{H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m+1]})} = p\varphi_{m+1}$. It is also clear that for any $s \in \mathrm{Fil}^m S$ and $x \in H_{\mathrm{crys}}^i(X/S)$ we have

$$\varphi_m(sx) = (c_1)^{-m} \varphi_m(s) \varphi_h(E(u)^m x).$$

Finally, the above subsection construct a connection $\nabla : H_{\mathrm{crys}}^i(X/S) \rightarrow H_{\mathrm{crys}}^i(X/S)$. By Proposition 5.4 and Lemma 5.3, we conclude that

- (1) $\nabla : H_{\mathrm{crys}}^i(X/S) \rightarrow H_{\mathrm{crys}}^i(X/S)$ is $W(k)$ -linear derivative satisfying

$$\nabla(f(u)x) = f'(u)x + f(u)\nabla(x)$$

- (2) (Griffiths Transversality) $\nabla(H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m]}))$ factors through $H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m-1]})$.

- (3) The following diagram commutes:

$$\begin{array}{ccc} H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m]}) & \xrightarrow{\varphi_m} & H_{\mathrm{crys}}^i(X/S) \\ E(u)\nabla \downarrow & & \downarrow c_1\nabla \\ H_{\mathrm{crys}}^i(X/S, \mathcal{I}_{\mathrm{crys}}^{[m]}) & \xrightarrow{u^{p-1}\varphi_m} & H_{\mathrm{crys}}^i(X/S) \end{array}$$

The last diagram follows that $pu^{p-1}\varphi \circ \nabla = \nabla \circ \varphi$ by Lemma 5.3 and that $\varphi(E) = pc_1$.

Now consider the p^n -torsion crystalline cohomology $H_{\mathrm{crys}}^i(X_n/S_n)$ together with filtration $H_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[m]})$. We claim that $H_{\mathrm{crys}}^i(X_n/S_n)$ admits all the above structures $\varphi_m : H_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[m]}) \rightarrow H_{\mathrm{crys}}^i(X_n/S_n)$ for $m \leq p-1$ and $\nabla : H_{\mathrm{crys}}^i(X_n/S_n) \rightarrow H_{\mathrm{crys}}^i(X_n/S_n)$ satisfying all the above properties. To see, note

that $\mathrm{R}\Gamma_{\mathrm{crys}}(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[m]}) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/S, \mathcal{I}_{\mathrm{crys}}^{[m]}) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$ where $\mathcal{I}_{\mathrm{crys}}^{[0]} = \mathcal{O}_{\mathrm{crys}}$ then all the above properties follow by taking $\otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$, except the Diagram (5.5) which requires torsion quasi-syntomic cohomology. For this, we define the following torsion cohomologies: For $m \geq 0$, $\mathrm{R}\Gamma_{\mathrm{dR}}(X_n/\mathfrak{S}_n, \mathrm{Fil}_{\mathrm{H}}^m) := \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathfrak{S}, \mathrm{Fil}_{\mathrm{H}}^m) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$, $\mathrm{R}\Gamma_{\mathrm{qSyn}}(X_n/\mathfrak{S}_n, \mathrm{Fil}_N^m \Delta^{(1)}) := \mathrm{R}\Gamma_{\mathrm{qSyn}}(X/\mathfrak{S}, \mathrm{Fil}_N^m \Delta_{-/\mathfrak{S}}^{(1)}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$, and finally $\mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) := \mathrm{R}\Gamma_{\Delta}(X/\mathfrak{S}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Then the derived modulo p^n version of Diagram (5.5) still holds by taking the original diagram and derived modulo p^n .

5.3. Galois action on torsion crystalline cohomology. Keep the notations as the above. Set \mathcal{X} to be the base change of X to $\mathrm{Spf} \mathcal{O}_{\mathbb{C}}$ and $\mathcal{X}_n := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Then $\mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/S_n)$ has an S -linear G_K -action when we define the G_K -action on S is trivial. Note that $\mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/A_{\mathrm{crys},n})$ also has A_{crys} -semi-linear G_K -action which is induced by G_K -actions on \mathcal{X} and A_{crys} . By Proposition 5.4 and its proof, we see that the natural map $W(k) \rightarrow \mathfrak{S} \rightarrow A_{\mathrm{inf}}$ induces the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/W_n(k)) & \xleftarrow{\alpha} & \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/S_n) & \xleftarrow{\tilde{\alpha}} & \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \\ & \searrow \beta & \downarrow \iota & & \downarrow \tilde{\iota} \\ & \sim & \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/A_{\mathrm{crys},n}) & \equiv & \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \otimes_{S_n} A_{\mathrm{crys},n} \end{array}$$

Note that the second row is an isomorphism because $\mathcal{X} = X \times_{\mathrm{Spec}(S)} \mathrm{Spec}(A_{\mathrm{crys}})$ and that $A_{\mathrm{crys},n}$ is flat over S_n . Thus $\tilde{\iota}$ is an injection. So is $\tilde{\alpha}$. Also we note that α and β are both compatible with G_K -actions because both the map $W(k) \rightarrow \mathfrak{S}$ and $W(k) \rightarrow A_{\mathrm{inf}}$ are G_K -compatible. But ι is not as $\mathfrak{S} \subset A_{\mathrm{inf}}$ is only stable under G_{∞} -action. It is also clear that $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \subset (\mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/S_n))^{G_K}$ via $\tilde{\alpha}$ and $\tilde{\alpha}$ is also compatible with connection on both sides. Now we claim the G_K -action on $\mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/A_{\mathrm{crys},n})$ is given by the following formula: For any $\sigma \in G_K$, any $x \otimes a \in \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \otimes_S A_{\mathrm{crys}} \simeq \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/A_{\mathrm{crys},n})$,

$$(5.6) \quad \sigma(x \otimes a) = \sum_{i=0}^{\infty} \nabla^i(x) \otimes \gamma_i(\sigma([\pi]) - [\pi]) \sigma(a).$$

To see this, for any $x \in \mathcal{M}^i := \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$, set

$$x^{\nabla} := \sum_{m=0}^{\infty} \nabla(x) \gamma_m([\pi] - u) \in \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/S_n).$$

Then we immediately see that $x^{\nabla} \in \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/W_n(k)) = \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/S_n)^{\nabla=0}$. Now we claim that $\mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/W_n(k))$ is generated by $\{x^{\nabla} | x \in \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)\}$ as an A_{crys} -module. If so then (5.6) follows the fact that β is G_K -equivariant and the construction of x^{∇} (note that both x and u are G_K -invariants).

To prove the claim, for any $y \in \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/W_n(k))$, suppose that $\beta(y) = \sum_j a_j \tilde{\iota}(x_j)$ with $a_j \in A_{\mathrm{crys}}$ and $x_j \in \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$. Then we see that $y^{\nabla} := \sum_j a_j x_j^{\nabla} \in \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/W_n(k))$. It suffices to that check $y = y^{\nabla}$. Since β is an isomorphism, it suffices to show that $\beta(y) = \beta(y^{\nabla})$. This follows that $\beta(x^{\nabla}) = \iota(x)$ for $x \in \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ as $\iota([\pi] - u) = [\pi] - [\pi] = 0$.

6. TORSION KISIN MODULE, BREUIL MODULE AND ASSOCIATED GALOIS REPRESENTATIONS

In this section, we set up the theory of generalized torsion Kisin modules which extends theory of Kisin modules, which is discussed, for example, [Liu07, §2]. The key point for the generalized Kisin modules is that it may have u -torsion, and it return to classical torsion Kisin modules when modulo u -torsion.

6.1. (Generalized) Kisin modules. Let $(\mathfrak{S}, E(u))$ be the Breuil–Kisin prism over \mathcal{O}_K with $d = E(u) = E$ the Eisenstein polynomial of fixed uniformizer $\pi \in \mathcal{O}_K$. A φ -module \mathfrak{M} over \mathfrak{S} is an \mathfrak{S} -module \mathfrak{M} together $\varphi_{\mathfrak{S}}$ -semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$. Write $\varphi^* \mathfrak{M} = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Note that $1 \otimes \varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is an \mathfrak{S} -linear map. A (generalized) Kisin module \mathfrak{M} of height h is a φ -module \mathfrak{M} of finite \mathfrak{S} -type so that there exists an \mathfrak{S} -linear map $\psi : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ so that $\psi \circ (1 \otimes \varphi) = E^h \mathrm{id}_{\varphi^* \mathfrak{M}}$ and $(1 \otimes \varphi) \circ \psi = E^h \mathrm{id}_{\mathfrak{M}}$. Maps between generalized Kisin modules are given by \mathfrak{S} -linear maps which are compatible with φ and ψ . We denote by $\mathrm{Mod}_{\mathfrak{S}}^{\varphi, h}$ the category of (generalized) Kisin module of height h .

In [Liu07], A *Kisin module* \mathfrak{M} of height h is defined to be an étale φ -module \mathfrak{M} of finite \mathfrak{S} -type so that $\text{coker}(1 \otimes \varphi)$ is killed by E^h . Here étale φ -module means that the natural map $\mathfrak{M} \rightarrow \widehat{\mathfrak{S}[\frac{1}{u}]} \otimes_{\mathfrak{S}} \mathfrak{M}$ is injective. Since $E(u)$ is a unit in $\widehat{\mathfrak{S}[\frac{1}{u}]}$, we easily see that the étale assumption implies that $(1 \otimes \varphi) : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is injective. Then existence and uniqueness of $\psi : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$, in definition of (generalized) Kisin modules of height h , then follows. That is, the Kisin module \mathfrak{M} of height h defined classically is (generalized) Kisin module of height h . So in the following, we drop “generalized” when we mention the object in $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$. If we need to emphasize \mathfrak{M} is also a Kisin modules of height m classically defined, we will mention that it is étale.

Lemma 6.1.

- (1) $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$ is an abelian category.
- (2) \mathfrak{M} is étale if and only if \mathfrak{M} has no u -torsion.
- (3) $\mathfrak{M}[\frac{1}{p}]$ is finite $\mathfrak{S}[\frac{1}{p}]$ -free.

Proof. (1) is easy to check because $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is faithfully flat. (2) It is clear from the definition that if \mathfrak{M} is étale then it has no u -torsion. Conversely, let $\mathfrak{M}[p^\infty] := \{x \in \mathfrak{M} | p^n x = 0 \text{ for some } n > 0\}$ and $\mathfrak{M}' := \mathfrak{M}/\mathfrak{M}[p^\infty]$. We get the short exact sequence $0 \rightarrow \mathfrak{M}[p^\infty] \rightarrow \mathfrak{M} \rightarrow \mathfrak{M}' \rightarrow 0$. It is clear that both $\mathfrak{M}[p^\infty]$ and \mathfrak{M}' are objects in $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$ and \mathfrak{M} has no u -torsion then both $\mathfrak{M}[p^\infty]$ and \mathfrak{M}' has no u -torsion. Since $\mathfrak{M}[p^\infty]$ is killed by some p -power, $\mathfrak{M} \otimes \widehat{\mathfrak{S}[\frac{1}{u}]} = \mathfrak{M}[\frac{1}{u}]$. So $\mathfrak{M}[p^\infty]$ has no u -torsion if and only if that $\mathfrak{M}[p^\infty]$ is étale. Now \mathfrak{M}' has no p -torsion, now we claim that $\mathfrak{M}'[\frac{1}{p}]$ is finite $\mathfrak{S}[\frac{1}{p}]$ -free, which will implies (3) and étaleness of \mathfrak{M}' . By [Fon90, §1.2.1], $\mathfrak{M}'[\frac{1}{p}] \simeq \bigoplus \mathfrak{S}[\frac{1}{p}]/P_i^{a_i}$ with $P_i \in W(k)[u]$, monic irreducible and $P_i \equiv u^{b_i} \pmod{p}$, or $P_i = 0$. Without loss of generality, we may assume that $P_i \neq 0$ and show such \mathfrak{M}' does not exist when $\mathfrak{M}' \in \text{Mod}_{\mathfrak{S}}^{\varphi, h}$. Consider wedge product \mathfrak{N} of $\mathfrak{M}'[\frac{1}{p}]$, then $\mathfrak{N} \simeq \mathfrak{S}[\frac{1}{p}]/f$ with $f = \prod P_i^{a_i}$ and write $\varphi^* := (1 \otimes \varphi)$. We also obtain $\varphi^* : \varphi^* \mathfrak{N} \rightarrow \mathfrak{N}$ and $\psi : \mathfrak{N} \rightarrow \varphi^* \mathfrak{N}$ so that $\psi \circ \varphi^* = E(u)^h \text{id}_{\varphi^* \mathfrak{N}}$ and $\varphi^* \circ \psi = E(u)^h \text{id}_{\mathfrak{N}}$ for some h . Since $\varphi^* \mathfrak{N} \simeq \mathfrak{S}[\frac{1}{p}]/\varphi(f)$, we can write the above maps explicitly as

$$\mathfrak{S}[\frac{1}{p}]/\varphi(f) \xrightarrow{\varphi^*} \mathfrak{S}[\frac{1}{p}]/f \xrightarrow{\psi} \mathfrak{S}[\frac{1}{p}]/\varphi(f)$$

Write $x = \varphi^*(1)$ and $y = \psi(1)$. We have $\varphi(f)x = fz'$ and $fy = \varphi(f)w'$ for some $z', w' \in \mathfrak{S}[\frac{1}{p}]$. The condition $\psi \circ \varphi^* = E(u)^h \text{id}_{\varphi^* \mathfrak{N}}$ and $\varphi^* \circ \psi = E(u)^h \text{id}_{\mathfrak{N}}$ implies that $\varphi(f)E(u)^h = fz$ and $fE(u)^h = \varphi(f)w$ with $z, w \in \mathfrak{S}[\frac{1}{p}]$. So $E(u)^{2h} = zw$. Since $E(u)$ is an Eisenstein polynomial, $z = z_0 E(u)^l$ with z_0 a unit in $\mathfrak{S}[\frac{1}{p}]$. Then $\varphi(f) = z_0 f E(u)^{l-h}$. We easily see $z_0 \in \mathfrak{S}^\times$ as both f and $E(u)$ monic. So $l - h > 0$ by mod p on the both sides. Let $a_0 = f(0)$ be the constant term of $f(u)$. Since $\varphi(f)(0) = \varphi(a_0) = z_0(0)a_0 p^{l-h}$. Comparing p -adic valuation on the both sides, we see that $a_0 = 0$. Then we may write $f = u^m g$ with $g(0) \neq 0$. But then we have $u^{pm-m} \varphi(g) = z_0 g E(u)^{l-h}$, which is impossible by comparing constant terms on the both sides. In summary, such \mathfrak{M}' can not exist and $\mathfrak{M}'[\frac{1}{p}]$ is finite $\mathfrak{S}[\frac{1}{p}]$ -free. \square

Let \mathfrak{M} be a Kisin module of height h and set $\mathfrak{M}[u^\infty] := \{x \in \mathfrak{M} | u^l x = 0 \text{ for some } l\}$. It is that both $(1 \otimes \varphi_{\mathfrak{M}})(\varphi^* \mathfrak{M}[u^\infty]) \subset \mathfrak{M}[u^\infty]$ and $\psi(\mathfrak{M}[u^\infty]) \subset \varphi^* \mathfrak{M}[u^\infty]$. The above lemma shows that $\mathfrak{M}[u^\infty] \subset \mathfrak{M}[p^\infty]$ and $\mathfrak{M}/\mathfrak{M}[u^\infty]$ is étale.

Lemma 6.2. *The following short exact sequence is in $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$*

$$0 \rightarrow \mathfrak{M}[u^\infty] \rightarrow \mathfrak{M} \rightarrow \mathfrak{M}/\mathfrak{M}[u^\infty] \rightarrow 0$$

with $\mathfrak{M}/\mathfrak{M}[u^\infty]$ being étale.

It turns out that étale Kisin module enjoys many nice properties. Let $\text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h}$ denote the full subcategory of $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$ whose object \mathfrak{M} is torsion, i.e., killed by p^n for some n . The following Lemma is a part of [Liu07, Proposition 2.3.2].

Lemma 6.3. *The following statements are equivalent for a torsion Kisin module $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h}$:*

- (1) \mathfrak{M} is étale.
- (2) \mathfrak{M} can be written as a successive quotient of \mathfrak{M}_i so that $\mathfrak{M}_i \in \text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h}$ and \mathfrak{M}_i is finite $k[[u]]$ -free.

(3) $\mathfrak{M} = \mathfrak{N}/\mathfrak{N}'$ where $\mathfrak{N}' \subset \mathfrak{N}$ are Kisin modules of height h and \mathfrak{N}' and \mathfrak{N} are finite free \mathfrak{S} -modules.

Corollary 6.4. *Give an étale Kisin module $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\varphi, h}$. There exists étale Kisin module $\mathfrak{M}_n \in \text{Mod}_{\mathfrak{S}}^{\varphi, h}$ killed by p^n satisfying $\mathfrak{M}/p^n \mathfrak{M}[\frac{1}{u}] = \mathfrak{M}_n[\frac{1}{u}]$ and $\mathfrak{M} = \varprojlim_n \mathfrak{M}_n$*

Proof. Let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}[\frac{1}{u}]}$. Consider the exact sequence $0 \rightarrow p^n M \rightarrow M \xrightarrow{q} M/p^n M \rightarrow 0$. Since \mathfrak{M} is étale, we see the natural map $\mathfrak{M} \rightarrow M$ is injective. Set $\mathfrak{M}_n = q(\mathfrak{M}) \subset M/p^n M$. It is easy to check that $\mathfrak{M}_n[\frac{1}{u}] = \mathfrak{M}/p^n \mathfrak{M}[\frac{1}{u}] = M/p^n M$, \mathfrak{M}_n has no u -torsion and $\mathfrak{M} = \varprojlim_n \mathfrak{M}_n$ (since \mathfrak{M} is p -adically closed in M). We just need to check that \mathfrak{M}_n has height h . This was proved by [Fon90, Proposition B 1.3.5]. \square

In general, the category of étale Kisin modules is not abelian but under some restrictions it could be abelian. Given $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h}$, let $M = \mathfrak{M}[u^\infty, p] := \{x \in \mathfrak{M}[u^\infty] \mid px = 0\}$.

Lemma 6.5. *If $eh < p - 1$ then $M = 0$ and if $eh < 2(p - 1)$ then $M \simeq \bigoplus k$ or 0 .*

Proof. So we have $\psi : M \rightarrow \varphi^* M$ so that $\psi \circ (1 \otimes \varphi) = d^h \text{id}_{\varphi^* M}$. We can write $M = \bigoplus_{j=1}^m k[u]/u^{a_j}$ with $a_j \geq 1$, and then $\varphi^* M \simeq \bigoplus_{j=1}^m k[u]/u^{pa_j}$. Assume that $a = \max_j \{a_j\}$ and let $x \in \varphi^* M$ so that $u^{pa} x = 0$ but $u^{pa-1} x \neq 0$. Since $\psi \circ (1 \otimes \varphi) = u^{eh} \text{id}_{\varphi^* M}$, we conclude that $u^{eh} x \in \psi(M)$. Note that $u^a M = \{0\}$ and ψ is $k[u]$ -linear, we have $u^{a+eh} x = 0$. This forces that $a + eh \geq pa$. That is, $a \leq \frac{eh}{p-1}$. Hence such a can not exist if $eh < p - 1$. If $eh < 2(p - 1)$ then $a = 1$ or 0 . This proves the Lemma. \square

Proposition 6.6. *If $eh < p - 1$ then $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$ is an abelian category.*

Proof. By Lemma 6.5, $\mathfrak{M}[u^\infty] = 0$. \square

Example 6.7. Let $E(u) = u - p$, $\mathfrak{M} = k \simeq k[u]/u$ and $\varphi(1) = 1$. Let $\psi : k[u]/u \rightarrow k[u]/u^p$ by $\psi(1) = u^{p-1}$. Then $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, p-1}$.

Let $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\varphi, h}$. Define *Breuil–Kisin filtration* on $\varphi^* \mathfrak{M}$ by

$$\text{Fil}_{\text{BK}}^h \varphi^* \mathfrak{M} := \text{Im}(\psi : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}).$$

In the case that \mathfrak{M} is étale then ψ is injective as explained above, and we have an identification

$$(6.8) \quad \text{Fil}_{\text{BK}}^h \varphi^* \mathfrak{M} \cong \{x \in \varphi^* \mathfrak{M} \mid (1 \otimes \varphi)(x) \in E(u)^h \mathfrak{M}\}$$

of submodules in $\varphi^* \mathfrak{M}$. Since there is only filtration considered for Kisin modules in this section, we drop BK from the notation for this section. Finally there is $\varphi_{\mathfrak{S}}$ -semi-linear map $\varphi := \varphi \otimes \varphi : \varphi^* \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$. It is clear that $\varphi(\text{Fil}^i \varphi^* \mathfrak{M}) \subset \varphi(E(u))^i \varphi^* \mathfrak{M}$. If \mathfrak{M} is étale, then we define $\varphi_i : \text{Fil}^i \varphi^* \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ via

$$\varphi_i(x) := \frac{\varphi(x)}{\varphi(E(u))^i}.$$

Lemma 6.9. *Suppose that $0 \rightarrow \mathfrak{M}' \rightarrow \mathfrak{M} \rightarrow \mathfrak{M}'' \rightarrow 0$ is an exact sequence inside $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$ and all modules are étale. Then the following sequence is exact:*

$$0 \rightarrow \text{Fil}^h \varphi^* \mathfrak{M}' \rightarrow \text{Fil}^h \varphi^* \mathfrak{M} \rightarrow \text{Fil}^h \varphi^* \mathfrak{M}'' \rightarrow 0$$

Proof. This easily follows that $\varphi^* : \text{Fil}^h \varphi^* \mathfrak{M} \rightarrow E^h \mathfrak{M}$ is bijective. \square

Remark 6.10. The above Lemma fails in general if $i < h$ or if the modules are not étale.

6.2. Galois representation attached to étale Kisin modules. Recall that we fix $\pi_n \in \overline{K}$ so that $\underline{\pi} := (\pi_n) \in \mathcal{O}_{\mathbb{C}}^b$ and $\pi_0 = \pi$; $K_\infty := \bigcup_{n \geq 0} K(\pi_n)$ and $G_\infty := \text{Gal}(\overline{K}/K_\infty)$. We embed $\mathfrak{S} \rightarrow A_{\text{inf}}$ via $u \mapsto [\underline{\pi}]$. This embedding is compatible with φ , but not with the G_K -action. We have $\mathfrak{S} \subset A_{\text{inf}}^{G_\infty}$.

For a Kisin module $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\varphi, h}$, we can associate a representation of G_∞ via

$$T_{\mathfrak{S}}(\mathfrak{M}) := \left(\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathbb{C}^b) \right)^{\varphi=1} = \left(\mathfrak{M}/\mathfrak{M}[u^\infty] \otimes_{\mathfrak{S}} W(\mathbb{C}^b) \right)^{\varphi=1}.$$

So the Galois representation attached to \mathfrak{M} is insensible to u -torsion parts because $\frac{1}{u} \in W(\mathbf{C}^b)$. It is well-known that $T_{\mathfrak{S}}$ is exact and there exists an $W(\mathbf{C}^b)$ -linear isomorphism

$$\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathbf{C}^b) \simeq T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} W(\mathbf{C}^b),$$

which is compatible with φ and G_{∞} -actions.

For many purposes, we define another variant $T_{\mathfrak{S}}^h$ of $T_{\mathfrak{S}}$: For an étale $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\varphi, h}$, we can naturally extend $\varphi_h : \text{Fil}^h \varphi^* \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ to $\varphi_h : \text{Fil}^h \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}} \rightarrow \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}}$.

$$T_{\mathfrak{S}}^h(\mathfrak{M}) := \left(\text{Fil}^h \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}} \right)^{\varphi_h=1} = \{x \in \text{Fil}^h \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}}, \varphi(x) = \varphi(E(u)^h)x\}.$$

Lemma 6.11. *Assume that $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\varphi, h}$ is étale. Then*

- (1) $T_{\mathfrak{S}}^h(\mathfrak{M}) \simeq T_{\mathfrak{S}}(\mathfrak{M})(h)$.
- (2) *The following sequence is short exact*

$$0 \longrightarrow T_{\mathfrak{S}}^h(\mathfrak{M}) \longrightarrow \text{Fil}^h \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}} \xrightarrow{\varphi_h-1} \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}} \longrightarrow 0$$

Proof. First it is clear that $T_{\mathfrak{S}}(\mathfrak{M}) = (\varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} W(\mathbf{C}^b))^{\varphi=1}$ because φ on $W(\mathbf{C}^b)$ is bijective. Let pa_0 be the constant term of $E(u)$. Let $\underline{\varepsilon} = (\zeta_{p^n})_{n \geq 0} \in \mathcal{O}_{\mathbf{C}}^b$ with ζ_{p^n} satisfying $\zeta_1 = 1$, $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ and $\zeta_p \neq 1$. By Example 3.2.3 in [Liu10], there exists nonzero $\mathfrak{t} \in A_{\text{inf}}$ so that $\mathfrak{t} \neq 0 \pmod{p}$, $\varphi(\mathfrak{t}) = a_0^{-1}E(u)\mathfrak{t}$ and $\mathfrak{t} := \log[\underline{\varepsilon}] = \mathfrak{c}\varphi(\mathfrak{t})$ with $\mathfrak{c} = \prod_{n=1}^{\infty} \varphi^n\left(\frac{a_0^{-1}E(u)}{p}\right) \in A_{\text{crys}}^*$. Write $\beta = \varphi(\mathfrak{t})$. Consider map $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \rightarrow T_{\mathfrak{S}}(\mathfrak{M})$ by $x \mapsto \frac{x}{\beta^h}$ for any $x \in \text{Fil}^h \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}}$. Since $\varphi(\beta) = \varphi(E(u))\beta$, and $\beta \in W(\mathbf{C}^b)$ is invertible as $\mathfrak{t} \neq 0 \pmod{p}$, ι makes sense. Note that $\mathfrak{c} \in (A_{\text{crys}})^{G_{\infty}}$. So $g(\beta)/\beta = g(\mathfrak{t})/\mathfrak{t}$ is cyclotomic character for any $g \in G_{\infty}$. So $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \rightarrow T_{\mathfrak{S}}(\mathfrak{M})(h)$ is a map compatible with G_{∞} -actions. We claim that $T_{\mathfrak{S}}^h$ is an exact functor. If so since $T_{\mathfrak{S}}$ is also exact, to show that ι is an isomorphism, we can reduce to the case that \mathfrak{M} is killed by p by Corollary 6.4. In this case, \mathfrak{M} is finite $k[[u]]$ -free. Picking a basis e_1, \dots, e_d of \mathfrak{M} , then $\varphi_{\mathfrak{M}}(e_1, \dots, e_d) = (e_1, \dots, e_d)A$ with a $k[[u]]$ -matrix A so that there exists a $k[[u]]$ -matrix B satisfying $AB = BA = (E(u))^h I_d$. Let us still regard e_i as a basis of $\varphi^* \mathfrak{M}$. Then it is easy to check that $(e_1, \dots, e_d)B$ is a basis of $\text{Fil}^h \varphi^* \mathfrak{M}$. Now for any $x = \sum_i e_i \otimes a_i \in \varphi^* \mathfrak{M} \otimes_{k[[u]]} \mathbf{C}^b$, the equation $\varphi(x) = x$ is equivalent to $\varphi(X) = \varphi(A)^{-1}X$ where $X = (a_1, \dots, a_d)^T$. The latter gives $\varphi(\beta^h X) = \varphi(E(u)^h A^{-1})(\beta^h X) = \varphi(B)(\beta^h X)$, which implies that $Y = \beta^h X$ is in $(\mathcal{O}_{\mathbf{C}}^b)^d$. That is $y = \beta^h x \in \varphi^* \mathfrak{M} \otimes_{k[[u]]} \mathcal{O}_{\mathbf{C}}^b$. Furthermore, consider $Z = B^{-1}\beta^h X$, since $\varphi(Z) = \varphi(B^{-1}A^{-1}E(u)^h)BZ = BZ$. We conclude that Z has all entries in $\mathcal{O}_{\mathbf{C}}^b$. Then $\beta^h x = (e_1, \dots, e_d)BZ$ is inside $\text{Fil}^h \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathbf{C}}^b$. This proves that ι is surjective. Since ι is clearly injective, we show that ι is an isomorphism.

Now we prove the claim that $T_{\mathfrak{S}}^h$ is exact. For this, it suffices to show that $\varphi_h - 1$ is surjective and we once again reduces to the case that \mathfrak{M} is killed by p . By writing the $k[[u]]$ -basis of \mathfrak{M} as the above, we need to solve the equation $\varphi(X) - BX = Y$ for any $Y = (a_1, \dots, a_d)^T$ for $a_i \in \mathcal{O}_{\mathbf{C}}^b$. Since \mathbf{C}^b is algebraic closed, we see X exists with entries in \mathbf{C}^b . It is easy to compare valuation of each entry by equation $\varphi(X) = BX + Y$ to show that all entries of X must be in $\mathcal{O}_{\mathbf{C}}^b$. □

6.3. Torsion Breuil modules. We fix $0 \leq h \leq p-2$ for this subsection. Recall that $S = \mathcal{A}$ is the p -adically completed PD-envelope of $\theta : \mathfrak{S} \rightarrow \mathcal{O}_K, u \mapsto \pi$, and for $i \geq 1$ write $\text{Fil}^i S \subseteq S$ for the (closure of the) ideal generated by $\{\gamma_n(E) = E^n/n!\}_{n \geq i}$. For $i \leq p-1$, one has $\varphi(\text{Fil}^i S) \subseteq p^i S$, so we may define $\varphi_i : \text{Fil}^i S \rightarrow S$ as $\varphi_i := p^{-i}\varphi$. We have $c_1 := \varphi(a_0^{-1}E(u))/p \in S^{\times}$.

Let $\text{Mod}_{\mathfrak{S}}^{\varphi, h}$ denote the category whose objects are triples $(\mathcal{M}, \text{Fil}^h \mathcal{M}, \varphi_h)$, consisting of

- (1) an S -module \mathcal{M}
- (2) an S -submodule $\text{Fil}^h \mathcal{M} \subset \mathcal{M}$ containing $\text{Fil}^h S \cdot \mathcal{M}$.
- (3) a φ -semi-linear map $\varphi_h : \text{Fil}^h \mathcal{M} \rightarrow \mathcal{M}$ such that for all $s \in \text{Fil}^h S$ and $x \in \mathcal{M}$ we have

$$\varphi_h(sx) = (c_1)^{-h} \varphi_h(s) \varphi_h(E(u)^h x).$$

(4) $\varphi_h(\mathrm{Fil}^h \mathcal{M})$ generates \mathcal{M} as S -modules.

Morphisms are given by S -linear maps preserving Fil^h 's and commuting with φ_h . A sequence is defined to be *short exact* if it is short exact as a sequence of S -module, and induces a short exact sequence on Fil^h 's. Let $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ denote the full subcategory of $'\mathrm{Mod}_S^{\varphi,h}$ so that \mathcal{M} is killed by a p -power and \mathcal{M} can be written as successive quotient of \mathcal{M}_i in $'\mathrm{Mod}_S^{\varphi}$ and each $\mathcal{M}_i \simeq \bigoplus S_1$ where $S_n := S/p^n S$.

For each object $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, we can extend φ_h and Fil^h to $A_{\mathrm{crys}} \otimes_S \mathcal{M}$ in the following: Since $A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}$ is faithfully flat over S/p^n by [CL19, Lem 5.6], $A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M} \rightarrow A_{\mathrm{crys}} \otimes_S \mathcal{M}$ is injective and so we can define $\mathrm{Fil}^h(A_{\mathrm{crys}} \otimes_S \mathcal{M}) := A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M}$ and then φ_h extends to $A_{\mathrm{crys}} \otimes_S \mathcal{M}$. This allows to define a representation of G_∞ via

$$T_S(\mathcal{M}) := (\mathrm{Fil}^h(A_{\mathrm{crys}} \otimes_S \mathcal{M}))^{\varphi_h=1}.$$

Now let us recall the relation of classical torsion Kisin modules and objects in $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h}$ and their relationship to torsion Galois representations. Let $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}\acute{\mathrm{e}}\mathrm{t}}^{\varphi,h}$ denote the category of étale torsion Kisin module of height h . In this subsection, all torsion Kisin modules are étale torsion Kisin modules, i.e., \mathfrak{M} is u -torsion free. For each such \mathfrak{M} , we construct an object $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ as the following: $\mathcal{M} := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and

$$\mathrm{Fil}^h \mathcal{M} := \{x \in \mathcal{M} \mid (1 \otimes \varphi)(x) \in \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M}\};$$

and $\varphi_h : \mathrm{Fil}^h \mathcal{M} \rightarrow \mathcal{M}$ is defined as the composite of following map

$$\mathrm{Fil}^h \mathcal{M} \xrightarrow{1 \otimes \varphi_{\mathfrak{M}}} \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_h \otimes 1} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}.$$

We write $\underline{\mathcal{M}}(\mathfrak{M})$ for $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ built from Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}\acute{\mathrm{e}}\mathrm{t}}^{\varphi,h}$ as the above. Note that $A_{\mathrm{crys}} \otimes_S \underline{\mathcal{M}}(\mathfrak{M}) = A_{\mathrm{crys}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

Proposition 6.12. *The above functor induces an exact equivalence between $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}\acute{\mathrm{e}}\mathrm{t}}^{\varphi,h}$ and $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. Furthermore, there exists short exact sequence*

$$(6.13) \quad 0 \longrightarrow T_S(\mathcal{M}) \longrightarrow A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M} \xrightarrow{\varphi_h^{-1}} A_{\mathrm{crys}} \otimes_S \mathcal{M} \longrightarrow 0$$

and an isomorphism of G_∞ -representations

$$T_S(\underline{\mathcal{M}}(\mathfrak{M})) \simeq T_{\mathfrak{S}}(\mathfrak{M})(h).$$

Proof. The equivalence of functor together with exactness is [CL09, Thm 2.2.1], which built on Breuil and Kisin's results (see [Liu08, Proposition 3.3.1]). Consider an exact sequence in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$,

$$0 \rightarrow \mathcal{M}'' \rightarrow \mathcal{M} \rightarrow \mathcal{M}' \rightarrow 0.$$

Then we have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_S(\mathcal{M}'') & \longrightarrow & T_S(\mathcal{M}) & \longrightarrow & T_S(\mathcal{M}') \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M}'' & \longrightarrow & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M} & \longrightarrow & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M}' \longrightarrow 0 \\ & & \downarrow \varphi_h^{-1} & & \downarrow \varphi_h^{-1} & & \downarrow \varphi_h^{-1} \\ 0 & \longrightarrow & A_{\mathrm{crys}} \otimes_S \mathcal{M}'' & \longrightarrow & A_{\mathrm{crys}} \otimes_S \mathcal{M} & \longrightarrow & A_{\mathrm{crys}} \otimes_S \mathcal{M}' \longrightarrow 0 \end{array}$$

By the definition of exactness in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and since A_{crys}/p^n is flat over S/p^n , we see that last two rows of the above diagram are exact. So to show $\varphi_h - 1$ is surjective on \mathcal{M} , we reduce to situation that \mathcal{M} is killed by p . Also the surjectivity of $\varphi_h - 1$ implies that the functor T_S is exact from the above diagram. So let us first accept that $\varphi_h - 1$ is surjective and postpone the proof in the end.

Now let us construct a natural map $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \rightarrow T_S(\underline{\mathcal{M}}(\mathfrak{M}))$. Write $\mathcal{M} := \underline{\mathcal{M}}(\mathfrak{M})$. It is clear that $\mathrm{Fil}^h \varphi^* \mathfrak{M} \subset \mathrm{Fil}^h \underline{\mathcal{M}}(\mathfrak{M})$ compatible with the injection $\varphi^* \mathfrak{M} \hookrightarrow \mathcal{M}$. But φ_h defined on Kisin modules are slightly different from that on Breuil modules. By chasing definitions, we see that for any $x \in \mathrm{Fil}^h \varphi^* \mathfrak{M}$,

$\varphi_{h,\mathcal{M}}(x) = c_1^h \varphi_{h,\varphi^*\mathfrak{M}}(x)$. Recall $\mathfrak{c} = \prod_{n=1}^{\infty} \varphi^n(\frac{a_0^{-1}E(u)}{p}) \in A_{\text{crys}}^*$ in the proof Lemma 6.11. Since $\varphi(\mathfrak{c}) = c_1 \mathfrak{c}$, the map $\iota : A_{\text{inf}} \otimes_{\mathfrak{S}} \text{Fil}^h \varphi^*\mathfrak{M} \rightarrow A_{\text{crys}} \otimes_S \mathcal{M}$ by $\iota(x) = \mathfrak{c}^h x$ induces a map $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \rightarrow T_S(\mathcal{M})$.

To show that ι is isomorphism, since $T_{\mathfrak{S}}, T_S$ and \mathcal{M} are all exact, we reduce to the case that \mathfrak{M} is killed by p where \mathfrak{M} is finite $k[[u]]$ -free. As the same argument in Lemma 6.11, there exists a basis e_1, \dots, e_d of $\varphi^*\mathfrak{M}$ so that $\text{Fil}^h \varphi^*\mathfrak{M}$ has basis $(e_1, \dots, e_d)B$, $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d)\varphi(A)$ and $AB = BA = E(u)^h I_d$. So any $x \in T_{\mathfrak{S}}^h(\mathfrak{M})$ corresponds to the solution of $\varphi(X) = BX$. Since $\mathcal{M} = \underline{\mathcal{M}}(\mathfrak{M})$, it is straightforward to compute that \mathcal{M} also has S_1 -basis e_1, \dots, e_d , $\text{Fil}^h \mathcal{M}$ is generated by $(e_1, \dots, e_d)B$ and $\text{Fil}^p S_1 \mathcal{M}$. Note that $c_1 \equiv 1 \pmod{(p, \text{Fil}^p S)}$. So $T_S(\mathcal{M})$ corresponds to solutions $\varphi(X) = BX \pmod{\text{Fil}^p A_{\text{crys},1}}$ where $A_{\text{crys},1} = A_{\text{crys}}/pA_{\text{crys}}$. Now it suffices to show the following map is bijective

$$\{X | \varphi(X) = BX, x_i \in \mathcal{O}_{\mathfrak{C}}^b\} \longrightarrow \{X | \varphi(X) = BX \pmod{\text{Fil}^p A_{\text{crys},1}}, x_i \in A_{\text{crys},1}\}$$

Let v denote the valuation on $\mathcal{O}_{\mathfrak{C}}^b$ which normalized by $v(u^e) = 1$. Suppose that X is in the kernel then $X \in E(u)^p \mathcal{O}_{\mathfrak{C}}^b$. So $v(x_j) \geq p, \forall j$. Let x_i be the entry with least valuation. Note that $v(\varphi(x_j)) = pv(x_j)$ for any j and $A\varphi(X) = u^{eh}X$. The mini possible of left side valuation is $pv(x_i)$, while the the right side is $h + v(x_i)$. This is impossible when $v(x_i) \geq p$ because $h \leq p - 2$. So this implies that $X = 0$. Indeed, if $v(x_i) \geq 2$ then the same proof show that $X = 0$. That is, if X_1, X_2 are two solution in the left side and $X_1 \equiv X_2 \pmod{E(u)^2}$ then $X_1 = X_2$

Conversely, let Z be the vector inside $A_{\text{crys},1}$ so that $\varphi(Z) = BZ \pmod{\text{Fil}^p A_{\text{crys},1}}$. Then there exists Z_0 with entries in $\mathcal{O}_{\mathfrak{C}}^b$ so that $\varphi(Z_0) = BZ_0 + E(u)^p C$ where C is a vector with entries in $\mathcal{O}_{\mathfrak{C}}^b$. Note that $E(u)^p = E(u)^{p-h}BA$. So we may write $\varphi(Z_0) = B(Z_0 + E(u)^{p-h}AC)$. Let $Z_1 = Z_0 + E(u)^{p-h}AC$. Then $\varphi(Z_1) = BZ_1 + u^{pe(p-h)}C_1$ with $C_1 = -\varphi(AC)$. Note that $pe(p-h) > pe > he$. we can write $BZ_1 + u^{pe(p-h)}C_1 = B(Z_1 + u^\alpha AC)$ with $\alpha = pe(p-h) - h$. Set $Z_2 = Z_1 + u^\alpha AC$ then $\varphi(Z_2) = BZ_2 + u^{p\alpha}C_2$. Continues this steps, we see that Z_n converges in $\mathcal{O}_{\mathfrak{C}}^b$ to Z' so that $\varphi(Z') = BZ'$ with $Z' \equiv Z_0 \pmod{E(u)^{p-h}}$. This settles the bijection of these two sets and completes the proof.

It remains to show that $\varphi_h - 1 : \text{Fil}^h \mathcal{M} \otimes_S A_{\text{crys}} \rightarrow \mathcal{M} \otimes_S A_{\text{crys}}$ is surjective and we may assume that $\mathcal{M} = \underline{\mathcal{M}}(\mathfrak{M})$ with \mathfrak{M} killed by p . Now that $\mathcal{M} \otimes_S A_{\text{crys}} = \varphi^*\mathfrak{M} \otimes_{k[[u]]} \mathcal{O}_{\mathfrak{C}}^b + \varphi^*\mathfrak{M} \otimes_{k[[u]]} \text{Fil}^p A_{\text{crys},1}$. By Lemma 6.11 (2), it suffices to show that for $y = m \otimes a$ with $m \in \varphi^*\mathfrak{M}$ and $a \in \text{Fil}^p A_{\text{crys},1}$ there exists a $x \in \text{Fil}^h \mathcal{M} \otimes_S A_{\text{crys}}$ so that $\varphi_h(x) - x = y$. Since $\varphi_h(a) = 0$ for $a \in \text{Fil}^p A_{\text{crys},1}$, then $y = -x$ is required. \square

Remark 6.14. If we combine the isomorphisms $\eta : T_{\mathfrak{S}}(\mathfrak{M})(h) \rightarrow T_{\mathfrak{S}}^h(\mathfrak{M}) \rightarrow T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ defined by $x \mapsto \beta^h x \mapsto (\beta\mathfrak{c})^h x = t^h x$. The isomorphism $\eta : T_{\mathfrak{S}}(\mathfrak{M})(h) \simeq T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ is natural in the following sense: Suppose that $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}}$ has a G_K -actions so that G_K -action is semi-linear on G_K -action on A_{inf} and commutes with $\varphi_{\mathfrak{M}}$. Then this G_K -action induces a G_K -actions on $\underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\text{crys}}$ compatible with Fil^h and φ . Then both $T_{\mathfrak{S}}(\mathfrak{M})(h)$ and $T_S(\mathcal{M})$ has G_K -actions and η is G_K -compatible isomorphism.

Regard both S as subring of $K_0[[u]]$. Define $I^+ S = S \cap uK_0[[u]]$ and $I^+ = u\mathfrak{S}$. Clearly we have a natural map $q : \mathfrak{M}/I^+ \rightarrow \underline{\mathcal{M}}(\mathfrak{M})/I^+ S$. By dévissage to the situation that \mathfrak{M} killed by p , we obtain

Corollary 6.15. *Let $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h}$. Then we have*

$$\text{length}_{W(k)}(\underline{\mathcal{M}}(\mathfrak{M})/I^+ S) = \text{length}_{W(k)}(\mathfrak{M}/u\mathfrak{M}) = \text{length}_{\mathbb{Z}}(T_S(\underline{\mathcal{M}}(\mathfrak{M}))) = \text{length}_{\mathbb{Z}}(T_{\mathfrak{S}}(\mathfrak{M})).$$

Now let us add one extra structure to $\text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h}$ to make $T_S(\mathcal{M})$ a G_K -representation. Let $\text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h, \nabla}$ denote the category of the object $(\mathcal{M}, \text{Fil}^h \mathcal{M}, \varphi_h, \nabla)$ where

- (1) $(\mathcal{M}, \text{Fil}^h \mathcal{M}, \varphi_h)$ is an object in $\text{Mod}_{\mathfrak{S}, \text{tor}}^{\varphi, h}$
- (2) $\nabla : \mathcal{M} \rightarrow \mathcal{M}$ is a connection satisfying the following:
 - (a) $E\nabla(\text{Fil}^h \mathcal{M}) \subset \text{Fil}^h \mathcal{M}$.

(b) the following diagram commutes:

$$(6.16) \quad \begin{array}{ccc} \mathrm{Fil}^h \mathcal{M} & \xrightarrow{\varphi_h} & \mathcal{M} \\ E(u)\nabla \downarrow & & \downarrow c_1 \nabla \\ \mathrm{Fil}^h \mathcal{M} & \xrightarrow{u^{p-1}\varphi_h} & \mathcal{M} \end{array}$$

Let us explain the relationship between objects in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$ and Breuil modules studied in work of Breuil and Caruso. Let $N_S : S \rightarrow S$ be $W(k)$ -linear differentiation so that $N_S(u) = u$. An object \mathcal{M} in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ is called a *Breuil module* if \mathcal{M} admits a $W(k)$ -linear morphism $N : \mathcal{M} \rightarrow \mathcal{M}$ such that :

- (1) for all $s \in S$ and $x \in \mathcal{M}$, $N(sx) = N_S(s)x + sN(x)$.
- (2) $E(u)N(\mathrm{Fil}^h \mathcal{M}) \subset \mathrm{Fil}^h \mathcal{M}$.
- (3) the following diagram commutes:

$$(6.17) \quad \begin{array}{ccc} \mathrm{Fil}^h \mathcal{M} & \xrightarrow{\varphi_h} & \mathcal{M} \\ E(u)N \downarrow & & \downarrow c_1 N \\ \mathrm{Fil}^h \mathcal{M} & \xrightarrow{\varphi_h} & \mathcal{M} \end{array}$$

Remark 6.18. Breuil and Caruso use convention $N_S(u) = -u$. In fact, there is almost no difference for entire theory by using $N_S(u) = u$ except for the formula (6.20) need to change sign comparing with the similar formula [Liu08, (5.1.1)]

Let $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,N}$ denote the category of Breuil modules. There is a natural functor $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla} \rightarrow \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,N}$ by define $N_{\mathcal{M}} = u\nabla$. It is easy to chase the diagram to see this functor makes sense. So we also call objects in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$ Breuil modules.

Now we can define a G_K -action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ as in: for any $\sigma \in G_K$, any $x \otimes a \in \mathcal{M} \otimes_S A_{\mathrm{crys}}$, define

$$(6.19) \quad \sigma(x \otimes a) = \sum_{i=0}^{\infty} \nabla^i(x) \otimes \gamma_i(\sigma([\pi]) - [\pi])\sigma(a).$$

We can also define a G_K -action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ as in [Liu10, §5.1]: for any $\sigma \in G_K$, recall $\underline{\varepsilon}(\sigma) = \frac{\sigma([\pi])}{[\pi]} \in A_{\mathrm{inf}}$. For any $x \otimes a \in \mathcal{M} \otimes_S A_{\mathrm{crys}}$, define

$$(6.20) \quad \sigma(x \otimes a) = \sum_{i=0}^{\infty} N^i(x) \otimes \gamma_i(\log(\underline{\varepsilon}(\sigma)))\sigma(a).$$

where $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. We claim that (6.19) and (6.20) are the same formula. Let us postpone the proof in §8.1 as the proof is just long combinatoric calculation.

Note that if $\sigma \in G_{\infty}$, then $\log(\underline{\varepsilon}(\sigma)) = 0$ and $\sigma(x \otimes a) = x \otimes \sigma(a)$. Thus G_K -action defined above (if it is well defined) is compatible with the natural G_{∞} -action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$.

Lemma 6.21. *The above action is well defined A_{crys} -semi-linear G_K -action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ and compatible with $\mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}})$ and φ_h .*

Proof. The proof of [Liu10, §5.1] essentially applies here. It is standard to check that (6.19) is well-defined map; it is A_{crys} -semi-linear-action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ and compatible with G_K -action on A_{crys} ; and G_{∞} -acts on $\mathcal{M} \otimes 1$ trivially. It is clear that $\log(\underline{\varepsilon}(\sigma)) \in \mathrm{Fil}^1 A_{\mathrm{crys}}$. So by that $E(u)N(\mathrm{Fil}^h \mathcal{M}) \subset \mathrm{Fil}^h \mathcal{M}$, we see that $\sigma(\mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}})) \subset \mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}})$. The only thing left to check is that φ_h commutes with G_K -action, which can be reduce to check the following: write $a = -\log(\underline{\varepsilon}(\sigma))$ and pick $x \in \mathrm{Fil}^h \mathcal{M}$, we have

$$\varphi_h(\gamma_i(a) \otimes N^i(x)) = \gamma_i(a) \otimes N^i(\varphi_h(x)).$$

It is clear that $\varphi(a) = pa$. So $\varphi(\gamma_i(a)) = \gamma_i(a)c_1^{-i}\varphi(E(u)^i)$. So the above equality is reduced to check $c_1^{-i}\varphi_h(E(u)^i N^i(x)) = N^i(\varphi_h(x))$ and this can be check by induction on i . \square

Corollary 6.22. *Given a Breuil module $\mathcal{M} \in \text{Mod}_{S, \text{tor}}^{\varphi, h, N}$, then $T_S(\mathcal{M})$ (as a G_∞ -representation) extends to a G_K -representation.*

To summarize our section, we return to the situation of §5.2 where $\mathcal{M}^i := H_{\text{crys}}^i(X_n/S_n)$ is proved to admit structures $\text{Fil}^i \mathcal{M}^i = H_{\text{crys}}^i(X_n/S_n, \mathcal{I}_{\text{crys}}^{[i]})$, $\varphi_i : \text{Fil}^i \mathcal{M}^i \rightarrow \mathcal{M}^i$ and $\nabla : \mathcal{M}^i \rightarrow \mathcal{M}^i$. Obviously, our axioms of $\text{Mod}_{S, \text{tor}}^{\varphi, h, \nabla}$ is aimed at describing these structures of $H_{\text{crys}}^i(X_n/S_n)$.

Definition 6.23. For $i \leq p-2$, we call that $H_{\text{crys}}^i(X_n/S_n)$ is a Breuil module if the quadruple

$$\left(H_{\text{crys}}^i(X_n/S_n), H_{\text{crys}}^i(X_n/S_n, \mathcal{I}_{\text{crys}}^{[i]}), \varphi_i, \nabla \right)$$

constructed in §5.2 is an object in $\text{Mod}_{S, \text{tor}}^{\varphi, i, \nabla}$, which is equivalent to the triple

$$\left(H_{\text{crys}}^i(X_n/S_n), H_{\text{crys}}^i(X_n/S_n, \mathcal{I}_{\text{crys}}^{[i]}), \varphi_i \right)$$

being an object in $\text{Mod}_{S, \text{tor}}^{\varphi, i}$.

Our main theorem is to show that $H_{\text{crys}}^i(X_n/S_n)$ together with these structures is indeed a Breuil module when $ei < p-1$.

7. TORSION COHOMOLOGY AND COMPARISON WITH ÉTALE COHOMOLOGY

In this section, we collect our previous preparations to understand the structures of torsion crystalline cohomology and its relationship with étale cohomology via torsion prismatic cohomology. In the end, we show that if $ei < p-1$ then p^n -th torsion crystalline cohomology $H_{\text{crys}}^i(X_n/S_n)$ has structure of torsion Breuil module to compare to $H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ via T_S , where $X_{\bar{\eta}}$ is a geometric generic fiber of X .

7.1. Prismatic cohomology and (generalized) Kisin modules. Let (A, I) be any prism. As in the end of §5.2, for any $n \geq 1$, we define torsion prismatic cohomology $\text{R}\Gamma_{\Delta}(X_n/A_n) := \text{R}\Gamma_{\Delta}(X/A, \mathcal{O}_{\Delta}/p^n\mathcal{O}_{\Delta}) = \text{R}\Gamma_{\Delta}(X/A) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$. We have $\text{R}\Gamma_{\Delta}(X_n/A_n) \simeq \text{R}\Gamma_{\text{qSyn}}(X, \Delta_{-/A}/p^n) \simeq \text{R}\Gamma_{\text{qSyn}}(X, \Delta_{-/A}) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$.

Warning 7.1. We warn readers that the notation $\text{R}\Gamma_{\Delta}(X_n/A_n)$ is misleading, as it might suggest that this cohomology theory only depends on the mod p^n reduction of X which is not true. See [BMS18, Remark 2.4] for a counterexample.

Proposition 7.2. *Assume that (A, I) is transversal and $\varphi : A \rightarrow A$ is flat. Then $H_{\Delta}^i(X_n/A_n)$ has height i .*

Proof. We follow the same idea of [BS19, Corollary 15.5] which proved that $H_{\Delta}^i(X/A)$ has height i . Examining the proof, it suffices to show that $\varphi^* \text{R}\Gamma_{\Delta}(X_n/A_n) \simeq L\eta_I \text{R}\Gamma_{\Delta}(X_n/A_n)$ when $X = \text{Spf}(R)$ is an affine smooth p -adic formal scheme over A/I . By Theorem 15.3 of *loc. cit.*, we have $\varphi^* \text{R}\Gamma_{\Delta}(X/A) \simeq L\eta_I \text{R}\Gamma_{\Delta}(X/A)$. Since $\varphi : A \rightarrow A$ is flat, it suffices to show that

$$(7.3) \quad (L\eta_I \text{R}\Gamma_{\Delta}(X/A)) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z} \simeq L\eta_I (\text{R}\Gamma_{\Delta}(X/A) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}).$$

Now we may apply [Bha18b, Lemma 5.16] to the above by $g = p^n$ and $f = d$. So we need to check that $H^*(\text{R}\Gamma_{\Delta}(X/A) \otimes_A^{\mathbb{L}} A/d)$ has no p^n -torsion. This follows from the Hodge–Tate comparison

$$H^i(\text{R}\Gamma_{\Delta}(X/A) \otimes_A^{\mathbb{L}} A/I) \simeq \Omega_{X/(A/I)}^i\{i\}.$$

□

Corollary 7.4. *For $n \in \mathbb{N} \cup \{\infty\}$, the φ -module $H_{\Delta}^i(X_n/\mathfrak{S}_n)$ is an object of $\text{Mod}_{\mathfrak{S}}^{\varphi, i}$, i.e., a (generalized) Kisin module of height i and $T_{\mathfrak{S}}(H_{\Delta}^i(X_n/\mathfrak{S}_n)) \simeq H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$.*

Proof. It suffices to prove that $T_{\mathfrak{S}}(H_{\Delta}^i(X_n/\mathfrak{S}_n)) \simeq H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$. Write $\mathfrak{M}_n^i := H_{\Delta}^i(X_n/\mathfrak{S}_n)$, $\mathcal{X} := \text{Spf } \mathcal{O}_{\mathbb{C}} \times_{\text{Spf } \mathcal{O}_K} X$. For $n \neq \infty$, by [BS19, Theorem 1.8 (4) (5)], we have

$$H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) \simeq \left(H^i(\text{R}\Gamma_{\Delta}(\mathcal{X}/A_{\text{inf}})/p^n) \left[\frac{1}{E(u)} \right] \right)^{\varphi=1} = (\mathfrak{M}_n^i \otimes_{\mathfrak{S}} W_n(\mathcal{O}_{\mathbb{C}}^b) \left[\frac{1}{u} \right])^{\varphi=1} = (\mathfrak{M}_n^i \otimes_{\mathfrak{S}} W_n(\mathbb{C}^b))^{\varphi=1},$$

which is just $T_{\mathfrak{S}}(\mathfrak{M}_n^i)$. The case of $n = \infty$ easily follows by taking inverse limits. □

Remark 7.5. The G_∞ -action on $T_\mathfrak{S}(\mathfrak{M}_n^i)$ discussed in §6.2 naturally extends to a G_K -action by isomorphism $\mathfrak{M}_n^i \otimes_\mathfrak{S} A_{\text{inf}} \simeq H_\Delta^i(X/A_{\text{inf}})$, which admits a natural G_K -action that commutes with φ . In this way $T_\mathfrak{S}(H_\Delta^i(X_n/\mathfrak{S}_n)) \simeq H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism of G_K -actions.

Let $X_k := X \times_{\text{Spf}(\mathcal{O}_K)} \text{Spf}(k)$ be the closed fiber of X .

Lemma 7.6. *If $\text{length}_{W(k)} H_{\text{crys}}^i(X_k/W_n(k)) = \text{length}_{\mathbb{Z}} H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ then \mathfrak{M}_n^j has no u -torsion for $j = i, i+1$.*

Proof. We claim that $\text{R}\Gamma_\Delta(X_n/\mathfrak{S}_n) \otimes_{\mathfrak{S}}^{\mathbb{L}} W(k) \simeq \text{R}\Gamma_{\text{crys}}(X_k/W_n(k))$. To see this, first note that $(\mathfrak{S}, E) \rightarrow (W(k), p)$ by $\text{mod } u$ is a map of prisms. So [BS19, Theorem 1.8 (5)] proves that $\text{R}\Gamma_\Delta(X/\mathfrak{S}) \otimes_{\mathfrak{S}}^{\mathbb{L}} W(k) \simeq \text{R}\Gamma_\Delta(X_k/W(k))$. Then Theorem 1.8 (1) *loc. cit.* shows that $\text{R}\Gamma_\Delta(X/\mathfrak{S}) \otimes_{\mathfrak{S}}^{\mathbb{L}} W(k) \simeq \text{R}\Gamma_{\text{crys}}(X_k/W(k))$. Then the claim follows by $\otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$ on both sides.

The claim immediately shows that the exact sequence

$$(7.7) \quad 0 \rightarrow \mathfrak{M}_n^i/u\mathfrak{M}_n^i \rightarrow H_{\text{crys}}^i(X_k/W_n(k)) \rightarrow \mathfrak{M}_n^{i+1}[u] \rightarrow 0$$

So $\text{length}_{W(k)} \mathfrak{M}_n^i/u\mathfrak{M}_n^i \leq \text{length}_{W(k)} H_{\text{crys}}^i(X_k/W_n(k))$. On the other hand, consider the exact sequence in Lemma 6.2 with $\mathfrak{M} := \mathfrak{M}_n^i$

$$0 \rightarrow \mathfrak{M}[u^\infty] \rightarrow \mathfrak{M} \rightarrow \mathfrak{M}/\mathfrak{M}[u^\infty] \rightarrow 0$$

Write $\mathfrak{M}^{\text{ét}} := \mathfrak{M}/\mathfrak{M}[u^\infty]$. Since $\mathfrak{M}^{\text{ét}}$ has no u -torsion, the above exact sequence remaining exact by modulo u . So we have $\text{length}_{W(k)}(\mathfrak{M}^{\text{ét}}/u\mathfrak{M}^{\text{ét}}) \leq \text{length}_{W(k)} \mathfrak{M}/u\mathfrak{M}$ and equality holds only when $\mathfrak{M}[u^\infty] = \{0\}$. Since $T_\mathfrak{S}(\mathfrak{M}) = T_\mathfrak{S}(\mathfrak{M}^{\text{ét}})$, and $T_\mathfrak{S}(\mathfrak{M}) \simeq H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ by Corollary 7.4, Corollary 6.15 proves the following inequalities

$$\text{length}_{\mathbb{Z}} H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) = \text{length}_{W(k)}(\mathfrak{M}^{\text{ét}}/u\mathfrak{M}^{\text{ét}}) \leq \text{length}_{W(k)}(\mathfrak{M}/u\mathfrak{M}).$$

Now combine with the exact sequence (7.7), we conclude that

$$\text{length}_{\mathbb{Z}} H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) \leq \text{length}_{W(k)} H_{\text{crys}}^i(X_k/W_n(k))$$

and equality holds only if all the above inequalities become equalities and \mathfrak{M}_n^i and \mathfrak{M}_n^{i+1} have no u -torsions. \square

7.2. Nygaard filtration and Breuil–Kisin filtration. By Corollary 7.4, $\mathfrak{M}_n^i := H_\Delta^i(X_n/\mathfrak{S}_n)$ is a Kisin module of height i . Then $\varphi^*\mathfrak{M}_n^i \simeq H_{\text{qSyn}}^i(X, \Delta_{-/ \mathfrak{S}}^{(1)} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z})$ admits two filtrations: Breuil–Kisin filtration defined in (6.8) and Nygaard filtration $H_{\text{qSyn}}^i(X, \text{Fil}_N^i \Delta_{-/ \mathfrak{S}}^{(1)} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z})$. The aim of this subsection is to compare these two filtrations.

This theme can be put in more general setting for a bounded prism (A, I) . Recall that in [BS19, §15] the authors studied $\Delta_{-/A}$ and $\Delta_{-/A}^{(1)} := A \widehat{\otimes}_{\varphi, A}^{\mathbb{L}} \Delta_{-/A}$ as sheaves on $\text{qSyn}_{A/I}$. Also constructed in *loc. cit.* is the so-called Nygaard filtration $\text{Fil}_N^j \Delta_{-/A}^{(1)}$, also discussed §4.2. For any $n \in \mathbb{N} \cup \{\infty\}$, set $\Delta_n^{(1)} := \Delta_{-/A}^{(1)} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$ and $\text{Fil}_N^j \Delta_n^{(1)} := \text{Fil}_N^j \Delta_{-/A}^{(1)} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$. Here and below, we adopt the convention that $n = \infty$ means we do not perform any base change.

Lemma 7.8. *Let (A, I) be a bounded prism. Let X be a smooth (p -adic) formal scheme over $\text{Spf}(A/I)$ of relative dimension n . Then we have:*

- (1) *The Nygaard filtration $\text{R}\Gamma(X_{\text{qSyn}}, \text{Fil}_N^\bullet)$ on $\text{R}\Gamma(X_{\text{qSyn}}, \Delta_{-/A}^{(1)})$ is complete.*
- (2) *The natural map*

$$\text{Fil}_N^i \otimes_A I^j \rightarrow \text{Fil}_N^{i+j}$$

of quasisyntomic sheaves induces a morphism

$$H^l(X_{\text{qSyn}}, \text{Fil}_N^i) \otimes_A I^j \rightarrow H^l(X_{\text{qSyn}}, \text{Fil}_N^{i+j})$$

which is an isomorphism when either $l \leq i$, and an injection when $l = i+1$. When $i \geq n$ this map induces an isomorphism

$$\text{R}\Gamma(X_{\text{qSyn}}, \text{Fil}_N^n) \otimes_A I^j \cong \text{R}\Gamma(X_{\text{qSyn}}, \text{Fil}_N^{n+j}).$$

(3) *The natural map*

$$\varphi: \mathrm{Fil}_N^i \rightarrow \Delta_{-/A} \otimes_A I^i$$

induces a map on cohomology

$$H^l(X_{\mathrm{qSyn}}, \mathrm{Fil}_N^i) \rightarrow H^l(X_{\mathrm{qSyn}}, \Delta_{-/A}) \otimes_A I^i$$

which is an isomorphism when $l \leq i$ and injective when $l = i + 1$.

Moreover their derived mod p^m counterparts hold true as well.

We thank Bhargav for pointing out the statement (3) above, which we did not realize can be proved so easily. This significantly simplifies an earlier draft.

Proof. (1) follows from (2). Indeed, (2) implies the Nygaard filtration on $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}_N^i)$ is simply the I -adic filtration, hence it is complete.

(2) follows from the following exact triangle of quasisyntomic sheaves:

$$\mathrm{Fil}_N^i \otimes_A I \rightarrow \mathrm{Fil}_N^{i+1} \rightarrow \mathrm{Fil}_H^{i+1} \mathrm{dR}_{-/A}^\wedge.$$

Observe that

$$\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}_H^l \mathrm{dR}_{-/A}^\wedge) \cong \mathrm{R}\Gamma(X, \mathrm{Fil}_H^l \mathrm{dR}_{-/A}^\wedge)$$

lives in $D^{\geq l}(A/I)$, and vanishes when $l > n$. An easy induction gives what we want.

As for (3): we look at the map of filtered complexes

$$\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}_N^i) \xrightarrow{\varphi} \mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \Delta_{-/A} \otimes_A I^i)$$

where the former is equipped with Nygaard filtration $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}_N^{i+*})$ and the latter is equipped with I -adic filtration $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \Delta_{-/A} \otimes_A I^{i+*})$. Notice that both filtrations are complete. Now [BS19, Theorem 15.2.(2)] implies that the cone of the $(i+*)$ -th graded piece lives in $D^{>(i+*)}(A/I)$. Hence we conclude that the cone of φ lives in $D^{>i}(A)$. Therefore the induced maps of degree at most i cohomology groups are isomorphisms, and the induced map in degree $i+1$ is injective.

Their derived mod p^m counterparts are proved in exactly the same way. \square

Now let us return to the situation of Breuil–Kisin prism $A = \mathfrak{S}$. Recall that $\Delta_n^{(1)} := \Delta_{-/ \mathfrak{S}}^{(1)} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}$ and $\mathrm{Fil}_N^i \Delta_n^{(1)} := \mathrm{Fil}_N^i \Delta_{-/ \mathfrak{S}}^{(1)} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}$. Recall that $\mathfrak{M}_n^i := H_{\Delta}^i(X_n/\mathfrak{S}_n)$ and recall that Breuil–Kisin-filtration on $\varphi^* \mathfrak{M}_n^i \cong H_{\mathrm{qSyn}}^i(X, \Delta_n^{(1)})$ is defined as the image of $\psi: \mathfrak{M}_n^i \rightarrow \varphi^* \mathfrak{M}_n^i$.

Corollary 7.9. *For any $i \in \mathbb{N}$ and any $n \in \mathbb{N} \cup \{\infty\}$, there is a functorial commutative diagram:*

$$\begin{array}{ccc} H_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^i \Delta_n^{(1)}) & \xrightarrow{\varphi_i} & \mathfrak{M}_n^i \\ & \searrow & \swarrow \psi \\ & & \varphi^* \mathfrak{M}_n^i \end{array}$$

with φ_i an isomorphism.

Proof. First let us justify the existence of the functorial commutative diagram. We may work with affine formal schemes $Y = \mathrm{Spf}(R)$. In this case, by the proof of [BS19, Theorem 15.3 and Corollary 15.5], we see ψ is constructed by the following (right-lower corner) diagram

$$\begin{array}{ccc} \tau^{\leq i} \mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, \mathrm{Fil}_N^i \Delta^{(1)}) & \dashrightarrow & \tau^{\leq i} \mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, \Delta) \otimes (E)^i \\ \downarrow & & \downarrow \\ \tau^{\leq i} \mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, \Delta^{(1)}) & \xrightarrow{\sim} \tau^{\leq i} L\eta_E \mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, \Delta) \xrightarrow{\psi} & \tau^{\leq i} \mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, \Delta) \\ & \searrow \varphi & \end{array}$$

Here the top row is the same as (truncation by $\leq i$ of) the following morphism

$$\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}_N^i) \xrightarrow{\varphi} \mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \Delta_{-/A} \otimes_A I^i)$$

appeared in Lemma 7.8. Derived mod p^n gives the desired functorial commutative diagram. By Lemma 7.8 (3) we know that φ_i is an isomorphism. \square

Remark 7.10. In the context of filtered derived infinity categories, a filtration is nothing but an arrow. Hence one could define two “quasi-filtrations”³: one being the Breuil–Kisin quasi-filtration: $\mathfrak{M}_n^i \xrightarrow{\psi} \varphi^* \mathfrak{M}_n^i$; another being the i -th Nygaard quasi-filtration: $\mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^i \Delta_n^{(1)}) \rightarrow \varphi^* \mathfrak{M}_n^i$. Then the above is saying that these two quasi-filtrations are canonically identified via φ_i .

Let us name the map

$$\iota_n^{i,j} : \mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^j \Delta_n^{(1)}) \rightarrow \mathrm{Fil}_{\mathrm{BK}}^j \mathrm{H}_{\mathrm{qSyn}}^i(X, \Delta_n^{(1)})$$

for any pair of natural numbers (i, j) and any $n \in \mathbb{N} \cup \{\infty\}$. We have the following knowledge of the image of $\iota_n^{i,j}$ when $i \leq j$.

Corollary 7.11. *Let $i \leq j$. Then we have an identification*

$$\mathrm{Im}(\iota_n^{i,j}) \cong \mathrm{Im}(\psi : \mathfrak{M}_n^i \rightarrow \varphi^* \mathfrak{M}_n^i) \cdot E^{j-i}.$$

In particular, define $\widetilde{\mathfrak{M}}_n^i := \mathfrak{M}_n^i/[u^\infty]$ and $\widetilde{\varphi^* \mathfrak{M}}_n^i := \varphi^* \mathfrak{M}_n^i/[u^\infty]$, we have an identification

$$\mathrm{Im}(\widetilde{\iota}_n^{i,j} : \mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^i \Delta_n^{(1)}) \rightarrow \mathrm{Fil}_{\mathrm{BK}}^i \widetilde{\varphi^* \mathfrak{M}}_n^i) \cong \{x \in \widetilde{\varphi^* \mathfrak{M}}_n^i \mid (1 \otimes \varphi)(x) \in E(u)^j \widetilde{\mathfrak{M}}_n^i\}.$$

Proof. The first statement follows from combining Lemma 7.8 (2) and Corollary 7.9. The second statement follows from the first statement and the fact that \mathfrak{M}_n^i has height i . \square

Below we make some primitive investigations of what happens without assuming $i \leq j$.

Proposition 7.12. *Let $A = \mathfrak{S}$ be the Breuil–Kisin prism. For any triple (i, j, n) , the kernel and cokernel of $\iota_n^{i,j}$ above are finite.*

Proof. Note that the kernel and cokernel of $\iota_n^{i,j}$ are finitely generated modules over $\mathfrak{S}/(p^n)$. We have a containment

$$E(u)^j \cdot \Delta^{(1)} \subset \mathrm{Fil}_N^j \Delta^{(1)} \subset \Delta^{(1)}$$

of sheaves on $\mathrm{qSyn}_{A/I}$. This shows that the map $\iota_n^{i,j}$ admits a section up to multiplication by $E(u)^j$, therefore the kernel and cokernel of $\iota_n^{i,j}$ are annihilated by $E(u)^j$. If $n \in \mathbb{N}$, the kernel and cokernel of $\iota_n^{i,j}$ are finitely generated modules over $\mathfrak{S}/(p^n, E(u)^j)$, hence finite.

If $n = \infty$, denote the map by $\iota^{i,j}$, we make the following

Claim 7.13. *The map $\iota^{i,j} : \mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^i \Delta^{(1)})[1/p] \rightarrow \mathrm{Fil}_{\mathrm{BK}}^i \varphi^* \mathfrak{M}^i[1/p]$ is an isomorphism.*

Granting this claim, the kernel and cokernel of $\iota^{i,j}$ are finitely generated modules over $\mathfrak{S}/(E(u)^j)$ annihilated by a power of p , hence finite. \square

Proof of Claim 7.13. First let us show the injectivity, which is the same as injectivity of

$$\mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^j \Delta^{(1)})[1/p] \rightarrow \mathrm{H}_{\mathrm{qSyn}}^i(X, \Delta^{(1)})[1/p].$$

To this end, we use the filtration $\mathrm{Fil}^{i,j}$ discussed in Section 4.2. We claim a slightly stronger statement: the maps

$$\mathrm{H}_{\mathrm{qSyn}}^m(X, \mathrm{Fil}^{i,j} \Delta^{(1)})[1/p] \rightarrow \mathrm{H}_{\mathrm{qSyn}}^m(X, \mathrm{Fil}^{i,0} \Delta^{(1)})[1/p]$$

are injective for all $i \geq 0$. The case of $i \geq j$ is trivial due to Proposition 4.10 (2). For the rest of i , we perform induction on descending i . By five Lemma and Proposition 4.10 (3), it suffices to know that the maps

$$\mathrm{H}^m(X, \mathrm{Fil}_{\mathrm{H}}^{j-i} \mathrm{dR}_{X/\mathcal{O}_K})[1/p] \rightarrow \mathrm{H}^m(X, \mathrm{dR}_{X/\mathcal{O}_K})[1/p]$$

³This terminology is suggested by S. Mondal.

are injective. This injectivity is equivalent to the degeneration of the Hodge-to-de Rham spectral sequence for the rigid space X_K , which is a result due to Scholze [Sch13, Theorem 1.8].

Next we show surjectivity by induction on j , the case of $j = 0$ being trivial. All we need to show is that the induced map

$$\mathrm{Coker}\left(\mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^{j+1} \Delta^{(1)})[1/p] \rightarrow \mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{Fil}_N^j \Delta^{(1)})[1/p]\right) \xrightarrow{\bar{\varphi}} \frac{E(u)^j \mathfrak{M}^i}{E(u)^{j+1} \mathfrak{M}^i}[1/p]$$

is injective. By the injectivity of $\iota^{i,j}[1/p]$ proved in the previous paragraph, we can rewrite the left hand side as $\mathrm{H}_{\mathrm{qSyn}}^i(X, \mathrm{gr}_N^j \Delta^{(1)})[1/p]$. Recall that $\mathfrak{M}^i[1/p]$ is finite free over $\mathfrak{S}[1/p]$ (see Lemma 6.1 (3)), therefore the right hand side can be rewritten as $\mathrm{H}_{\mathrm{qSyn}}^i(X, \bar{\Delta})[1/p]\{j\}$, the j -th Breuil–Kisin twist of the i -th Hodge–Tate cohomology of X_K . By [BS19, Theorem 15.2], we can identify the left hand side further as the j -th conjugate filtration of the right hand side. Now it follows from the degeneration of Hodge–Tate spectral sequence [BMS18, Theorem 13.3] that $\bar{\varphi}$ is always injective. \square

Below we exhibit an example illustrating the necessity of the $i \leq j$ assumption in Corollary 7.11.

Example 7.14 (see [Li20, Section 4]). Let K be a ramified quadratic extension of \mathbb{Q}_p and let G be a lift of α_p over \mathcal{O}_K . Denote the classifying stack of G by BG . Below we summarize previous study of various cohomologies of BG as documented in [Li20, 4.6-4.10], following notation thereof.

- (1) The Breuil–Kisin prismatic cohomology ring of BG is given by

$$\mathrm{H}_{\Delta}^*(BG/\mathfrak{S}) \cong \mathfrak{S}[\tilde{u}]/(p \cdot \tilde{u})$$

where \tilde{u} has degree 2.

- (2) The Hodge–Tate spectral sequence does not degenerate at E_2 page, but does degenerate at E_3 page, giving rise to short exact sequences:

$$0 \rightarrow \mathrm{H}^{i+1}(BG, \wedge^{i-1} \mathbb{L}_{BG/\mathcal{O}_K}) \simeq \mathbb{F}_p \rightarrow \mathrm{H}_{\mathrm{HT}}^{2i}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K/(p) \rightarrow \mathrm{H}^i(BG, \wedge^i \mathbb{L}_{BG/\mathcal{O}_K}) \simeq \mathbb{F}_p \rightarrow 0$$

for all $i > 0$.

- (3) The Hodge-to-de Rham spectral sequence does not degenerate at E_1 page, but does degenerate at E_2 page, giving rise to short exact sequences:

$$0 \rightarrow \mathrm{H}^{2i-1}(BG, \mathbb{L}_{BG/\mathcal{O}_K}) \simeq \mathbb{F}_p \rightarrow \mathrm{H}_{\mathrm{dR}}^{2i}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K/(p) \rightarrow \mathrm{H}^{2i}(BG, \mathcal{O}_{BG}) \simeq \mathbb{F}_p \rightarrow 0$$

for all $i > 0$.

By [BS19, Theorem 15.2], we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta^{(1)}) & \xrightarrow{\varphi} & \mathrm{R}\Gamma_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{dR}}(BG/\mathcal{O}_K) & \longrightarrow & \mathrm{R}\Gamma(BG, \mathcal{O}_{BG}) \longrightarrow \mathrm{R}\Gamma_{\mathrm{HT}}(BG/\mathcal{O}_K) \end{array}$$

where φ is the Frobenius on prismatic cohomology, vertical maps are derived modulo $E(u)$ reductions, the two arrows on the bottom row are natural arrows appearing in Hodge-to-de Rham and Hodge–Tate spectral sequences respectively. Looking at the degree 2 cohomology together with (2) and (3) above, we see that φ on $\mathrm{H}_{\Delta}^2(BG/\mathfrak{S})$ is given by, up to a unit in \mathfrak{S}/p , multiplication by $u \in \mathfrak{S}/p$. Since φ is a map of \mathbb{E}_{∞} -algebras, using (1) we see that φ on $\mathrm{H}_{\Delta}^4(BG/\mathfrak{S})$ is given by, up to a unit in \mathfrak{S}/p , multiplication by $u^2 = E(u) \in \mathfrak{S}/p$. In particular, we see that $\mathrm{Fil}_{\mathrm{BK}}^1 \mathrm{H}_{\mathrm{qSyn}}^4(BG/\mathfrak{S}, \Delta^{(1)}) = \mathrm{H}_{\mathrm{qSyn}}^4(BG/\mathfrak{S}, \Delta^{(1)})$ is the whole cohomology group.

On the other hand, we claim that the map

$$\mathrm{H}_{\mathrm{qSyn}}^4(BG/\mathfrak{S}, \mathrm{Fil}_N^1 \Delta^{(1)}) \rightarrow \mathrm{H}_{\mathrm{qSyn}}^4(BG/\mathfrak{S}, \Delta^{(1)})$$

is not surjective. Indeed we have a long exact sequence coming from the exact triangle $\mathrm{Fil}_N^1 \Delta^{(1)} \rightarrow \Delta^{(1)} \rightarrow \mathcal{O}_{BG}$ with the second arrow being the composition of derived modulo $E(u)$ followed by projection modulo first Hodge filtration. Hence (3) above shows that the cokernel is exactly of length 1. This shows that BG is a

smooth proper stack counterexample for $(i, j, n) = (4, 1, \infty)$. Since all these cohomology groups are p -torsion, we see that this also provides a stacky counterexample for $(i, j, n) = (3, 1, 1)$.

Finally let us use an approximation of BG to get a smooth proper scheme counterexample. By [Li20, Subsection 4.3] there is a smooth projective fourfold X over \mathcal{O}_K together with a map $f: X \rightarrow BG$ such that the induced pullback map of Hodge cohomology is injective when total degree is no larger than 4. By functoriality of the formation of Breuil–Kisin filtrations, we know that

$$\mathrm{Im}\left(f^*: \mathrm{H}_{\mathrm{qSyn}}^4(BG/\mathfrak{S}, \Delta^{(1)}) \rightarrow \mathrm{H}_{\mathrm{qSyn}}^4(X/\mathfrak{S}, \Delta^{(1)})\right) \subset \mathrm{Fil}_{\mathrm{BK}}^1 \mathrm{H}_{\mathrm{qSyn}}^4(X/\mathfrak{S}, \Delta^{(1)}).$$

Lastly we claim that $f^*(\tilde{u}^2) \in \mathrm{H}_{\mathrm{qSyn}}^4(X/\mathfrak{S}, \Delta^{(1)})$ is not in the image of $\mathrm{H}_{\mathrm{qSyn}}^4(X/\mathfrak{S}, \mathrm{Fil}_{\mathrm{N}}^1 \Delta^{(1)})$. To see this it suffices to compare the two exact sequences:

$$\begin{array}{ccccc} \mathrm{H}_{\mathrm{qSyn}}^4(BG/\mathfrak{S}, \mathrm{Fil}_{\mathrm{N}}^1 \Delta^{(1)}) & \longrightarrow & \mathrm{H}_{\mathrm{qSyn}}^4(BG/\mathfrak{S}, \Delta^{(1)}) & \longrightarrow & \mathrm{H}_{\mathrm{qSyn}}^4(BG, \mathcal{O}_{BG}) \\ \downarrow & & \downarrow & & \downarrow f^* \\ \mathrm{H}_{\mathrm{qSyn}}^4(X/\mathfrak{S}, \mathrm{Fil}_{\mathrm{N}}^1 \Delta^{(1)}) & \longrightarrow & \mathrm{H}_{\mathrm{qSyn}}^4(X/\mathfrak{S}, \Delta^{(1)}) & \longrightarrow & \mathrm{H}_{\mathrm{qSyn}}^4(X, \mathcal{O}_X) \end{array}$$

and invoke the fact that f^* is injective by our choice of X . This gives us smooth projective fourfold over \mathcal{O}_K violating the conclusion of Corollary 7.11 for $(i, j, n) = (4, 1, \infty)$ or $(i, j, n) = (3, 1, 1)$.

7.3. Torsion crystalline cohomology. Now we are ready to discuss the structure of $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ via prismatic cohomology. First, we provide an application of the comparison $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S}, \varphi} S \cong \mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{X}/S)$, which concerns the module structure of the cohomology of the latter. We need some preparations.

Lemma 7.15. *The rings S/p^n are coherent for all $n \in \mathbb{N}$.*

We do not know if the ring S itself is coherent.

Proof. We make an induction on n . The starting case $n = 1$: since S is given by p -completely adjoin the divided powers of the Eisenstein polynomial $E(u)$ to \mathfrak{S} , we see that S/p is obtained by adjoining divided powers of $E(u) \equiv u^e$ to $\mathfrak{S}/p = k[[u]]$. It is well-known that the result is $S/p \cong k[[u]/u^{pe} \otimes_k k[[u_1, u_2, \dots]]/(u_i^p)$ where u_i is the image of the p^i -th divided powers of $E(u)$. One checks this explicit algebra is coherent by noting that any finitely generated ideal is generated by polynomials involving only finitely many variables.

Now we do the induction, which largely relies on [BMS18, Lemma 3.26]. Indeed the cited Lemma reduces us to showing the ideal $(p^n)/(p^{n+1})$ in S/p^{n+1} , when viewed as an S/p -module, is finitely presented. But in fact S is p -torsion free, hence the ideal $(p^n)/(p^{n+1})$ is free when viewed as an S/p -module with generator p^n . \square

Lemma 7.16. *Suppose that C^\bullet is a perfect \mathfrak{S}_n -complex. Then there exists an exact sequence of S -modules*

$$0 \longrightarrow \mathrm{H}^i(C^\bullet) \otimes_{\mathfrak{S}} S \longrightarrow \mathrm{H}^i(C^\bullet \otimes_{\mathfrak{S}}^{\mathbb{L}} S) \longrightarrow \mathrm{Tor}_1^{\mathfrak{S}}(\mathrm{H}^{i+1}(C^\bullet), S) \longrightarrow 0.$$

In particular S has Tor-amplitude 1 over \mathfrak{S} and the functor $M \mapsto \mathrm{Tor}_1^{\mathfrak{S}}(M, S)$ is left exact.

Proof. For the first claim, see the argument before the proof of Theorem 5.4 in [CL19] and replace A_{inf} (resp. A_{crys}) there by \mathfrak{S} (resp. S). The fact that S has Tor-amplitude 1 over \mathfrak{S} follows from the Auslander–Buchsbaum formula and torsion-freeness of S . \square

Proposition 7.17. *Let M be a finitely generated Kisin module, then $\mathrm{Tor}_1^{\mathfrak{S}}(M, \varphi_* S)$ is a finitely presented S module.*

Proof. Denote $N := M[u^\infty]$ which is the maximal finite length \mathfrak{S} -submodule inside M .

We first show $\mathrm{Tor}_1^{\mathfrak{S}}(N, \varphi_* S) \rightarrow \mathrm{Tor}_1^{\mathfrak{S}}(M, \varphi_* S)$ is an isomorphism. Since S has Tor-amplitude 1 over \mathfrak{S} by Lemma 7.16, it suffices to show the vanishing of $\mathrm{Tor}_1^{\mathfrak{S}}(M/N, \varphi_* S)$. Note that M/N is an étale Kisin module, we have a sequence

$$0 \rightarrow (M/N)_{\mathrm{tor}} \rightarrow M/N \rightarrow (M/N)_{\mathrm{tf}} \rightarrow 0,$$

where $(M/N)_{\text{tor}}$ is a successive extension of $k[[u]] = \mathfrak{S}/p$ as M/N is étale, and $(M/N)_{\text{tf}}$ is torsion-free. Next observe that both these two structures are preserved under base change along the Frobenius on \mathfrak{S} . Therefore it suffices to show $\text{Tor}_1^{\mathfrak{S}}(-, S) = 0$ whenever the inputting \mathfrak{S} -module is \mathfrak{S}/p or torsion-free. For the former case, it follows from the fact that S has no p -torsion. For the latter, consider the reflexive hull $M'^{\vee\vee}$ of the inputting module $M' \subset M'^{\vee\vee}$, which is finite free as \mathfrak{S} is a regular Noetherian of dimension 2. Finally the desired vanishing of $\text{Tor}_1^{\mathfrak{S}}(M', S)$ follows from the left exactness of Tor_1 against S over \mathfrak{S} : see Lemma 7.16.

It suffices to show $\text{Tor}_1^{\mathfrak{S}}(N', S)$ is finitely presented for any finite length \mathfrak{S} -module, which is the content of the next lemma. \square

Lemma 7.18. *Let N be a finite length \mathfrak{S} -module, then $N \otimes_{\mathfrak{S}} S$ and $\text{Tor}_1^{\mathfrak{S}}(N, S)$ are finitely presented S -modules.*

Proof. When $N = k \cong \mathfrak{S}/(p, u)$, the statement for $k \otimes_{\mathfrak{S}} S = S/(p, u) = (S/p)/u$ and $\text{Tor}_1^{\mathfrak{S}}(k, S) \cong S/p[u]$ follows from the fact that S/p is coherent (Lemma 7.15) and [Sta20, Tag 05CW (3)].

Next we make an induction on the length of N . By considering $N \rightarrow N/(p, u) \simeq k^{\oplus r}$, we have a short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow k \rightarrow 0$, which induces a long exact sequence:

$$0 \rightarrow \text{Tor}_1^{\mathfrak{S}}(N', S) \rightarrow \text{Tor}_1^{\mathfrak{S}}(N, S) \rightarrow \text{Tor}_1^{\mathfrak{S}}(k, S) \rightarrow N' \otimes_{\mathfrak{S}} S \rightarrow N \otimes_{\mathfrak{S}} S \rightarrow k \otimes_{\mathfrak{S}} S \rightarrow 0.$$

Induction hypotheses imply that all terms except of $N \otimes_{\mathfrak{S}} S$ and $\text{Tor}_1^{\mathfrak{S}}(N, S)$ are finitely presented S -modules. Note that the finite length assumption implies all modules are S/p^N -modules for some sufficiently large N . The coherence of S/p^N (Lemma 7.15) and [Sta20, Tag 05CW (3)] shows the boundary map $\text{Tor}_1^{\mathfrak{S}}(k, S) \rightarrow N' \otimes_{\mathfrak{S}} S$ has finitely presented kernel and cokernel. Now we use [Sta20, Tag 0519] to finish the proof. \square

Proposition 7.19. *Let \mathcal{X} be a smooth proper p -adic formal scheme over $\text{Spf}(\mathcal{O}_K)$. The S/p^n -module $\text{H}_{\text{crys}}^i(\mathcal{X}_n/S_n)$ is finitely presented, for any integer i and any $n \in \mathbb{N} \cup \{\infty\}$.*

Let us stress again that this already follows from [BS19, Theorem 5.2].

Proof. The case of finite n follows from Lemma 7.15: the prismatic cohomology complex is a perfect complex, hence the comparison [BS19, Theorem 5.2] or Theorem 3.5 shows the crystalline cohomology complex is also perfect over the coherent ring S/p^n . Therefore all of its cohomology modules are finitely presented as S/p^n -modules.

Now we turn to the case $n = \infty$. By Lemma 7.16 there is a short exact sequence:

$$0 \rightarrow \text{H}_{\Delta}^i(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S}, \varphi} S \rightarrow \text{H}_{\text{crys}}^i(\mathcal{X}/S) \rightarrow \text{Tor}_1^{\mathfrak{S}}(\text{H}_{\Delta}^{i+1}(\mathcal{X}/\mathfrak{S}), \varphi_* S) \rightarrow 0.$$

Since prismatic cohomology complex is a perfect complex and the ring \mathfrak{S} is Noetherian, we know the term $\text{H}_{\Delta}^i(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S}, \varphi} S$ is finitely presented. Using [Sta20, Tag 0519] we are reduced to showing the term $\text{Tor}_1^{\mathfrak{S}}(\text{H}_{\Delta}^{i+1}(\mathcal{X}/\mathfrak{S}), \varphi_* S)$ is finitely presented. This in turn follows from Proposition 7.17 and the fact that $\text{H}_{\Delta}^{i+1}(\mathcal{X}/\mathfrak{S})$ is a Kisin module: see Corollary 7.4. \square

Now we turn to the main result of our paper, which concerns the Breuil-module structure of the crystalline cohomology. Write $\mathfrak{M}_n^j := \text{H}_{\Delta}^j(X_n/\mathfrak{S}_n)$ and $\mathcal{M}_n^j := \text{H}_{\text{crys}}^j(X_n/S_n)$.

Lemma 7.20. *The following sequence*

$$0 \rightarrow \mathfrak{M}_n^i/u\mathfrak{M}_n^i \rightarrow \mathcal{M}_n^i/I^+S \rightarrow \text{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_* S)/I^+S \rightarrow 0$$

is exact.

Proof. By derived mod p^n version of Theorem 3.11, we have $S \otimes_{\varphi, \mathfrak{S}}^{\mathbb{L}} \text{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) \simeq \text{R}\Gamma_{\text{crys}}(X_n/S_n)$. So Lemma 7.16 yields an exact sequence

$$(7.21) \quad 0 \longrightarrow S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_n^i \longrightarrow \mathcal{M}_n^i \longrightarrow \text{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_* S) \longrightarrow 0$$

as φ on \mathfrak{S} is finite flat.

We only need to show the above exact sequence remains left exact after modulo I^+S . To see this, note that $\text{R}\Gamma_{\text{crys}}(X_k/W_n(k)) \simeq \text{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) \otimes_{\mathfrak{S}}^{\mathbb{L}} W(k) \simeq \text{R}\Gamma_{\text{crys}}(X_n/S_n) \otimes_S^{\mathbb{L}} W(k)$, where in the last identification we

use the fact that Frobenius on $W(k)$ is an isomorphism. Using the exact sequence (7.7), then we have the following commutative diagram

$$\begin{array}{ccccccc} S/I^+S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_n^i & \longrightarrow & \mathcal{M}_n^i/I^+S & \longrightarrow & \mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_* S_n)/I^+S & \longrightarrow & 0 \\ \downarrow \wr & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{M}_n^i/u\mathfrak{M}_n^i & \longrightarrow & \mathrm{H}^i(X_k/W_n(k)) & \longrightarrow & \mathfrak{M}_n^{i+1}[u] \longrightarrow 0. \end{array}$$

Since the left column is an isomorphism, we conclude that the top row is left exact as desired. \square

Recall in Definition 6.23, $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ is defined to be a Breuil module if the quadruple

$$\left(\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n), \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[i]}), \varphi_i, \nabla \right)$$

constructed in §5.2 is an object of $\mathrm{Mod}_{S, \mathrm{tor}}^{\varphi, i, \nabla}$. This condition is equivalent to the triple

$$\left(\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n), \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[i]}), \varphi_i \right)$$

being an object of $\mathrm{Mod}_{S, \mathrm{tor}}^{\varphi, i}$.

Theorem 7.22. *Let $n \in \mathbb{N}$ and assume $i \leq p-2$. Then $\mathrm{H}_{\Delta}^j(X_n/\mathfrak{S}_n)$ has no u -torsion for $j = i, i+1$ if and only if $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ is an Breuil module. When that happens we have $\underline{\mathcal{M}}(\mathrm{H}_{\Delta}^i(X_n/\mathfrak{S}_n)) \simeq \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ inside $\mathrm{Mod}_{S, \mathrm{tor}}^{\varphi, i}$.*

Proof. Write $\mathfrak{M}_n^j := \mathrm{H}_{\Delta}^j(X_n/\mathfrak{S}_n)$. Suppose that it has no u -torsion for $j = i, i+1$. So \mathfrak{M}_n^i is an étale Kisin module of height i by Proposition 7.2. By the discussion of §6.3, we know $\mathcal{M}_n^i := \underline{\mathcal{M}}(\mathfrak{M}_n^i)$ is an object of $\mathrm{Mod}_{S, \mathrm{tor}}^{\varphi, i}$. By derived mod p^n version of Theorem 3.11, we have $S \otimes_{\varphi, \mathfrak{S}}^{\mathbb{L}} \mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_n/S_n)$. So Lemma 7.16 yields

$$(7.23) \quad 0 \longrightarrow S \otimes_{\varphi, \mathfrak{S}} \mathrm{H}_{\Delta}^i(X_n/\mathfrak{S}_n) \longrightarrow \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \longrightarrow \mathrm{Tor}_1^{\mathfrak{S}}(\mathrm{H}_{\Delta}^{i+1}(X_n/\mathfrak{S}_n), \varphi_* S_n) \longrightarrow 0.$$

Our assumption that \mathfrak{M}_n^{i+1} has no u -torsion gives an isomorphism $\iota : S \otimes_{\varphi, \mathfrak{S}} \mathrm{H}_{\Delta}^j(X_n/\mathfrak{S}_n) \simeq \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$.

Now we claim that ι induces a natural map $\iota^i : \mathrm{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i) \rightarrow \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[i]})$ and both the source and target are natural submodules inside $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$. In particular, the ι^i is an injection. To see this, we note that ι is induced by natural map $\varphi^* \mathfrak{M}_n^i \rightarrow \mathcal{M}_n^i$ which we still denote by ι . By Theorem 4.21, we have the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}_{\mathrm{qSyn}}^{i-1}(X_n, \Delta_{-/\mathfrak{S}}^{(1)} / \mathrm{Fil}_{\mathbb{N}}^i \Delta_{-/\mathfrak{S}}^{(1)}) & \xrightarrow{\alpha} & \mathrm{H}_{\mathrm{qSyn}}^i(X_n, \mathrm{Fil}_{\mathbb{N}}^i \Delta_{-/\mathfrak{S}}^{(1)}) & \xrightarrow{\beta} & \mathrm{H}_{\mathrm{qSyn}}^i(X_n, \Delta_{-/\mathfrak{S}}^{(1)}) & \longrightarrow & \dots \\ \downarrow \wr & & \downarrow & & \downarrow \iota & & \\ \mathrm{H}_{\mathrm{qSyn}}^{i-1}(X_n, \mathrm{dR}_{-/\mathfrak{S}}^{\wedge} / \mathrm{Fil}_{\mathbb{H}}^i \mathrm{dR}_{-/\mathfrak{S}}^{\wedge}) & \xrightarrow{\alpha'} & \mathrm{H}_{\mathrm{qSyn}}^i(X_n, \mathrm{Fil}_{\mathbb{H}}^i \mathrm{dR}_{-/\mathfrak{S}}^{\wedge}) & \xrightarrow{\beta'} & \mathrm{H}_{\mathrm{qSyn}}^i(X_n, \mathrm{dR}_{-/\mathfrak{S}}^{\wedge}) & \longrightarrow & \dots \end{array}$$

with both rows being exact. By Theorem 4.21 (4), the left column is an isomorphism. As \mathfrak{M}_n^i is assumed to have no u -torsion, Corollary 7.11 shows that β is an injection. Thus α and hence α' are zero maps. So β' is an injection. Therefore, Theorem 4.1 gives the following commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}_{\mathrm{BK}}^i \varphi^* \mathfrak{M}_n^i & \hookrightarrow & \varphi^* \mathfrak{M}_n^i \\ \downarrow & & \downarrow \iota \\ \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[i]}) & \hookrightarrow & \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n). \end{array}$$

Since $\iota : \underline{\mathcal{M}}(\mathfrak{M}_n^i) \xrightarrow{\cong} \mathcal{M}_n^i = \mathbf{H}_{\text{crys}}^i(X_n/S_n)$ is an isomorphism and $\text{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i)$ is the S -submodule of \mathcal{M}_n^i generated by the image of $\text{Fil}^i \varphi^* \mathfrak{M}_n^i$ and $\text{Fil}^i S \cdot \mathcal{M}_n^i$, we see $\text{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i) \subset \mathbf{H}_{\text{crys}}^i(X_n/S_n, \mathcal{I}_{\text{crys}}^{[i]})$ via ι . This shows ι induces an injection $\iota : \text{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i) \hookrightarrow \mathbf{H}_{\text{crys}}^i(X_n/S_n, \mathcal{I}_{\text{crys}}^{[i]})$.

Next we claim that ι^i is an isomorphism. After faithfully flat base changing along $S_n \rightarrow A_{\text{crys},n} := A_{\text{crys}}/p^n$, we are now working with $\mathcal{X} := X_{\mathcal{O}_{\mathbb{C}}}$. Now we need some facts about the sheaf $\mathbb{Z}_p(h)$ on $\mathcal{X}_{\text{qSyn}}$ defined in [BMS19, §7.4]. First according to [BMS19, Theorem 10.1], we have $\mathbb{Z}/p^n \mathbb{Z}(h) \simeq \tau^{\leq h} \mathbf{R}\psi_*(\mathbb{Z}/p^n \mathbb{Z}(h))$ where $\psi : (\mathcal{X}_{\mathbb{C}})_{\text{ét}} \rightarrow \mathcal{X}_{\text{ét}}$ is the natural map of étale sites. By [AMMN20, Theorem F], when $h \leq p-2$, we have

$$\mathbb{Z}_p(h) \simeq \text{fib}(\varphi_h - 1 : \text{Fil}_{\mathbb{H}}^h \mathbf{dR}_{-/ \mathbb{Z}_p}^{\wedge} \rightarrow \mathbf{dR}_{-/ \mathbb{Z}_p}^{\wedge}).$$

Now Proposition 5.4 implies

$$\mathbb{Z}_p(h) \simeq \text{fib}(\varphi_h - 1 : \text{Fil}_{\mathbb{H}}^h \mathbf{dR}_{-/ A_{\text{inf}}}^{\wedge} \rightarrow \mathbf{dR}_{-/ A_{\text{inf}}}^{\wedge}).$$

Since fib commutes with derived mod p^n , we may apply $\otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n \mathbb{Z}$ to this equation. Finally by Theorem 4.1, we get the following exact sequence for $i \leq h \leq p-2$:

$$(7.24) \quad \cdots \mathbf{H}_{\text{crys}}^{i-1}(\mathcal{X}_n/A_{\text{crys},n}) \rightarrow \mathbf{H}_{\text{ét}}^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}/p^n \mathbb{Z}(h)) \rightarrow \mathbf{H}_{\text{crys}}^i(\mathcal{X}_n/A_{\text{crys},n}, \mathcal{I}_{\text{crys}}^{[h]}) \xrightarrow{\varphi_h^{-1}} \mathbf{H}_{\text{crys}}^i(\mathcal{X}_n/A_{\text{crys},n})$$

By Equation (7.24) and Proposition 6.12, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_S(\underline{\mathcal{M}}(\mathfrak{M}_n^i)) & \longrightarrow & A_{\text{crys}} \otimes_S \text{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i) & \longrightarrow & A_{\text{crys}} \otimes_S \underline{\mathcal{M}}(\mathfrak{M}_n^i) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow 1 \otimes \iota^i & & \downarrow \iota \\ \mathbf{H}_{\text{ét}}^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}/p^n \mathbb{Z})(i) & \xrightarrow{s} & \mathbf{H}_{\text{crys}}^i(\mathcal{X}_n/A_{\text{crys},n}, \mathcal{I}_{\text{crys}}^{[i]}) & \longrightarrow & \mathbf{H}_{\text{crys}}^i(\mathcal{X}_n/A_{\text{crys},n}) & & \end{array}$$

with both rows being exact. Since $1 \otimes \iota^i$ is an injection, we see that the map α is also injective. Then α must be an isomorphism because $T_S(\underline{\mathcal{M}}(\mathfrak{M}_n^i)) \simeq T_{\mathfrak{S}}(\mathfrak{M}_n^i)(i) \simeq \mathbf{H}_{\text{ét}}^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}/p^n \mathbb{Z})(i)$ due to Proposition 6.12 and [BS19, Theorem 1.8.(4)]. Therefore s is also injective. Now by the snake lemma, we see that $\text{coker}(1 \otimes \iota) = 0$ as required.

Conversely, assume that $\mathcal{M}_n^i := \mathbf{H}_{\text{crys}}^i(X_n/S_n)$ is an object in $\text{Mod}_{S, \text{tor}}^{\varphi, i}$ with $\text{Fil}^i \mathcal{M}_n^i = \mathbf{H}_{\text{crys}}^i(X_n/S_n, \mathcal{I}_{\text{crys}}^{[i]})$. As before, we consider the base change $\mathcal{X} := X_{\mathcal{O}_{\mathbb{C}}}$ and we still have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_S(\mathcal{M}_n^i) & \longrightarrow & A_{\text{crys}} \otimes_S \text{Fil}^i \mathcal{M}_n^i & \longrightarrow & A_{\text{crys}} \otimes_S \mathcal{M}_n^i \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \iota & & \downarrow \iota \\ \mathbf{H}_{\text{ét}}^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}/p^n \mathbb{Z})(i) & \xrightarrow{s} & \mathbf{H}_{\text{crys}}^i(\mathcal{X}_n/A_{\text{crys},n}, \mathcal{I}_{\text{crys}}^{[i]}) & \longrightarrow & \mathbf{H}_{\text{crys}}^i(\mathcal{X}_n/A_{\text{crys},n}). & & \end{array}$$

The difference here is that the middle column is now an isomorphism, whereas the first column α is not known to be an isomorphism.

First it is easy to see that α is an injection by chasing the diagram. Now by Corollary 6.15, we have an inequality

$$\text{length}_{W(k)}(\mathcal{M}_n^i/I^+S) = \text{length}_{\mathbb{Z}}(T_S(\mathcal{M}_n^i)) \leq \text{length}_{\mathbb{Z}}(\mathbf{H}_{\text{ét}}^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}/p^n \mathbb{Z})).$$

On the other hand, by the proof of Lemma 7.6 and Lemma 7.20, we see that

$$\text{length}_{\mathbb{Z}}(\mathbf{H}_{\text{ét}}^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}/p^n \mathbb{Z})) \leq \text{length}_{W(k)}(\mathfrak{M}_n^i/u\mathfrak{M}_n^i) \leq \text{length}_{W(k)}(\mathcal{M}_n^i/I^+S).$$

Combining the above two inequalities, we see

$$\text{length}_{\mathbb{Z}}(\mathbf{H}_{\text{ét}}^i(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}/p^n \mathbb{Z})) = \text{length}_{W(k)}(\mathfrak{M}_n^i/u\mathfrak{M}_n^i) = \text{length}_{W(k)}(\mathcal{M}_n^i/I^+S).$$

Now the proof of Lemma 7.6 implies that \mathfrak{M}_n^i has no u -torsion. By the length equality, the injection $\mathfrak{M}_n^i/u\mathfrak{M}_n^i \hookrightarrow \mathcal{M}_n^i/I^+S$ in Lemma 7.20 is in fact an isomorphism. and hence $\text{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_*S)/I^+S = 0$. It is easy to see that $\text{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_*S)$ is a finitely generated S module, applying Nakayama's lemma yields $\text{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_*S) = 0$. Therefore \mathfrak{M}_n^{i+1} has no u -torsion by the following claim.

Claim: If \mathfrak{M} is a p^n -torsion \mathfrak{S} -module and $\text{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, \varphi_*S) = 0$ then \mathfrak{M} has no u -torsion. To prove this, we first note that $\text{Tor}_1^{\mathfrak{S}}(-, \varphi_*S)$ is a left exact functor by Lemma 7.16. Secondly note that \mathfrak{M} has no u -torsion

if and only if it has no (u, p) -torsion. Let $\mathfrak{M}' \subset \mathfrak{M}$ be the submodule of (u, p) -torsions in \mathfrak{M} . The above discussion implies that $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}', \varphi_* S) = 0$. Now by definition, we have $\mathfrak{M}' \cong \bigoplus_{\Lambda} k$ as an \mathfrak{S} -module, where Λ is an indexing set. One computes directly that

$$\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}', \varphi_* S) = \bigoplus_{\Lambda} \mathrm{Tor}_{\mathfrak{S}}^1(\mathfrak{S}/(p, u), \varphi_* S) = \bigoplus_{\Lambda} \mathrm{Tor}_{\mathfrak{S}}^1(\mathfrak{S}/(p, u^p), S) = \bigoplus_{\Lambda} \ker(S/p \xrightarrow{\cdot u^p} S/p).$$

Since $\ker(S/p \xrightarrow{\cdot u^p} S/p)$ is nonzero ($u^{pe} = 0$ in S/p), the above computation implies $\Lambda = \emptyset$, as claimed. \square

Corollary 7.25. *If $ei < p - 1$ then $H_{\Delta}^j(X_n/\mathfrak{S}_n)$ has no u -torsion for $j = i, i + 1$, and $H_{\mathrm{crys}}^i(X_n/S_n)$ is a Breuil module.*

Proof. By Lemma 6.5 and Proposition 7.2, we know that $H_{\Delta}^i(X_n/\mathfrak{S}_n)$ has no u -torsion. To show that $H_{\Delta}^{i+1}(X_n/\mathfrak{S}_n)$ has no u -torsion, we first consider the case that $n = 1$. The main theorem of [Car08] shows that $H_{\mathrm{crys}}^i(X_1/S_1)$ is a Breuil module when $n = 1$ and $ei < p - 1$. Then Theorem 7.22 shows that $H_{\Delta}^{i+1}(X_1/\mathfrak{S}_1)$ has no u -torsion.

Let us prove by induction that $\mathfrak{M}_n^{i+1} := H_{\Delta}^{i+1}(X_n/\mathfrak{S}_n)$ has no u -torsion. We use the long exact sequence relating various $\mathfrak{M}_n^{i+1} := H_{\Delta}^{i+1}(X_n/\mathfrak{S}_n)$:

$$\cdots \rightarrow \mathfrak{M}_{n-1}^i \xrightarrow{f} \mathfrak{M}_1^{i+1} \rightarrow \mathfrak{M}_n^{i+1} \rightarrow \mathfrak{M}_{n-1}^{i+1} \cdots$$

By induction, we may assume that \mathfrak{M}_{n-1}^{i+1} has no u -torsion. It suffices to prove that $\mathfrak{M}_1^{i+1}/f(\mathfrak{M}_{n-1}^i)$ has no u -torsion. To that end, write $\mathfrak{N} := f(\mathfrak{M}_{n-1}^i)$ which has height i , $\mathfrak{M} := \mathfrak{M}_1^{i+1}$ which has height $i + 1$ and $\mathfrak{L} := \mathfrak{M}_1^{i+1}/\mathfrak{N}$. By construction we have the following exact sequence

$$0 \rightarrow \mathfrak{N} \xrightarrow{f} \mathfrak{M} \xrightarrow{g} \mathfrak{L} \rightarrow 0.$$

Let $\mathfrak{M}' = g^{-1}(\mathfrak{L}[u^\infty])$. Then we obtain two exact sequences

$$0 \rightarrow \mathfrak{N} \rightarrow \mathfrak{M}' \rightarrow \mathfrak{L}[u^\infty] \rightarrow 0 \text{ and } 0 \rightarrow \mathfrak{M}' \rightarrow \mathfrak{M} \rightarrow \mathfrak{L}/\mathfrak{L}[u^\infty] \rightarrow 0.$$

The second sequence has all terms being étale Kisin modules. Since \mathfrak{M} has height $i + 1$, we conclude that both \mathfrak{M}' and $\mathfrak{L}/\mathfrak{L}[u^\infty]$ has height $i + 1$. Since both \mathfrak{N} and \mathfrak{M}' are étale and hence finite free over $k[[u]]$. This allows us to choose a basis e_1, \dots, e_d of \mathfrak{N} and a basis e'_1, \dots, e'_d of \mathfrak{M}' so that $(e_1, \dots, e_d) = (e'_1, \dots, e'_d)\Lambda$, where $\Lambda = \mathrm{diag}(u^{a_1}, \dots, u^{a_d})$ a diagonal matrix with u^{a_j} on the main diagonal such that $a_1 \leq a_2 \leq \dots \leq a_d$. Let A and A' be the matrices of Frobenius for the corresponding basis. We easily see that following relation

$$\Lambda A = A' \varphi(\Lambda).$$

Hence the last column of $A' \varphi(\Lambda)$ is divisible by u^{pa_d} . Consequently, the last column of A is divisible by $u^{(p-1)a_d}$. But \mathfrak{N} has height i , which means there exists a matrix B with entries in $k[[u]]$ so that $AB = BA = u^{ei} I_d$. But this is impossible as $ei < p - 1$, unless $a_d = 0$. This forces that $\Lambda = I_d$ and hence a posteriori \mathfrak{L} has no u -torsion as desired. \square

Remark 7.26. Let T be the largest integer satisfying $T \cdot e < p - 1$, and let $n \in \mathbb{N}$. It is a result of Min [Min20, Lemma 5.1] that $H_{\Delta}^i(X/\mathfrak{S})$ has no u -torsion when $0 \leq i \leq T + 1$. By a similar argument, one can also show that $H_{\Delta}^i(X_n/\mathfrak{S}_n)$ has no u -torsion for $0 \leq i \leq T$. The slight improvement along this direction in Corollary 7.25 is the statement that $H_{\Delta}^{T+1}(X_n/\mathfrak{S}_n)$ is also u -torsion free. This would imply Min's result. As far as we can tell, Min's strategy does not give u -torsion freeness of $H_{\Delta}^{T+1}(X_n/\mathfrak{S}_n)$.

Proposition 7.27. *Let $i \leq p - 2$ be an integer. Suppose that $H_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[i]}) \rightarrow H_{\mathrm{crys}}^i(X_n/S_n) =: \mathcal{M}_n^i$ is injective, and denote its image as $\mathrm{Fil}^i \mathcal{M}_n^i$. Assume furthermore that \mathcal{M}_n^i together with*

$$(\mathrm{Fil}^i \mathcal{M}_n^i = H_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[i]}), \varphi_i, \nabla)$$

is an object of $\mathrm{Mod}_{S_{\mathrm{tor}}}^{\varphi_i, \nabla}$. Then $T_S(\mathcal{M}_n^i) \simeq H_{\mathrm{et}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n \mathbb{Z})(i)$ as G_K -representations.

Proof. Theorem 7.22 together with Proposition 6.12 have already shown the following isomorphisms

$$T_S(\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)) \xrightarrow{\iota_1} T_{\mathfrak{S}}(\mathrm{H}_{\Delta}^i(X_n/\mathfrak{S}_n))(i) \xrightarrow{\iota_2} \mathrm{H}_{\mathrm{et}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i).$$

The main point here is to check these two isomorphisms ι_1, ι_2 here are compatible with G_K -actions. Let $\mathcal{X} := X_{\mathcal{O}_{\mathbb{C}}}$.

First we have $A_{\mathrm{inf}} \otimes_{\mathfrak{S}} \mathfrak{M}_n^i \simeq \mathrm{H}_{\Delta}^i(\mathcal{X}_n/A_{\mathrm{inf},n})$ which admits natural G_K -action. Since A_{inf} is a perfect prism, [BS19, Theorem 1.8.(4)] proves that $T_{\mathfrak{S}}(\mathfrak{M}_n^i) = (\mathrm{H}_{\Delta}^i(\mathcal{X}_n/A_{\mathrm{inf},n}))^{\varphi=1} \simeq \mathrm{H}_{\mathrm{et}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ is compatible with G_K -action, as explained in Remark 7.5. This concludes that ι_2 is compatible with G_K -actions.

Now Theorem 3.11 shows that the comparison isomorphism

$$\bar{\iota} : \mathrm{H}_{\Delta}^i(\mathcal{X}_n/A_{\mathrm{inf},n}) \otimes_{A_{\mathrm{inf},\varphi}} A_{\mathrm{crys}} \simeq \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/A_{\mathrm{crys},n})$$

is functorial. So $\bar{\iota}$ is compatible with natural G_K -actions on the both sides. Also $\bar{\iota}$ is compatible with the isomorphism $\iota : \underline{\mathcal{M}}(\mathfrak{M}_n^i) \simeq \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$. Applying Remark 6.14 here then concludes that

$$\iota_1 : T_S(\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)) \simeq T_{\mathfrak{S}}(\mathrm{H}_{\Delta}^i(X_n/\mathfrak{S}_n))(i)$$

is compatible with G_K -actions if we define the G_K -action on $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \otimes_S A_{\mathrm{crys}}$ via the identification $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \otimes_S A_{\mathrm{crys}} = \mathrm{H}_{\mathrm{crys}}^i(\mathcal{X}_n/A_{\mathrm{crys},n})$. Recall the G_K -action on $\mathrm{H}_{\mathrm{crys}}^i(X_n/S_n) \otimes_S A_{\mathrm{crys}}$ is defined via Formula (6.19), and we have showed in §5.3 that these two G_K -actions are the same. This proves that ι_1 is also compatible with G_K -actions. \square

In the end of this subsection, we explain how our results are related to Fontaine–Messing theory in [FM87] (see also [Kat87]) for a proper smooth *formal* scheme X over $W(k)$. For any $n \geq 1$, the scheme X_n is smooth proper over $\mathrm{Spec}(W_n(k))$. So when $0 \leq j \leq i \leq p-1$, the triple $M^i := (\mathrm{H}_{\mathrm{crys}}^i(X_n/W_n(k)), \mathrm{H}_{\mathrm{crys}}^i(X_n/W_n(k), \mathcal{I}_{\mathrm{crys}}^{[j]}), \varphi_j)$ is known to be a Fontaine–Laffaille data.

Now let $i \leq p-2$ and one wants to show that $T_{\mathrm{crys}}(M^i) \simeq \mathrm{H}_{\mathrm{et}}^i(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i)$. We recall the construction of $T_{\mathrm{crys}}(M^i)$ in the following: write $\mathrm{Fil}^j M^i := \mathrm{H}_{\mathrm{crys}}^i(X_n/W_n(k), \mathcal{I}_{\mathrm{crys}}^{[j]})$ and let

$$\mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_{W(k)} M^i) = \sum_{j=0}^i \mathrm{Fil}^j A_{\mathrm{crys}} \otimes_{W(k)} \mathrm{Fil}^{i-j} M^i \subset A_{\mathrm{crys}} \otimes_{W(k)} M^i.$$

Then one can define $\varphi_i : \mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_{W(k)} M^i) \rightarrow A_{\mathrm{crys}} \otimes_{W(k)} M^i$ by $\varphi_i := \sum_{j=0}^i \varphi_j|_{\mathrm{Fil}^j A_{\mathrm{crys}} \otimes \varphi_{i-j}|_{\mathrm{Fil}^{i-j} M^i}}$.

Now $T_{\mathrm{crys}}(M^i) := (\mathrm{Fil}^i(A_{\mathrm{crys}} \otimes M^i))^{\varphi_i=1}$.

Let $\mathcal{M}^i := \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n)$ which is an object of $\mathrm{Mod}_{S, \mathrm{tor}}^{\varphi_i, \nabla}$ by Corollary 7.25. It is clear that the base change map $\iota : S \otimes_{W(k)} M^i \rightarrow \mathcal{M}^i$ is an isomorphism as $W(k) \rightarrow S$ is flat. Define $\mathrm{Fil}^i(S \otimes_{W(k)} M^i) := \sum_{j=0}^i \mathrm{Fil}^j S \otimes_{W(k)} \mathrm{Fil}^{i-j} M^i \subset S \otimes_{W(k)} M^i$. Since $\mathrm{Fil}^j M^i$ is direct summand of $\mathrm{Fil}^{j-1} M^i$, the natural map $\mathrm{Fil}^i(S \otimes_{W(k)} M^i) \rightarrow \mathrm{H}_{\mathrm{crys}}^i(X_n/S_n, \mathcal{I}_{\mathrm{crys}}^{[i]})$ induced by ι is injective. Therefore, we obtain the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_{\mathrm{crys}}(M^i) & \longrightarrow & \mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_{W(k)} M^i) & \xrightarrow{\varphi_i-1} & A_{\mathrm{crys}} \otimes_{W(k)} M^i \\ & & \downarrow & & \downarrow & & \downarrow \iota \\ 0 & \longrightarrow & T_S(\mathcal{M}^i) & \longrightarrow & \mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_S \mathcal{M}^i) & \xrightarrow{\varphi_i-1} & A_{\mathrm{crys}} \otimes_S \mathcal{M}^i. \end{array}$$

It is well-known from Fontaine–Laffaille theory that $\mathrm{length}_{\mathbb{Z}} T_{\mathrm{crys}}(M^i) = \mathrm{length}_{W(k)} M^i$. By Corollary 6.15, we know $\mathrm{length}_{\mathbb{Z}}(T_S(\mathcal{M}^i)) = \mathrm{length}_{W(k)}(\mathcal{M}^i/I^+S) = \mathrm{length}_{W(k)} M^i$. Therefore, the left column must be bijective. By Proposition 7.27, it remains to check that the isomorphism $T_{\mathrm{crys}}(M^i) \rightarrow T_S(\mathcal{M}^i)$ is compatible with G_K -actions. Since the G_K -action on $T_S(\mathcal{M}^i)$ is the G_K -action on $A_{\mathrm{crys}} \otimes_S \mathcal{M}^i$ via (6.19), it suffices to show that $M^i \subset (\mathcal{M}^i)^{\nabla=0}$, which follows from Proposition 5.4 (1).

Corollary 7.28. *Fontaine-Messing theory in [FM87] and [Kat87] accommodates X being proper smooth formal scheme over $W(k)$.*

8. SOME CALCULATIONS ON T_S

8.1. **Identification on (6.20) and (6.19).** In this section, we show that (6.20) and (6.19) are the same.

Lemma 8.1. *If we write $N^n = \sum_{i=1}^n A_{i,n} u^i \nabla^i$ then $A_{i,n+1} = A_{i-1,n} + iA_{i,n}$ and $A_{1,n} = A_{n,n} = 1$*

Proof. An easy induction on n by $N = u\nabla$. □

Recall that $\gamma_i(t)$ denote the i -divided power of t .

Lemma 8.2. $\sum_{n \geq i} A_{i,n} \gamma_n(t) = \gamma_i(e^t - 1)$.

Proof. It suffices to show that Taylor's expansion as functions of t on both sides are equal. This is clear to see that the coefficients of t^n , which is first nonzero term, coincide on the both sides. If we write $\gamma_i(e^t - 1) = \sum_{n \geq i} B_{i,n} \gamma_n(t)$ then it suffices to show that $B_{i,n}$ satisfies the recursive formula: $B_{i,n+1} = B_{i-1,n} + iB_{i,n}$ for $n \geq i$. Note that

$$\gamma_i(e^t - 1) = \frac{1}{i!} \left(\sum_{m=0}^i \binom{i}{m} (-1)^{i-m} e^{mt} \right).$$

Therefore, $B_{i,n} = \frac{1}{i!} \left(\sum_{m=0}^i \binom{i}{m} (-1)^{i-m} m^n \right)$. So to check that $B_{i-1,n} + iB_{i,n} = B_{i,n+1}$ is equivalent to check that

$$\frac{1}{(i-1)!} \left(\sum_{m=0}^{i-1} \binom{i-1}{m} (-1)^{i-1-m} m^n + \sum_{m=0}^i \binom{i}{m} (-1)^{i-m} m^n \right) = \frac{1}{i!} \sum_{m=0}^i \binom{i}{m} (-1)^{i-m} m^{n+1}.$$

This follows that $i \left(\binom{i}{m} - \binom{i-1}{m} \right) = \binom{i}{m} m$. □

Now by the above Lemmas,

$$\sum_{n=0}^{\infty} N^n(x) \gamma_n(\log(\underline{\epsilon}(\sigma))) = \sum_{n=0}^{\infty} \nabla^n(x) u^n \gamma_n(e^{\log(\underline{\epsilon}(\sigma))} - 1) = \sum_{n=0}^{\infty} \nabla^n(x) \gamma_n(u(\underline{\epsilon}(\sigma) - 1)) = \sum_{n=0}^{\infty} \nabla^n(x) \gamma_n(\sigma(u) - u).$$

This proves that (6.20) and (6.19) are the same.

8.2. T_S and $T_{\text{st},\star}$. In this subsection, we explain our functor T_S and functor $T_{\text{st},\star}$ used in [Car08] are the same. For this purpose, we have to review period ring \widehat{A}_{st} from [Car08]. Let $\widehat{A}_{\text{st}} = A_{\text{crys}}\langle X \rangle$ be the p -adic completion of PD algebra of A_{crys} . We extend Frobenius φ and filtration of A_{crys} to \widehat{A}_{st} as follows: Let $\varphi(X) = (1 + X)^p - 1$ and

$$\text{Fil}^i \widehat{A}_{\text{st}} := \left\{ \sum_{j=0}^{\infty} a_j \gamma_j(X), a_j \in \text{Fil}^{\max\{i-j, 0\}} A_{\text{crys}}, \lim_{j \rightarrow \infty} a_j \rightarrow 0 \text{ } p\text{-adically} \right\}.$$

It is easy to see that we can define $\varphi_r : \text{Fil}^r \widehat{A}_{\text{st}} \rightarrow \widehat{A}_{\text{st}}$ similar to that of A_{crys} . To extend G_K -action to \widehat{A}_{st} , for any $g \in G_K$, recall that $\underline{\epsilon}(g) = \frac{g([\pi])}{[\pi]} \in A_{\text{inf}}$ defined before (6.20). Set $g(X) = \underline{\epsilon}(g)X + \underline{\epsilon}(g) - 1$. Finally define an A_{crys} -linear monodromy by set $N(X) = -(1 + X)$. We embed S inside \widehat{A}_{st} via $u \mapsto [\pi](1 + X)^{-1}$. At this point, we have two embeddings $S \rightarrow \widehat{A}_{\text{st}}$: $\iota_1 : S \hookrightarrow A_{\text{crys}} \subset \widehat{A}_{\text{st}}$ via $u \mapsto [\pi] \in A_{\text{inf}}$ and $\iota_2 : S \hookrightarrow \widehat{A}_{\text{st}}$ via $u \mapsto [\pi](1 + X)^{-1}$. We will use both embeddings in the following. Notice that there is an A_{crys} -linear projection $q : \widehat{A}_{\text{st}} \rightarrow A_{\text{crys}}$ by sending $\gamma_i(X) \mapsto 0$. It is easy to check that q is compatible with filtration, Frobenius, G_K -actions, and both embeddings $\iota_i : S \hookrightarrow \widehat{A}_{\text{st}}$. Set $\beta := \log(1 + X) \in \widehat{A}_{\text{st}}$.

Remark 8.3. Breuil-Caruso's theory set $N(1 + X) = 1 + X$. Our setting is different by -1 sign to fit our setting $\nabla(u) = 1$. There is no difference for these two different settings up to changing some signs.

Given a Breuil module $\mathcal{M} \in \text{Mod}_{S, \text{tor}}^{\varphi, N, h}$, we extend filtration, φ_h , monodromy and G_K -actions to $\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ as follows

$$\text{Fil}^h \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M} = \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \text{Fil}^h \mathcal{M} + \text{Fil}^h \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}.$$

For $a \otimes m \in \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \text{Fil}^h \mathcal{M}$, set $\varphi_h(a \otimes m) = \varphi(a) \otimes \varphi_h(m)$, and for $a \otimes m \in \text{Fil}^h \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$, set $\varphi_h(a \otimes m) = \varphi_h(a) \otimes \varphi_h(E^h m)$. It is easy to check these φ_h are compatible with intersection so that φ_h extends to $\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$. We extend G_K -action from \widehat{A}_{st} to $\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ by acting on \mathcal{M} -trivially, and $N(a \otimes m) = N(a) \otimes m + a \otimes N(m), \forall a \in \widehat{A}_{\text{st}}, m \in \mathcal{M}$. Now set

$$T_S(\mathcal{M}) := (\text{Fil}^h(\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}))^{\varphi_h=1, N=0}.$$

Proposition 8.4. *There is an isomorphism $T_S(\mathcal{M}) \simeq T_{\text{st}, \star}(\mathcal{M})$ as G_K -representations.*

Proof. For $m \in \mathcal{M} \subset \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$, set $m^\nabla := \sum_{i=0}^{\infty} N^i(m) \gamma_i(\beta)$ and $\mathcal{M}^\nabla = \{m^\nabla | m \in \mathcal{M}\} \subset \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$. To understand the map $\alpha : \mathcal{M} \rightarrow \mathcal{M}^\nabla$, consider the following diagram induced by $q : \widehat{A}_{\text{st}} \twoheadrightarrow A_{\text{crys}}$:

$$\begin{array}{ccc} & \mathcal{M}^\nabla & \\ \alpha \nearrow & & \searrow j \\ \mathcal{M} & \xrightarrow{\alpha'} & \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M} \\ \parallel \downarrow & & \downarrow q \\ \mathcal{M} & \xrightarrow{\quad} & A_{\text{crys}} \otimes_S \mathcal{M}. \end{array}$$

Where $\alpha' : \mathcal{M}^\nabla \rightarrow q(\mathcal{M}) = \mathcal{M}$ is induced by q . By definition of α , it is easy to show that α and α' is bijective. Also α is an isomorphism of S -modules in the sense of $\alpha(\iota_2(s)m) = \iota_1(s)\alpha(m)$ for $s \in S$ and $m \in \mathcal{M}$. Using that N satisfies Griffiths transversality and diagram(6.17) together with facts that $\beta \in \text{Fil}^1 \widehat{A}_{\text{st}}$ and $\varphi(\beta) = p\beta$, a similar argument in Lemma 6.21 (by replacing a with β) show that for any $m \in \text{Fil}^h \mathcal{M}$ we have $m^\nabla \in \text{Fil}^h(\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M})$ and $\varphi_h(m^\nabla) = (\varphi_h(m))^\nabla$. In summary, $\alpha : \mathcal{M} \rightarrow \mathcal{M}^\nabla$ is an isomorphism in $\text{Mod}_{S, \text{tor}}^{\varphi, h}$ and injections $\mathcal{M} \xrightarrow{\alpha} \mathcal{M}^\nabla \subset \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ are compatible with with filtrations and φ_h .

Now consider the natural map $A_{\text{crys}} \otimes_S \mathcal{M}^\nabla \subset \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ induced by inclusion $j : \mathcal{M}^\nabla \subset \widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ which is still denoted by \tilde{j} . Since $q \circ \tilde{j}$ is an isomorphism (as $A_{\text{crys}} \otimes_S (q \circ \alpha)$ is an isomorphism), we conclude that $A_{\text{crys}} \otimes_S \mathcal{M}^\nabla$ is an A_{crys} -submodule of $\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ which is compatible with filtration and φ_h .

Using that $N(\beta) = -1$, we easily see that $\mathcal{M}^\nabla \subset (\widehat{A}_{\text{st}} \otimes_S \mathcal{M})^{N=0}$. In particular, we have an injection $\tilde{j} : A_{\text{crys}} \otimes_S \mathcal{M}^\nabla \rightarrow (\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M})^{N=0}$ compatible with filtration and φ_h . Therefore \tilde{j} induces an injection

$$T_S(\mathcal{M}) = (\text{Fil}^h(A_{\text{crys}} \otimes_S \mathcal{M}))^{\varphi_h=1} \xrightarrow{\alpha} (\text{Fil}^h(A_{\text{crys}} \otimes_S \mathcal{M}^\nabla))^{\varphi_h=1} \subset (\text{Fil}^h(\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}))^{\varphi_h=1, N=0} = T_{\text{st}, \star}(\mathcal{M}).$$

To see this this injection is an isomorphism, we can dévissage to the case that \mathcal{M} is killed by p because both T_S and $T_{\text{st}, \star}$ are exact functors ([Car08, Corollary 2.3.10]). In this case, it also well-known that $\dim_{\mathbb{F}_p} T_{\text{st}, \star} = \text{rank}_{S_1} \mathcal{M} = \dim_{\mathbb{F}_p} T_S(\mathcal{M})$. This establish the isomorphism $T_S(\mathcal{M}) \simeq T_{\text{st}, \star}(\mathcal{M})$. Finally, we check this isomorphism is compatible with G_K -actions. Note that $T_S(\mathcal{M})$ has G_K -action via (6.20), while $T_{\text{st}, \star}$ has G_K -action from that on $\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ with trivial G_K -action on \mathcal{M} . We have to show that $A_{\text{crys}} \otimes_S \mathcal{M}^\nabla$ has G_K -action as the subspace of $\widehat{A}_{\text{st}} \otimes_{\iota_2, S} \mathcal{M}$ is the same as that defined as (6.20). But this easily follows from the formula that $m^\nabla := \sum_{i=0}^{\infty} N^i(m) \gamma_i(\beta)$ and $g(\beta) = \log(\underline{\varepsilon}(g)) + \beta$. \square

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