

ON TUTTE'S CHROMATIC INVARIANT

SABIN CAUTIS* AND DAVID M. JACKSON†

ABSTRACT. For a simple connected planar graph G with a contractible circuit J and a partition ϕ of the vertex set of J we denote by $P_{(G,\phi)}(t)$ the number of ways of colouring the vertices of G with at most t colours so that vertices in the same block of ϕ have the same colour. Tutte showed that this polynomial may be expressed uniquely as a linear combination of $P_{(G,\pi)}(t)$ over all planar partitions π , with scalars $\vartheta_{\phi,\pi}(t)$ that are independent of G . We show that the (chromatic) invariants $\vartheta_{\phi,\pi}$ have a natural algebraic setting in terms of the orthogonal projection from the Partition Algebra $\mathbb{P}_r(t)$ to the Temperley-Lieb Subalgebra $\text{TL}_r(t, 1)$.

We define the genus of a partition and give an extension of the invariants to arbitrary genus g . We consider a graded filtration of $\mathbb{P}_r(t)$ which serves as a natural setting for partitions of genus at most g . We also introduce a lift of the Partition Algebra, which we call the Ribbon Algebra, an algebra that is worthy of further study.

Finally, we summarise the rôle of the genus 0 invariants in the algebraic approach of Birkhoff and Lewis to the Four Colour Theorem.

CONTENTS

1. Introduction	3
2. Tutte's Invariants	3
2.1. Planar and non-planar partitions of J	3
2.2. Free chromials	3
2.3. Tutte's invariants	4
2.4. Completeness of Tutte's Relations	5
3. The Relation Between Tutte's Invariants, the Temperley-Lieb Algebra and the Partition Algebra	6
3.1. The Temperley-Lieb Algebra	6
3.2. Construction of an orthonormal basis for \mathbb{K}_{2r}	9
3.3. The Partition Algebra	12
4. Higher genus Tutte invariants	13
4.1. The genus of a partition	13
4.2. Generalisation of the chromatic invariant	14
4.3. The higher genus chromatic join matrix	14
5. The Ribbon Algebra $\mathbb{RA}_r(t)$	15
5.1. A natural bijection between rooted hypermonopoles and permutations	15
5.2. Counting Hypermonopoles According to Genus	17
6. The Connexion Between the Tutte Invariants, the Birkhoff-Lewis equations and the Four-Colour Theorem	21
6.1. Constrained chromials	21

Date: 14 October 2005.

Key words and phrases. Chromatic invariant, non-crossing partitions, Temperley-Lieb Algebra, Partition Algebra, Birkhoff-Lewis Equations, hypermonopoles, ribbon graphs.

*Department of Mathematics, Harvard University, Cambridge, MA;
 scautis@math.harvard.edu.

†Department of Combinatorics and Optimization, University of Waterloo, Ontario, Canada;
 dmjackson@math.uwaterloo.ca.

6.2. Examples	22
Acknowledgments	26
References	26

1. INTRODUCTION

The result of Tutte we shall discuss contains objects that we have termed Tutte's (chromatic) invariants. These invariants, which are defined in Section 2, encode relations between (free) chromatic polynomials. Our aim is to provide a richer context within which to study these Tutte invariants. We do this in Section 3 by realizing them as coefficients of the orthogonal projection from the partition algebra \mathbb{P}_r to the Temperley-Lieb algebra $\mathbb{TL}_r \subset \mathbb{P}_r$. This, in Section 4, suggests a higher genus generalization of Tutte invariants. This in turn leads us to the Ribbon Algebra \mathbb{RA}_r which we describe in Section 5. In Section 6 we summarize the relationship between the invariants and the Four Colour Theorem. We hope to convince the reader that the invariants themselves lead to a rich mathematical setting, regardless of their application to the Four Colour Theorem.

2. TUTTE'S INVARIANTS

2.1. Planar and non-planar partitions of J . Let J be a circuit of length r of a graph G . We shall refer to J as an r -ring. A *partition* ϕ of J is a partition of the vertex set of J into subsets called *blocks*. Let $\ell(\pi)$ denote the number of blocks of π . Denote by $\Phi(J)$ the set of all partitions of the vertex set of J . A partition π of J is said to be *planar* if lines in the interior of J can be drawn between each pair of vertices which are part of the same block of π such that they do not meet. We represent each block by a region in the interior of J as illustrated in Figure 1. We denote the set of planar partitions of J by $\Pi(J)$. It is well known that the number of planar partitions of J is the Catalan number $C_r = \frac{1}{r+1} \binom{2r}{r}$. For example, the number of partitions of J_4 is 15 but $|\Pi(J_4)| = C_4 = 14$, so there is a unique non-planar partition in $\Phi(J_4)$, namely $\{[1, 3], [2, 4]\}$.

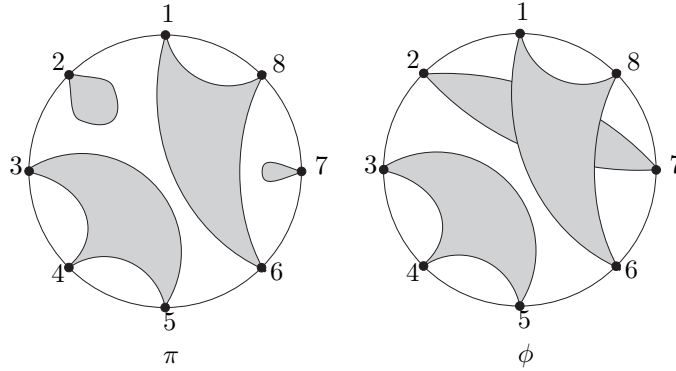


FIGURE 1. A planar partition $\pi = \{[1, 6, 8], [3, 4, 5], [2], [7]\}$ and a non-planar partition $\phi = \{[1, 6, 8], [3, 4, 5], [2, 7]\}$.

Define a partial order \preceq on $\Phi(J)$ by $\phi' \preceq \phi$ if ϕ' is obtained from ϕ by refinement (see *e.g.* [S, §3.1.1]). Clearly, $\ell(\phi') \geq \ell(\phi)$ for all $\phi' \preceq \phi$. The poset $(\Phi(J), \preceq)$ is a *lattice*. For every pair of partitions $\phi, \phi' \in \Phi(J)$, the *join* $\phi \vee \phi'$ is defined by $\phi \vee \phi' = \min_{\sigma \in \Phi(J)} \{\sigma \mid \sigma \succ \phi, \phi'\}$. For example, if $\phi = \{[1, 3], [2], [4]\}$ and $\phi' = \{[1], [2, 4], [3]\}$ then $\ell(\phi) = 3 = \ell(\phi')$ with $\phi \vee \phi' = \{[1, 3], [2, 4]\}$. Notice that even though ϕ and ϕ' are planar partitions their join $\phi \vee \phi'$ is not, so the sublattice of planar partitions is not closed under join. This restricts the effectiveness of lattice theoretic approaches to understanding the lattice of planar partitions.

2.2. Free chromials. We consider throughout only undirected graphs G . A *planar* graph is a graph which admits a topological embedding into the plane with no crossing edges. A circuit J of length r in the planar

graph G is *contractible* if G can be embedded into the plane so that its interior contains no vertices. We will consider pairs (G, ϕ) where $\phi \in \Phi(J)$ is a partition of the vertex set of the circuit J .

A *(vertex) colouring* of a graph G with t colors is a colouring of vertices of G with colors $\{1, \dots, t\}$ such that no edge joins vertices of the same colour. We denote by $P_G(t)$ the number of such colourings. It is called the *chromial* of G . The *free chromial* $P_{(G, \phi)}(t)$ of (G, ϕ) is the number of ways of colouring (G, ϕ) with at most t colours such that the vertices of G in the same block of ϕ receive the same colour. Clearly, the free chromial $P_{(G, \phi)}(t)$ specialises to the chromial $P_G(t)$ when $\phi = \varepsilon$, the partition of $V(J)$ into singleton blocks.

2.3. Tutte's invariants. The starting point is the following theorem due to Tutte. It provides relations between free chromials. Remarkably, these relations are independent of the graph G . Hence the coefficients $\vartheta_{\phi, \pi}$ in the relations are fundamental invariants which we call *Tutte's Chromatic Invariants*. It is our contention that these invariants are interesting objects to study in themselves.

Theorem 2.1 (Tutte's Invariants [T1]). *Let G be a planar graph and J a contractible circuit of G of length $r < \infty$. Suppose the interior of J is empty and let $\phi \in \Phi(J)$ (any partition of the vertices in J). Then there exist invariants $\vartheta_{\phi, \pi}(t) \in \mathbb{Q}((t))$ independent of G , where $(\phi, \pi) \in \Phi(J) \times \Pi(J)$, such that*

$$(1) \quad P_{(G, \phi)}(t) = \sum_{\pi \in \Pi(J)} \vartheta_{\phi, \pi}(t) P_{(G, \pi)}(t).$$

Moreover, $\vartheta_{\phi, \pi}(t)$ can be determined from the equation

$$\left[t^{\ell(\phi \vee \sigma)} \right]_{B_r \times C_r} = [\vartheta_{\phi, \pi}]_{B_r \times C_r} \left[t^{\ell(\pi \vee \sigma)} \right]_{C_r \times C_r}.$$

Proof. We use induction on the number of edges of G . Suppose e is an edge of G without both ends on the ring J . By deletion and contraction, $P_{(G, \phi)}(t) = P_{(G-e, \phi)}(t) - P_{(G/e, \phi)}(t)$. By the induction hypothesis, Tutte's relation holds for both $G - e$ and G/e while $\vartheta_{\phi, \pi}$ is independent of $G - e$ and G/e so

$$\begin{aligned} P_{(G, \phi)}(t) &= P_{(G-e, \phi)}(t) - P_{(G/e, \phi)}(t) \\ &= \sum_{\pi \in \Pi(J)} \vartheta_{\phi, \pi}(t) (P_{(G-e, \pi)}(t) - P_{(G/e, \pi)}(t)) = \sum_{\pi \in \Pi(J)} \vartheta_{\phi, \pi}(t) P_{(G, \pi)}(t). \end{aligned}$$

Tutte's relation also stays true if we remove isolated vertices introduced by the delete and contract procedure since an isolated vertex contributes a factor of t to both sides of (1). Thus we are reduced to proving the theorem for the case where G contains vertices only on J (this is the base case of the induction). In this base case the graph G is planar and the edges join vertices on J .

Consider an arbitrary way T of deleting and contracting all the remaining edges of G . This identifies vertices on J into blocks which form the blocks of a partition σ_T of the vertices of J . This partition is planar since delete and contract operations preserve planarity. Now $P_{(G, \phi)}$ can be expressed as a sum over all such T . The contribution of one such T to $P_{(G, \phi)}$ is $(-1)^{c(T)} t^{\ell(\phi \vee \sigma_T)}$ where $c(T)$ is the number of edges contracted by T . Thus $P_{(G, \phi)} = \sum_T (-1)^{c(T)} t^{\ell(\phi \vee \sigma_T)}$. Similarly we get $P_{(G, \pi)} = \sum_T (-1)^{c(T)} t^{\ell(\pi \vee \sigma_T)}$. So equation (1) becomes

$$\sum_T (-1)^{c(T)} t^{\ell(\phi \vee \sigma_T)} = \sum_{\pi \in \Pi(J)} \vartheta_{\phi, \pi} \sum_T (-1)^{c(T)} t^{\ell(\pi \vee \sigma_T)} = \sum_T (-1)^{c(T)} \sum_{\pi \in \Pi(J)} \vartheta_{\phi, \pi} t^{\ell(\pi \vee \sigma_T)}.$$

Hence it suffices to show that there exist $\vartheta_{\phi, \pi}$ such that for any $(\phi, \pi) \in (\Phi(J), \Pi(J))$ we have

$$t^{\ell(\phi \vee \sigma)} = \sum_{\pi \in \Pi(J)} \vartheta_{\phi, \pi} t^{\ell(\pi \vee \sigma)}.$$

Let $\mathbf{B}_r = [t^{\ell(\phi \vee \sigma)}]_{B_r \times C_r}$, $\mathbf{A}_r = [\vartheta_{\phi, \pi}]_{B_r \times C_r}$, and $\mathbf{M}_r = [t^{\ell(\pi \vee \sigma)}]_{C_r \times C_r}$. Then the relation above can be rewritten as $\mathbf{B}_r = \mathbf{A}_r \mathbf{M}_r$. Hence such $\vartheta_{\phi, \pi}$ exist if and only if \mathbf{M}_r is invertible.

We now prove the invertibility of \mathbf{M}_r . Denote the planar partitions by π_1, \dots, π_m (where $m = C_r$). A general term in the expansion of $\det \mathbf{M}_r$ is $\pm t^{\ell(\pi_1 \vee \pi_{i_1})} \dots t^{\ell(\pi_m \vee \pi_{i_m})}$. Note that $\ell(\pi_1 \vee \pi_{i_1}) \leq \ell(\pi_1 \vee \pi_1)$ with equality if and only if $\pi_{i_1} \preceq \pi_1$. Consequently

$$\ell(\pi_1 \vee \pi_{i_1}) + \dots + \ell(\pi_m \vee \pi_{i_m}) \leq \ell(\pi_1 \vee \pi_1) + \dots + \ell(\pi_m \vee \pi_m)$$

with equality if and only if $\pi_{i_1} \preceq \pi_1, \dots, \pi_{i_m} \preceq \pi_m$. By Lemma 2.2, which follows below, this means that equality holds if and only if $i_1 = 1, \dots, i_m = m$. Thus there is a unique term of highest degree in the expansion of $\det \mathbf{M}_r$ (namely the product of the diagonal elements of \mathbf{M}_r). This element occurs with non-zero coefficient, whence \mathbf{M}_r is invertible. \square

Lemma 2.2. *Let (\mathcal{P}, \preceq) be a finite poset with elements p_1, \dots, p_m . If i_1, \dots, i_m is a permutation of $1, \dots, m$ such that $p_{i_1} \preceq p_1, \dots, p_{i_m} \preceq p_m$ then $i_1 = 1, \dots, i_m = m$.*

Proof. Without loss of generality assume p_1 is a minimal element. Then $p_{i_1} \preceq p_1$ implies that $i_1 = 1$. Now delete p_1 from the poset and repeat the argument. This process terminates since (\mathcal{P}, \preceq) is finite. \square

Since $\vartheta_{\phi, \pi}(t)$ is independent of G we shall call it Tutte's *chromatic invariant*. If ϕ happens to be planar then $\vartheta_{\phi, \pi} = \delta_{\phi, \pi}$ and Tutte's relation gives no information. The matrix $\mathbf{M}_r = [t^{\ell(\pi \vee \sigma)}]_{C_r \times C_r}$ is usually called the *matrix of chromatic joins*.

Remark 2.3. Similarly, it can be shown that for any planar graph G with circuit J and arbitrary exterior as well as interior we have

$$P_{(G, \phi)}(t) = \sum_{\rho} \vartheta'_{\rho, \phi} P_{(G, \rho)}(t),$$

for some $\vartheta'_{\rho, \phi}$ where the sum is over all partitions ρ that can be expressed as $\rho = \sigma_1 \vee \sigma_2$ for some planar partitions σ_1 and σ_2 . This follows because the base case in the proof of this involves a circuit J with $\text{ext}_J G$ and $\text{int}_J G$ encoded by planar partitions σ_1 and σ_2 . Thus $P_{(G, \phi)} = t^{\ell(\sigma_1 \vee \sigma_2 \vee \phi)}$ and the σ in the proof of Theorem 2.1, which was planar, is now replaced by $\sigma_1 \vee \sigma_2$. The matrix which is analogous to \mathbf{M}_r is $[t^{\ell(\rho_1 \vee \rho_2)}]$ where both ρ_1 and ρ_2 can be expressed as the join of two planar partitions. All that remains to prove such a result is to show that the determinant of this matrix is not identically zero. This can be proven using the same ideas we used above to show that \mathbf{M}_r is invertible.

2.4. Completeness of Tutte's Relations. Next we show that Tutte's relations (1) account for all the linear relations between free chromials. By such a linear relation we mean an equation of the form

$$\sum_{\phi \in \Phi(J)} f_{\phi}(t) P_{(G, \phi)}(t) = 0$$

where the $f_{\phi}(t)$ are independent of G .

Theorem 2.4. *Tutte's relations (1) form a basis over $\mathbb{R}(t)$ for the set of all linear relations between free chromials.*

Proof. Suppose we have such a linear relation $\sum_{\phi \in \Phi(J)} f_{\phi}(t) P_{(G, \phi)}(t) = 0$. Using equation (1) we can replace each occurrence of $P_{(G, \phi)}(t)$ where ϕ is non-planar by a linear combination involving terms $P_{(G, \pi)}(t)$ where the π are all planar. Whence we get a relation $\sum_{\pi \in \Pi(J)} f_{\pi}(t) P_{(G, \pi)}(t) = 0$ where the sum is over *planar* partitions. It is enough to show that in such a relation we have $f_{\pi}(t) = 0$ for all $\pi \in \Pi(J)$.

Take G to have no vertices outside of J so that it is given by some planar partition $\sigma \in \Pi(J)$. Then $P_{(G, \pi)}(t) = t^{\ell(\pi \vee \sigma)}$ and the relation reads $\sum_{\pi \in \Pi(J)} f_{\pi}(t) t^{\ell(\pi \vee \sigma)} = 0$. Since this holds for any planar partition σ and the chromatic join matrix $[t^{\ell(\pi \vee \sigma)}]$ is invertible it follows that $f_{\pi}(t) = 0$ for all planar partitions π . \square

3. THE RELATION BETWEEN TUTTE'S INVARIANTS, THE TEMPERLEY-LIEB ALGEBRA AND THE PARTITION ALGEBRA

The bivariate Temperley-Lieb algebra $\mathbb{TL}_r(x, y)$ will enable us to gain further insight into Tutte's invariants. We introduce this algebra and then use it to determine the determinant of the matrix M_r of chromatic joins. Part of the motivation for doing this is to find further properties of M_r . The Temperley-Lieb algebra is used to obtain a natural decomposition of M_r^{-1} into the product of an upper triangular matrix P_r and its transpose. Further details regarding the structure of $\mathbb{TL}_r(x, y)$ and its relationship to M_r are given in [CJ].

3.1. The Temperley-Lieb Algebra.

3.1.1. *Planar partitions and strand diagrams.* There is an elementary bijection between the set Π_r of all planar partitions of J_r and the set \mathcal{S}_r of rooted, 2-coloured, planar strand diagrams. The elementwise action of this bijection is best described by an example that generalises in the obvious way. Let $\pi_0 = \{[1, 2, 6], [3, 4], [5]\}$, a planar partition in Π_6 . Figure 2 shows how the planar diagram for π_0 is transformed via vertex splitting into a strand diagram representing an element in the Temperley-Lieb algebra.

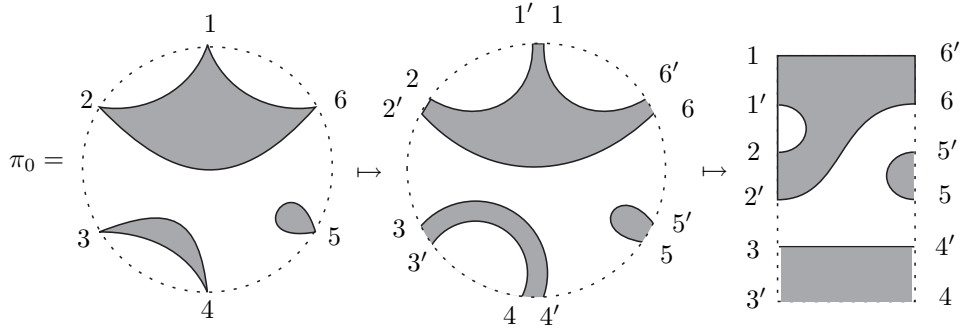


FIGURE 2. Construction of the strand diagram for the planar partition $\pi_0 = \{[1, 2, 6], [3, 4], [5]\}$ via vertex splitting.

This construction is clearly reversible and extends naturally to all r .

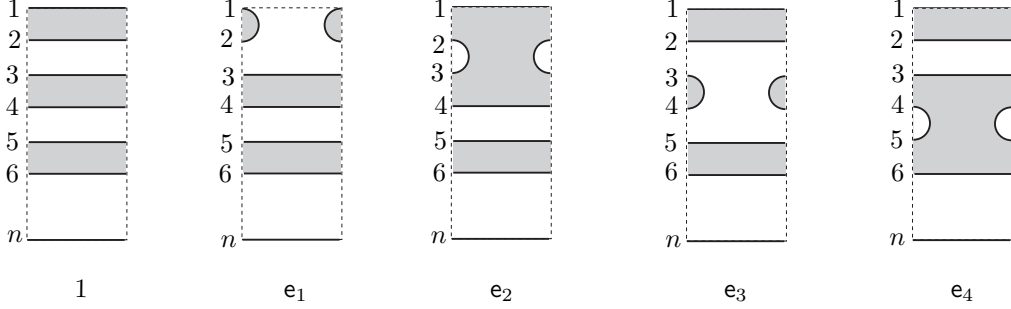
3.1.2. *Strand diagrams and the bivariate Temperley-Lieb Algebra.* The bivariate Temperley-Lieb Algebra $\mathbb{TL}_r(x, y)$ is the free additive algebra over $\mathbb{R}(x, y)$ with multiplicative generators $1, e_1, \dots, e_{r-1}$ subject to the relations

- S1 $e_i^2 = x_i e_i$ for $i = 1, \dots, r - 1$, where $x_i = x$ if i odd, and $x_i = y$ if i even,
- S2 $e_i e_j = e_j e_i$ if $|i - j| > 1$,
- S3 $e_i e_{i\pm 1} e_i = e_i$ for $i = 1, \dots, r - 1$,

where x and y commute with all elements, and 1 is the multiplicative identity. Thus

$$\mathbb{TL}_r(x, y) = \mathbb{R}(x, y)\langle 1, e_1, \dots, e_{r-1} \rangle / (S1, S2, S3).$$

The set \mathcal{B}_1 of all distinct monomials in the generators modulo the relations S1, S2 and S3 affords a linear basis of $\mathbb{TL}_r(x, y)$ over $\mathbb{R}(x, y)$. For example, $\mathcal{B}_1 = (1, e_1, e_2, e_1 e_2, e_2 e_1)$ is a linear basis for $\mathbb{TL}_3(x, y)$ where the ordering of the basis elements is, for the moment, arbitrary.

FIGURE 3. Strand diagrams corresponding to generators of $1, e_1, e_2, e_3, e_4 \in \mathbb{T}\mathbb{L}_n$

The connexion of $\mathbb{T}\mathbb{L}_r(x, y)$ to planar strand diagrams is through the combinatorial presentation of the generators $1, e_1, \dots, e_{n-1}$ as properly 2-coloured *planar strand diagrams* which are shown in Figure 3.

The product in $\mathbb{T}\mathbb{L}_r(x, y)$ corresponds to the concatenation of strand diagrams with the convention that each black loop so formed is marked by x and deleted, and each white loop is marked by a y and deleted. Note that concatenation of strand diagrams preserves their planarity and respects colouring. It is a good exercise to check the three relations ($S1, S2, S3$) diagrammatically. It is also not hard to see that every planar strand diagram can be obtained by concatenating together strand diagrams corresponding to the generators e_1, \dots, e_{r-1} . Hence there is a bijection between monomials in $\mathbb{T}\mathbb{L}_r(x, y)$ and planar strand diagrams.

We summarize the correspondences as follows. To every planar partition $\pi \in \Pi_r$ there corresponds a planar 2-coloured strand diagram henceforth denoted by p_π . Moreover, each such strand diagram corresponds to a monomial $p_\pi \in \mathbb{T}\mathbb{L}_r(x, y)$. With a minor abuse of notation, we denote a monomial $p_\pi \in \mathbb{T}\mathbb{L}_r(x, y)$ and the corresponding strand diagram by the same symbol.

3.1.3. *A bilinear form on $\mathbb{T}\mathbb{L}_r(x, y)$.* For $\pi \in \Pi_r$ the closure $\bar{\pi}$ of π is obtained by joining the ends of the strands of π by arcs. Figure 4 illustrates the closure of p_{π_0} .

If $\pi \in \Pi_r$, let $\text{sh}(\pi)$ and $\text{ush}(\pi)$ be, respectively, the number of (finite) shaded and unshaded regions in the closure $\bar{\pi}$ of π . Let

$$\text{tr}: \mathbb{T}\mathbb{L}_r(x, y) \rightarrow \mathbb{R}(x, y): p_\pi \mapsto x^{\text{sh}(\pi)} y^{\text{ush}(\pi)},$$

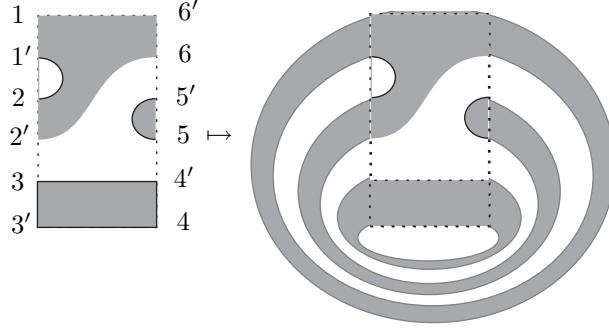
extended linearly to $\mathbb{T}\mathbb{L}_r(x, y)$. For example, $\text{tr}(\pi_0) = x^2 y^2$. Notice that $\text{tr}(ab) = \text{tr}(ba)$ for all $a, b \in \mathbb{T}\mathbb{L}_r(x, y)$, so tr is a trace function in $\mathbb{T}\mathbb{L}_r(x, y)$. It is worthwhile noting that the obvious generalization of this trace is no longer well-defined if the bivariate Temperley-Lieb algebra is replaced by the obvious multivariate generalization (where $e_i^2 = x_i e_i$ for some indeterminates x_i).

The *transpose* π^t of a strand diagram for π is obtained by flipping the diagram about a vertical axis. If $a = p_\pi \in \mathbb{T}\mathbb{L}_r(x, y)$ is a monomial corresponding to some partition $\pi \in \Pi_r$ then we define the transpose of a by $a^t = p_{\pi^t}$, extending linearly to all elements of $\mathbb{T}\mathbb{L}_r(x, y)$. Hence we have:

- $e_i^t = e_i$ for $i = 1, \dots, r-1$,
- $(ab)^t = b^t a^t$ for all $a, b \in \mathbb{T}\mathbb{L}_r(x, y)$ and
- $(c_1 a + c_2 b)^t = c_1 a^t + c_2 b^t$ for all $a, b \in \mathbb{T}\mathbb{L}_r(x, y)$ and all $c_1, c_2 \in \mathbb{R}(x, y)$.

We now define a symmetric bilinear form on $\mathbb{T}\mathbb{L}_r(x, y)$ as follows:

$$(2) \quad \langle \cdot, \cdot \rangle: \mathbb{T}\mathbb{L}_r(x, y) \times \mathbb{T}\mathbb{L}_r(x, y) \rightarrow \mathbb{R}(x, y): (a, b) \mapsto \text{tr}(b^t a).$$

FIGURE 4. Loop diagram for the closure of \mathfrak{p}_{π_0} .

The Gram matrix of $\langle \cdot, \cdot \rangle$ is denoted by $\mathbf{M}_r(x, y)$. In other words, the rows and columns of $\mathbf{M}_r(x, y)$ are indexed by planar partitions with the entry in position (π_1, π_2) being $\langle \mathfrak{p}_{\pi_1}, \mathfrak{p}_{\pi_2} \rangle$. The key point is that that $\mathbf{M}_r(t, 1)$ is precisely the matrix of chromatic joins. This is because if we set $x = t$ and $y = 1$ then $\langle \mathfrak{p}_{\pi_1}, \mathfrak{p}_{\pi_2} \rangle = t^{\ell(\pi_1 \vee \pi_2)}$.

Thus we have transported the problem of computing the determinant of the chromatic join matrix to one of computing the determinant of the Gram matrix of $\mathbb{T}\mathbb{L}_r(x, y)$. This is a more attainable goal once we know more about the structure of $\mathbb{T}\mathbb{L}_r(x, y)$. We include the following result about $\mathbb{T}\mathbb{L}_r(x, y)$ taken from [CJ](Thm 3.7 and Cor 4.4).

Theorem 3.1.

$$\mathbb{T}\mathbb{L}_r(x, y) \cong \mathcal{M}_{l_1}(\mathbb{R}(x, y)) \oplus \mathcal{M}_{l_2}(\mathbb{R}(x, y)) \oplus \cdots \oplus \mathcal{M}_{l_r}(\mathbb{R}(x, y)),$$

where $\mathcal{M}_i(\mathbb{R}(x, y))$ is the ring of $i \times i$ matrices over $\mathbb{R}(x, y)$ and $l_i = \binom{r}{(r-i)/2} - \binom{r}{(r-i-2)/2}$. Moreover, if $e_{ij}^k \in \mathcal{M}_{l_k} \subset \mathbb{T}\mathbb{L}_r(x, y)$ denotes the elementary matrix with 1 in position (i, j) and 0 everywhere else then $\text{tr}(e_{ij}^k) = V_k \delta_{i,j}$.

3.1.4. The ideal \mathbb{K}_{2r} of $\mathbb{T}\mathbb{L}_{2r}$ and the chromatic join matrix. In [CJ] we computed $\det \mathbf{M}_r(x, y)$ by working in $\mathbb{T}\mathbb{L}_r(x, y)$. That argument can be simplified by working in an ideal $\mathbb{K}_{2r} \subset \mathbb{T}\mathbb{L}_{2r}(x, y)$ rather than in $\mathbb{T}\mathbb{L}_r(x, y)$ itself. Let \mathbb{K}_{2r} be the left ideal of $\mathbb{T}\mathbb{L}_{2r}(x, y)$ generated by the monomials corresponding to the set $\widehat{\Pi}_r$ of partitions in Π_r that are augmented by singletons; i.e. $\widehat{\Pi}_r = \{\pi \cup \{[r+1], \dots, [2r]\} : \pi \in \Pi_r\}$. If $\pi \in \Pi_r$ then the corresponding augmented element $\pi \cup \{[r+1], \dots, [2r]\}$ is denoted by $\widehat{\pi}$. Thus we get a bijection between $\Pi_r \rightarrow \mathbb{K}_{2r}$ given by $\pi \mapsto \mathfrak{p}_{\widehat{\pi}}$. Figure 5 illustrates this bijection for the partition $\pi_0 \in \Pi_6$.

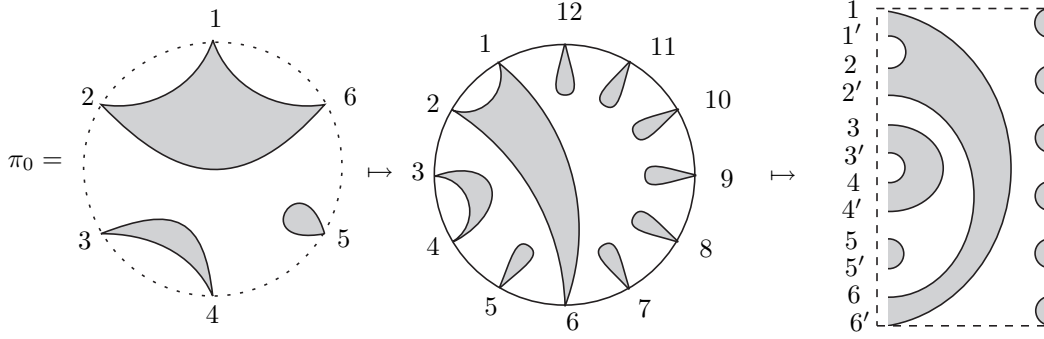


FIGURE 5. The strand diagram $\hat{a} = p_{\hat{\pi}_0} \in \mathbb{K}_{12}$ corresponding to $\pi_0 = \{[1, 2, 6], [3, 4], [5]\}$.

Similarly consider $\hat{b} = p_{\hat{\gamma}_0}$ where $\gamma_0 = \{[1, 4], [2, 3], [5], [6]\}$. The left hand diagram of Figure 6 shows the strand diagram of the transpose \hat{b}^t . This diagram is juxtaposed with that of $\hat{a} = p_{\hat{\pi}_0}$. The right hand diagram gives the strand diagram for the closure of $\hat{b}^t \hat{a}$ and therefore determines the trace of $\hat{b}^t \hat{a}$ and thence the inner product $\langle \hat{a}, \hat{b} \rangle$.

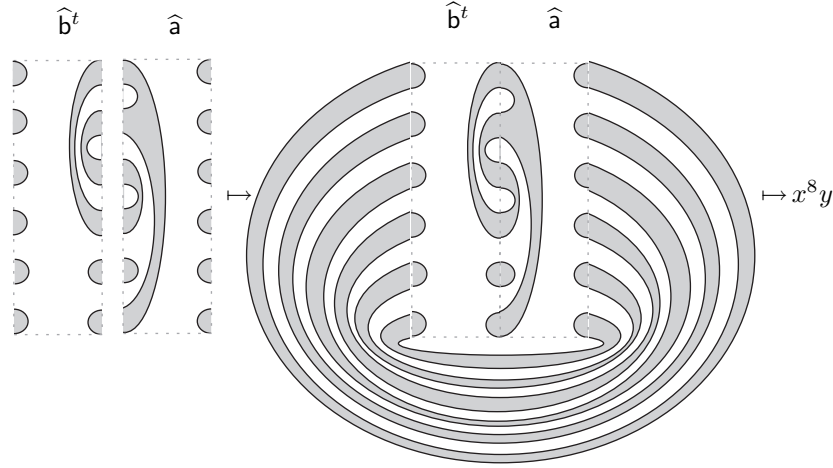


FIGURE 6. Strand diagrams for $\hat{b}^t = p_{\hat{\gamma}_0^t}$ and $\hat{a} = p_{\hat{\pi}_0}$ in the left ideal \mathbb{K}_{12} as well as the inner product of \hat{a} and \hat{b} on the right.

From this example we conclude the following key fact: if $\pi, \gamma \in \Pi_r$ then $\hat{a} = p_{\hat{\pi}}$ and $\hat{b} = p_{\hat{\gamma}}$ satisfy $\langle \hat{a}, \hat{b} \rangle = t^{r+\ell(\pi \vee \gamma)}$ and hence $M_r(t, 1) = t^{-r} [\langle \hat{a}, \hat{b} \rangle]_{\hat{a}, \hat{b} \in \mathbb{K}_{2r}}$.

For example, the chromatic join of π_0 and γ_0 is $\pi_0 \vee \gamma_0 = \{[1, 2, 3, 4, 6], [5]\}$ so $\ell(\pi_0 \vee \gamma_0) = 2$. On the other hand $\langle p_{\hat{\pi}_0}, p_{\hat{\gamma}_0} \rangle = x^{2+r} y$ where $r = 6$. So at $x = t$ and $y = 1$ we have $t^{-r} \langle p_{\hat{\pi}_0}, p_{\hat{\gamma}_0} \rangle = t^2 = t^{\ell(\pi_0 \vee \gamma_0)}$.

3.2. Construction of an orthonormal basis for \mathbb{K}_{2r} . We construct an orthonormal basis \mathcal{B}_2 for \mathbb{K}_{2r} by a process that can be viewed precisely as Gram-Schmidt Orthogonalisation in the ideal $\mathbb{K}_{2r}(x, y)$. To do this we make use of the Temperley-Lieb Algebra to determine the orthogonal projections. The change of basis

matrix $\widehat{\mathbf{P}}_r$ from the original standard basis to \mathcal{B}_2 will be seen to be upper triangular where the entries on the diagonal are given explicitly. Since $\mathbf{M}_r(x, y)^{-1} = \mathbf{P}_r \mathbf{P}_r^t$, this will allow us to determine $\det \mathbf{M}_r(x, y)$.

3.2.1. *Strand diagrams and Dyck paths.* A path is a sequence of displacements, or *steps*, $(1, \pm 1)$, starting from the origin of the non-negative integer sublattice \mathbb{Z}^2 of \mathbb{R}^2 . A *Dyck path* is a path terminating on the x -axis and having no points below the x -axis, and the set of all Dyck paths of length $2r$ is denoted by \mathcal{D}_r . The displacements $(1, 1)$ and $(1, -1)$ are represented by \nearrow and \searrow , respectively. There is an elementary bijection between planar strand diagrams with r strands and Dyck paths in \mathcal{D}_r . Consequently there is also a bijection between planar strand diagrams corresponding to partitions in $\widehat{\Pi}_r$ and elements of $\widehat{\mathcal{D}}_r$ where $\widehat{\mathcal{D}}_r = \mathcal{D}_r \cdot (\nearrow \searrow)^r$, with “.” denoting concatenation. For example, the Dyck Path corresponding to $\widehat{\mathbf{p}}_{\pi_0} \in \mathbb{K}_{12}(x, y)$ where $\pi_0 = \{[1, 2, 6], [3, 4], [5]\}$ is given in Figure 7.

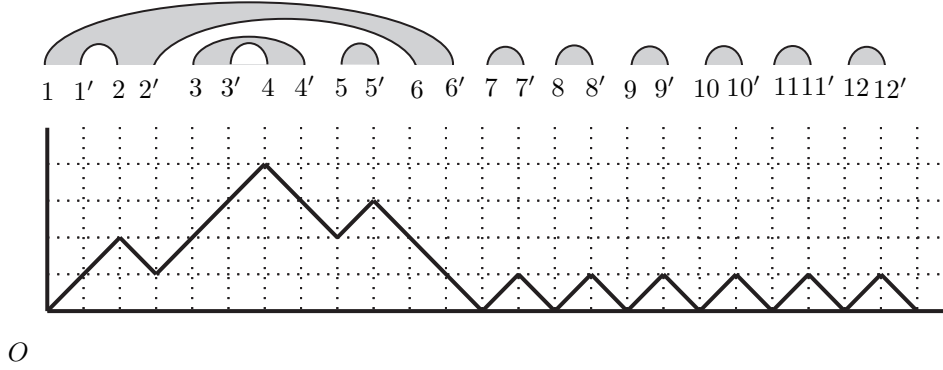


FIGURE 7. Dyck path corresponding to $\widehat{\mathbf{a}} = \widehat{\mathbf{p}}_{\pi_0} \in \mathbb{K}_{12}$.

Let $[a]$ denote the Dyck path corresponding to a monomial $a \in \mathbb{T}\mathbb{L}_r(x, y)$. For example, given the trivial partition $\varepsilon_r = \{[1], \dots, [r]\}$ we have $\widehat{\mathbf{p}}_{\varepsilon_r} = \mathbf{e}_1 \mathbf{e}_3 \cdots \mathbf{e}_{2r-1} \in \mathbb{K}_{2r}$ and its corresponding Dyck path is $[\widehat{\mathbf{p}}_{\varepsilon_r}] = (\nearrow \searrow)^{2r} \in \widehat{\mathcal{D}}_r$.

It is clear that any path $\mathbf{p} \in \widehat{\mathcal{D}}_r$ can be constructed iteratively from $\widehat{\varepsilon}_r$ by inserting unit boxes \diamond snugly into minima $\searrow \nearrow$ (such that at each stage the inserted box has its two lower sides in contact with the two sides of the minimum). Consequently we shall call $[\widehat{\mathbf{p}}_{\varepsilon_r}]$ the *base path* for \mathbb{K}_{2r} . Let $\mathbf{p} \boxplus \diamond_i$ be the Dyck path obtained from \mathbf{p} by inserting a box into a minimum at position i (that is, the x -coordinate of the bottom of the minimum is i). Let $h_i(\mathbf{p})$, the *height* of \mathbf{p} at position i , be the y -coordinate of the point at position i of \mathbf{p} .

The fundamental observation that connects the algebra to the diagrammatics is that

$$(3) \quad (\mathbf{p} \boxplus \diamond_i)_1 = \mathbf{e}_i(\mathbf{p})_1 \quad \text{if } i < r \text{ and } \mathbf{p} \text{ has a minimum at position } i,$$

where $(\mathbf{p})_1 \in \mathbb{K}_{2r}$ denotes the monomial corresponding to the Dyck path $\mathbf{p} \in \widehat{\mathcal{D}}_r$.

3.2.2. *The basis \mathcal{B}_2 of \mathbb{K}_{2r} .* To construct the basis \mathcal{B}_2 of the ideal \mathbb{K}_{2r} with respect to which the Gram matrix of $\langle \cdot, \cdot \rangle$ is diagonal, we shall need $\mu_i = V_{i-1}/V_i \in \mathbb{Q}(x, y)$, for $i \geq 1$, where V_i are the *bivariate Chebyshev polynomials*. These are defined using the linear recurrence

$$V_{k+1} = x_{k+1} V_k - V_{k-1}, \quad k \geq 1,$$

with initial conditions $V_0(x, y) = 1$, $V_1(x, y) = x$ and where x_k is x if k is odd and y if k is even. Thus $V_k(t, t)$ is a normalised Chebyshev polynomial of the second kind. Let $(\cdot)_2: \widehat{\mathcal{D}}_r \rightarrow \mathbb{K}_{2r}$ be the mapping defined recursively by

$$(\mathfrak{p} \boxplus \diamond_i)_2 = \sqrt{\frac{\mu_{j+1}}{\mu_j}}(e_i - \mu_j \mathbf{1})(\mathfrak{p})_2 \quad i < r,$$

where $j = h_i(\mathfrak{p}) + 1$ is the height of the centre of \diamond_i . The initial condition is $(\nearrow \searrow)^{2r} \mapsto e_1 e_3 \dots e_{2r-1} = ((\nearrow \searrow)^{2r})_1$. Finally, we let $\mathcal{B}_2 = \{(\mathfrak{p})_2 : \mathfrak{p} \in \widehat{\mathcal{D}}_r\}$.

For example, the basis \mathcal{B}_2 for \mathbb{K}_4 is given by

$$\mathfrak{p}_1 = (\nearrow \searrow \nearrow \searrow \nearrow \searrow \nearrow \searrow)_2 = e_1 e_3$$

and

$$\mathfrak{p}_2 = (\nearrow \nearrow \searrow \searrow \nearrow \searrow \nearrow \searrow)_2 = \sqrt{\frac{\mu_2}{\mu_1}}(e_2 - \mu_1)e_1 e_3.$$

Notice that $\mu_1 = \frac{1}{x}$ so that

$$e_3 e_1 (e_2 - \mu_1) e_1 e_3 = e_3 e_1 e_2 e_1 e_3 - \frac{1}{x} e_3 e_1 e_1 e_3 = e_3 e_1 e_3 - e_3 e_1 e_3 = 0.$$

Hence $\mathfrak{p}_1^t \cdot \mathfrak{p}_2 = e_3 e_1 \cdot \sqrt{\frac{\mu_2}{\mu_1}}(e_2 - \mu_1)e_1 e_3 = 0$ so that $\langle \mathfrak{p}_1, \mathfrak{p}_2 \rangle = \text{tr}(\mathfrak{p}_1^t \cdot \mathfrak{p}_2) = 0$ as expected.

Note that this construction can be extended to give an orthonormal basis for $\mathbb{T}\mathbb{L}_r(x, y)$. This was done in [CJ] but requires the construction of Jones-Wenzl projectors. Alternatively, one could also use the construction above together with a bijective map $\mathbb{K}_{2r} \rightarrow \mathbb{T}\mathbb{L}_r$ to avoid having to build Jones-Wenzl projectors (in fact, this gives an alternate way to understand the Jones-Wenzl projectors). We choose to work in $\mathbb{K}_{2r}(x, y)$ in order to avoid this extra complication.

3.2.3. Properties of \mathcal{B}_2 . The next result shows that \mathcal{B}_2 is an orthogonal basis for $\mathbb{K}_{2r}(x, y)$.

Theorem 3.2. *If $\pi, \gamma \in \Pi_r$ then $\langle [\mathfrak{p}_\pi]_2, [\mathfrak{p}_\gamma]_2 \rangle = x^r \delta_{\pi, \gamma}$.*

Proof. This is a special case of corollary Cor 4.4 in [CJ]. □

Finally, we relate all this to $\mathbf{M}_r(x, y)$.

Lemma 3.3. *Let \mathbf{P}_r be the matrix that expresses the elements of \mathcal{B}_2 in terms of the elements of \mathcal{B}_1 . Then $\mathbf{M}_r^{-1}(x, y) = x^{-r} \mathbf{P}_r \mathbf{P}_r^t$.*

Proof. The Gram matrix of $\langle \cdot, \cdot \rangle$ with respect to the orthogonal basis \mathcal{B}_2 of \mathbb{K}_{2r} is $x^{2r} \mathbf{I}_r$. It follows that $\mathbf{P}_r^t \mathbf{M}_r(x, y) \mathbf{P}_r = x^{2r} \mathbf{I}_r$ so \mathbf{M}_r is invertible and $\mathbf{M}_r^{-1}(x, y) = x^{-2r} \mathbf{P}_r \mathbf{P}_r^t$, completing the proof. □

Lemma 3.4 (See Cor 4.6 [CJ]). *Let \preceq_{gi} be the partial ordering defined on $\widehat{\mathcal{D}}_r$ such that $\mathfrak{p} \preceq_{\text{gi}} \mathfrak{p}'$ if \mathfrak{p}' may be obtained from \mathfrak{p} by box addition. Then \mathbf{P}_r is upper triangular with respect to this ordering.*

Corollary 3.5.

$$(4) \quad \det(\mathbf{M}_r(x, y)) = \prod_{i=1}^r V_i^{a_{r,i}} \quad \text{where} \quad a_{r,i} = b_{r,i} - b_{r,i-1} \geq 0 \quad \text{and} \quad b_{r,i} = \binom{2r}{r-i} - \binom{2r}{r-i-1}.$$

Proof. The terms on the diagonal of \mathbf{P}_r may be read directly and grouped to obtain expressions of the powers of V_i , which are then readily simplified. The details are given in the proof of Theorem 5.6 [CJ]. □

We note that $b_{r,i} = \binom{2r}{r-i} - \binom{2r}{r-i-1}$ is the number of Dyck paths of length $2r$ with the modified condition that they end at a point with height $2i \geq 0$.

The following is now immediate.

Corollary 3.6. $M_r(t, 1)$ is invertible.

One of the reasons for this exercise was to convince the reader that the Temperley-Lieb algebra is indeed a natural setting for studying planar partitions. The bivariable Chebyshev polynomials seen above are inately related to the structure of the Temperley-Lieb algebra. This relationship explains how $\det M_r(t, 1)$ factors into a product of these Chebyshev polynomials (evaluated at $x = t$ and $y = 1$). Without this connexion the reason for the factorization of $\det M_r(t, 1)$ is quite mysterious.

3.3. The Partition Algebra. In this section we show that Tutte's chromatic invariant $\vartheta_{\phi, \pi}(t)$ can be understood in terms of the orthogonal projection from the Partition Algebra \mathbb{P}_r to the Temperley-Lieb Algebra $\mathbb{TL}_r \hookrightarrow \mathbb{P}_r$. An account of this algebra is given in [R]

We have seen that $\mathbb{TL}_r(x, y)$ is an algebra whose standard basis is in bijection with the set Π_r of planar partitions of $\{1, \dots, r\}$. This idea can be extended naturally to include all partitions of $\{1, \dots, r\}$ and gives us the Partition Algebra $\mathbb{P}_r(t)$. The standard basis of $\mathbb{P}_r(t)$ is in bijection with all partitions of $\{1, \dots, r\}$. As before, we draw partitions as 2-coloured strand diagrams except that now the strands can cross (see, for example, Figure 8). Multiplication corresponds to concatenation of strand diagrams. The variable t is used to mark shaded components. Unfortunately it is not possible to have a bivariate version of $\mathbb{P}_r(t)$ so we shall have only an inclusion $\mathbb{TL}_r(t, 1) \hookrightarrow \mathbb{P}_r(t)$. Given a partition $\phi \in \Phi_r$ of $\{1, \dots, r\}$, we shall denote by p_ϕ the corresponding monomial in $\mathbb{P}_r(t)$.

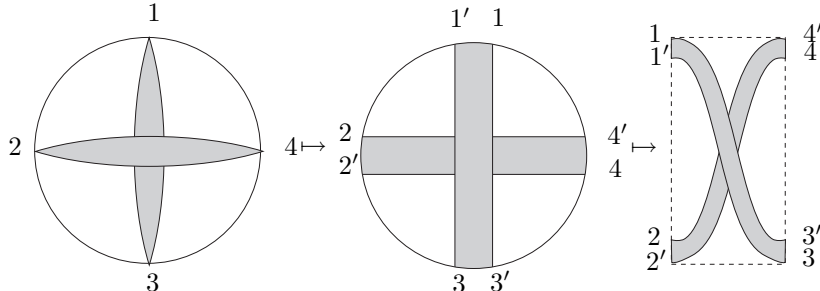


FIGURE 8. The non-planar partition $\phi = \{[1, 3], [2, 4]\}$ and the corresponding element $p_\phi \in \mathbb{P}_4$.

The trace tr and the bilinear form $\langle \cdot, \cdot \rangle$ can be extended from $\mathbb{TL}_r(t, 1)$ to $\mathbb{P}_r(t)$ by extending our old definitions. That is to say, the trace is obtained by closing up a diagram and counting the number of shaded regions which are marked by t . The inner product is $\langle a, b \rangle = \text{tr}(ab^t) = \text{tr}(b^t a)$ where b^t denotes the transpose of b (the diagram obtained from b by flipping).

3.3.1. An orthogonal projection.

Lemma 3.7. The restriction of the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{P}_r(t)$ to $\mathbb{TL}_r(t, 1)$ is non-degenerate.

Proof. The inner product $\langle \cdot, \cdot \rangle$ is non-degenerate if the determinant of its Gram matrix is non-zero. But the Gram matrix for $\mathbb{TL}_r(t, 1)$ is $M_r(t, 1)$ whose determinant is calculated in (4) to be non-zero. \square

Lemma 3.7 justifies the following definition.

Definition 3.8. $\text{proj}^\perp : \mathbb{P}_r(t) \rightarrow \mathbb{TL}_r(t, 1)$ is the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$.

The following theorem recovers the chromatic invariant $\vartheta_{\phi, \pi}$ from proj^\perp .

Theorem 3.9. Let $\phi \in \Phi_r$. Then $\text{proj}^\perp(\mathbf{p}_\phi) = \sum_{\pi \in \Pi_r} \vartheta_{\phi, \pi} \mathbf{p}_\pi$.

Proof. From Theorem 2.1, $P_{(\text{ext}_J G, \phi)}(t) = \sum_{\pi \in \Pi_r} \vartheta_{\phi, \pi}(t) P_{(\text{ext}_J G, \pi)}(t)$, where $\vartheta_{\phi, \pi}(t)$ is independent of G . Select $\text{ext}_J G$ to be a planar graph with no vertices in the exterior of J . This graph induces a planar partition π' on $V(J)$. Moreover, $P_{(\text{ext}_J(G, \phi))}(t) = t^{\ell(\pi' \vee \phi)}$ and $P_{(\text{ext}_J(G, \pi))}(t) = t^{\ell(\pi' \vee \pi)}$. Thus

$$(5) \quad t^{\ell(\pi' \vee \phi)} = \sum_{\pi \in \Pi_r} \vartheta_{\phi, \pi}(t) t^{\ell(\pi' \vee \pi)}$$

for any planar partition π' and any partition ϕ . Now $\text{proj}^\perp(\mathbf{p}_\phi) \in \mathbb{TL}_r$, so $\text{proj}^\perp(\mathbf{p}_\phi) = \sum_{\pi \in \Pi_r} b_{\phi, \pi} \mathbf{p}_\pi$ for some scalars $b_{\phi, \pi}$. Then $\langle \mathbf{p}_\phi, \mathbf{p}_{\pi'} \rangle = \langle \text{proj}^\perp(\mathbf{p}_\phi), \mathbf{p}_{\pi'} \rangle = \sum_{\pi \in \Pi_r} b_{\phi, \pi} \langle \mathbf{p}_\pi, \mathbf{p}_{\pi'} \rangle$ since proj^\perp is an orthogonal projection. So $t^r t^{\ell(\phi \vee \pi')} = \sum_{\pi \in \Pi_r} b_{\phi, \pi} t^r t^{\ell(\pi \vee \pi')}$ whence $t^{\ell(\pi' \vee \phi)} = \sum_{\pi \in \Pi_r} b_{\phi, \pi} t^{\ell(\pi' \vee \pi)}$. It follows from (5) that $b_{\phi, \pi} = \vartheta_{\phi, \pi}(t)$. \square

Note that if $\phi = \pi'$ is planar, then from Theorem 3.9, $\vartheta_{\pi, \pi'}(t) = \delta_{\pi, \pi'}$ as expected.

4. HIGHER GENUS TUTTE INVARIANTS

4.1. The genus of a partition. Let $\phi \in \Phi_r$ be an arbitrary partition and \mathbf{p}_ϕ its corresponding diagram in the Partition Algebra $\mathbb{P}_r(t)$. From \mathbf{p}_ϕ one can contract the boundary to a point to obtain a rooted hypermonopole (i.e. a monopole whose faces are two coloured) which we denote by \mathbf{h}_ϕ . For instance, Figure 9 shows how to obtain a rooted hypermonopole from the non-planar partition $\phi = \{[1, 3], [2, 4]\}$.

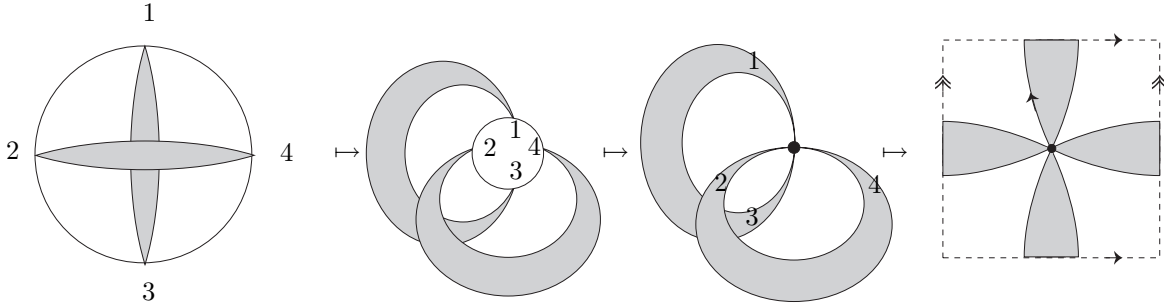


FIGURE 9. The non-planar partition $\phi = \{[1, 3], [2, 4]\}$ and the corresponding hypermonopole \mathbf{h}_ϕ .

The *genus* $g(\phi)$ of a partition ϕ is defined to be the genus of \mathbf{h}_ϕ . For example, $g(\phi) = 0$ if and only if ϕ is planar. We denote by Φ_r^g and $\Phi_r^{\leq g}$ the partitions of $\{1, \dots, r\}$ of genus g and genus less than or equal to g , respectively. For example, for $g \gg 0$ we have $\Phi_r^g = \Phi_r^{\leq g} = \Phi_r$.

Even though we can associate to a rooted hypermonopole to a partition as done above, it is not true that partitions are in bijection with them. In fact, it is permutations which are in bijection with rooted hypermonopoles, a point that is discussed further in Section 5

4.2. Generalisation of the chromatic invariant. A graph G has genus g if it can be embedded without crossing edges on a surface of genus g while it cannot be embedded on a surface of genus less than g . For example, planar graphs have genus 0. The following theorem shows that there exist Tutte invariants for higher genus graphs.

Theorem 4.1. *Let G be a graph of genus g and let J be a contractible circuit of G of degree $r < \infty$. Suppose the interior of J is empty. Then for every partition $\phi \in \Phi_r$ there exist invariants $\vartheta_{\phi, \pi}^g(t) \in \mathbb{Q}((t))$ where $\pi \in \Phi^{\leq g}$ which are independent of G and satisfy*

$$P_{(G, \phi)}(t) = \sum_{\pi \in \Phi_r^{\leq g}} \vartheta_{\phi, \pi}^g(t) P_{(G, \pi)}(t).$$

Moreover, $\vartheta_{\phi, \pi}^g(t)$ can be determined from the equation

$$\left[t^{\ell(\phi \vee \sigma)} \right]_{B_r \times C_r^g} = [\vartheta_{\phi, \pi}]_{B_r \times C_r^g} \left[t^{\ell(\pi \vee \sigma)} \right]_{C_r^g \times C_r^g}$$

where C_r^g is the number of partitions of $\{1, \dots, r\}$ of genus less than or equal to g .

Proof. The proof is the same as that of Theorem 2.1 except for the base case of the induction in which the circuit J is now on a surface of genus g . The same argument as before goes through except that now we can express free chromials $P_{(G, \phi)}(t)$ as a sum over partitions of genus less than or equal to g instead of a sum over partitions of genus 0. \square

Let $\mathbb{TL}_r^g(t)$ be the span of \mathfrak{p}_ϕ for all $\phi \in \Phi^{\leq g}$. Note that $\mathbb{TL}_r^g(t) \subset \mathbb{P}_r(t)$ is not a subalgebra (unless $g = 0$ or $g \gg 0$). Nevertheless, they do form a graded filtration $\mathbb{TL}_r^0 \subset \mathbb{TL}_r^1 \subset \dots \subset \mathbb{TL}_r^{g \gg 0} = \mathbb{P}_r$ since $\mathbb{TL}_r^{g_1} \cdot \mathbb{TL}_r^{g_2} \subseteq \mathbb{TL}_r^{g_1 + g_2}$.

Lemma 4.2. *The restriction of the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{P}_r(x, y)$ to \mathbb{TL}_r^g is non-degenerate.*

Proof. To show that the determinant of the Gram matrix of \mathbb{TL}_r^g is non-zero one proceeds as in the proof of Theorem 2.1 (which deals with the case of \mathbb{TL}_r^0). \square

This allows us to define, as before, an orthogonal projection $\text{proj}_{\mathbb{TL}_r^g}^\perp : \mathbb{P}_r \rightarrow \mathbb{TL}_r^g$. The following generalizes Theorem 3.9.

Theorem 4.3. *Let ϕ be a partition. Then*

$$\text{proj}_{\mathbb{TL}_r^g}^\perp(\mathfrak{p}_\phi) = \sum_{\pi \in \Phi^{\leq g}} \vartheta_{\phi, \pi}^g \mathfrak{p}_\pi.$$

Proof. Analogous to the proof of Theorem 3.9. \square

4.3. The higher genus chromatic join matrix. The *higher genus chromatic join matrix* $\mathbf{M}_r^g(t)$ is defined by $[\mathbf{M}_r^g(t)]_{\pi_1, \pi_2} = \langle \mathfrak{p}_{\pi_1}, \mathfrak{p}_{\pi_2} \rangle$ for $\pi_1, \pi_2 \in \Phi^{\leq g}$. Clearly, $[\mathbf{M}_r^g(t)]_{\pi_1, \pi_2}$ is t to the power which is the number of parts in the join of partitions π_1 and π_2 . Note that $\mathbf{M}_r^0(t) = \mathbf{M}_r(t, 1)$.

Using the structure of the Temperley-Lieb algebra we were able to calculate the determinant of $\mathbf{M}_r^0(t)$, which factored as products of bivariate Chebyshev polynomials (evaluated at $y = 1$). These polynomials reflected the structure of $\mathbb{TL}_r(t)$. In the case of higher genus join matrices we do not have a similar algebra within which to work since (as noted before) $\mathbb{TL}_r^g(t)$ for $g > 0$ are only subspaces of $\mathbb{P}_r(t)$ and are not closed under multiplication. However, it is still interesting to compute $\det \mathbf{M}_r^g(t)$ for general g . If $\det \mathbf{M}_r^g(t)$ were to factor well then this may shed light, for instance, on the existence of yet another algebra which serves as a natural setting for partitions of genus $\leq g$. Unfortunately, we were unable to compute explicitly $\det \mathbf{M}_r^g(t)$ for general g except in the cases $g = 0$ (Corollary 3.5) and $g \gg 0$ which is done below. The case $g \gg 0$ is

where we consider all partitions. The method used below is unlikely to generalize but we consider it is worthwhile mentioning nevertheless.

4.3.1. *The case $g \gg 0$.* The key to computing $\det \mathbf{M}_r^\infty(t)$ is Lindström's theorem. We can apply it since the lattice of all partitions is closed under join. This is no longer true if we look at the sublattice of partitions of genus greater or equal to g . For example, the join of planar partitions $\{[1, 3], [2], [4]\}$ and $\{[1], [3], [2, 4]\}$ is $\{[1, 3], [2, 4]\}$ which has genus 1. Thus we cannot use the same method to compute $\det \mathbf{M}_r^g(t)$ in general.

Theorem 4.4 (Lindström [L]). *Let \mathcal{L} be a finite lattice (or \vee -semilattice) and let $F: \mathcal{L} \rightarrow \mathbb{A}$ where \mathbb{A} is a ring. Let $F(u) = \sum_{w \geq u} f(w)$ where u is in \mathcal{L} . Then $\det[F(u \vee v)]_{u,v \in \mathcal{L}} = \prod_{w \in \mathcal{L}} f(w)$.*

Proposition 4.5. $\det \mathbf{M}_r^\infty(t) = \prod_{i=0}^r (x - i)^{b(r,i)}$

Here $b(r, i) = B(r) - \sum_{j=0}^i S(r, j)$ where $S(r, j) = \begin{bmatrix} r \\ j \end{bmatrix} \frac{(e^t - 1)^j}{j!}$ (Stirling number of the second kind) is the number of partitions of $\{1, \dots, r\}$ into j blocks and $B(r) = \begin{bmatrix} r \\ r \end{bmatrix} \exp(e^t - 1)$ (Bell number) is the total number of partitions. For a complete proof of the proposition see [J] Theorem 2.1.

5. THE RIBBON ALGEBRA $\mathbb{R}\mathbb{A}_r(t)$

In this section we introduce the Ribbon Algebra which is a lifting of the Partition Algebra. This is an algebra that we believe is worthy of further study.

5.1. A natural bijection between rooted hypermonopoles and permutations. In Section 4 we saw that we can represent a partition as a hypermonopole on a surface of genus g . However, not all hypermonopoles correspond to a partition. Consider the two hypermonopoles in Figure 10. The one on the left has genus 0 and corresponds to the partition $\{[1, 2, 3]\}$. The one on the right has genus 1 and does not correspond to any partition. In fact, hypermonopoles are in bijection with permutations. For example, the hypermonopole on the left encodes the permutation (123) and the one on the right encodes permutation (132).

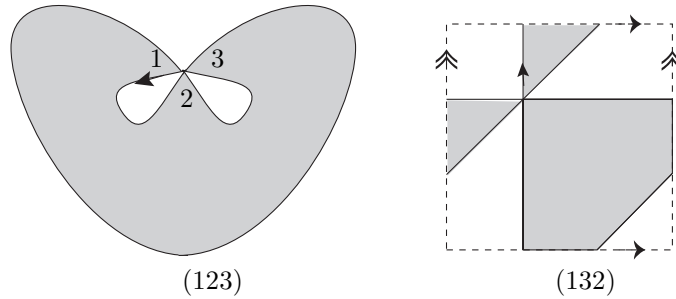


FIGURE 10. Rooted hypermonopoles corresponding to permutations (123) (left) and (132) (right) of genera 0 and 1 respectively.

Theorem 5.1. *There is a natural bijection between rooted hypermonopoles with r edges and the permutation group \mathfrak{S}_r acting on r symbols. The number of shaded regions in the hypermonopole equals the number of cycles in the permutation.*

Proof. The idea is to generalize the picture in Figure 10 which encodes the permutations (123) and (132). Each cycle of a permutation corresponds to a shaded face. □

With the partition algebra serving as a model, one can define a multiplication operator (\bullet) on the set of all hypermonopoles. This leads to the Ribbon Algebra which we now describe. The easiest way to define the multiplication is to view the hypermonopoles as ribbon graphs (whence the name Ribbon Algebra), which carry the embedding. The basis elements of the Ribbon Algebra $\mathbb{R}\mathbb{A}_r(t)$ are rooted ribbon hypermonopoles, and typical basis elements are displayed as the left hand diagrams of Figures 11 and 12.

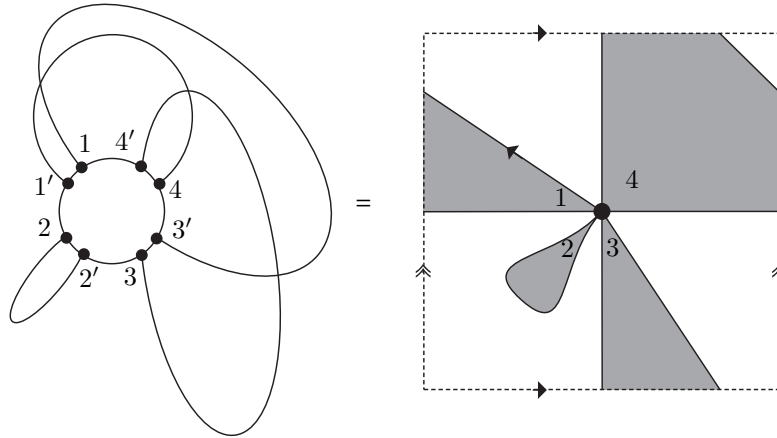


FIGURE 11. A ribbon graph and the corresponding rooted hypermonopole.

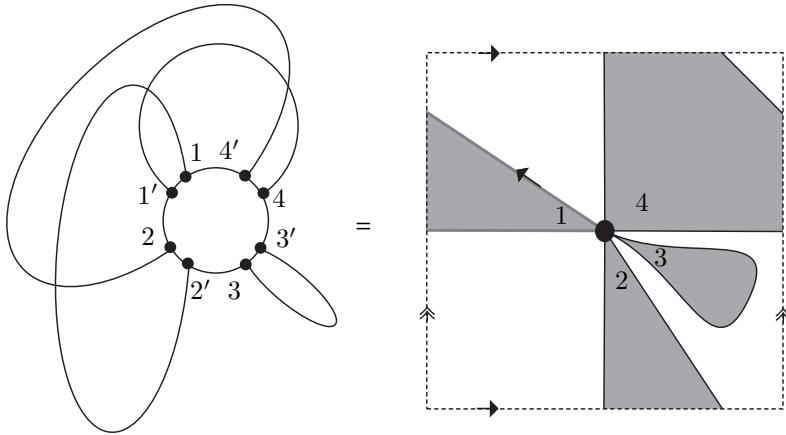


FIGURE 12. Another ribbon graph and the corresponding rooted hypermonopole.

Such elements correspond bijectively to hypermonopoles since the ribbon structure gives a unique embedding of the monopole into a surface. This correspondence is depicted in Figures 11 and 12. The genus of this 2-cell embedding is the genus of the ribbon hypermonopole.

Using the rooting we can concatenate two ribbon hypermonopoles so that the root edges coincide. This way we obtain a new rooted ribbon hypermonopole, thereby providing $\mathbb{R}\mathbb{A}_r(t)$ with a multiplication operator. Figure 13 illustrates an example of concatenation. Each additional component which arises from concatenation (as in Figure 13) is marked by a factor of t . Multiplication in the Ribbon Algebra mimics

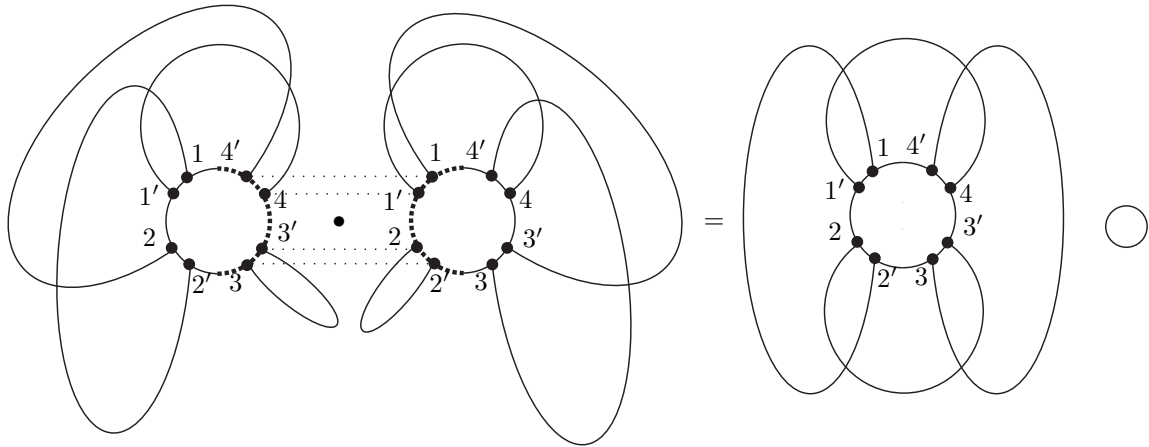


FIGURE 13. The product of two ribbon hypermonopoles.

that of \mathbb{P}_r and \mathbb{TL}_r . It is associative for the same reasons that \mathbb{P}_r and \mathbb{TL}_r are associative. There is also a morphism of algebras $\psi : \mathbb{RA}_r(t) \rightarrow \mathbb{P}_r(t)$. This map is the analogue of the usual forgetful map from permutations to partitions which forgets the structure within each cycle of a permutation (for instance, both permutations (123) and (132) are mapped to the partition $\{1, 2, 3\}$).

The Ribbon Algebra comes with a graded filtration. If $\mathbb{RA}_r^g(t)$ denotes the subspace spanned by the ribbon hypermonopoles of genus less than or equal to g then $\mathbb{RA}_r^0(t) \subset \mathbb{RA}_r^1(t) \subset \dots \subset \mathbb{RA}_r^{g \gg 0}(t) = \mathbb{RA}_r(t)$. The morphism ψ is compatible with the filtration in the sense that $\psi : \mathbb{RA}_r^g \rightarrow \mathbb{P}_r^g$. Moreover, just as with $\mathbb{P}_r(t)$, we have $\mathbb{RA}_r^0(t) \cong \mathbb{TL}_r(t, 1)$.

Although, using $\mathbb{P}_r(t)$ again as a model, there is an obvious candidate for a trace on $\mathbb{RA}_r(t)$, this trace is degenerate since $\text{tr}(\mathbf{a}) = \text{tr}(\psi(\mathbf{a}))$. This is a reason that $\mathbb{P}_r(t)$ rather than $\mathbb{RA}_r(t)$ is the correct algebra for working with higher genus Tutte Invariants. However, there is also a trace obtained by counting strands in the closure of the ribbon rather than the number of shaded regions. We believe it would be fruitful to understand in greater detail the structure of $\mathbb{RA}_r(t)$ and consequently suggest the following questions:

- Is $\mathbb{RA}_r(t)$ a semisimple algebra and, if so, what are the irreducible modules (how many are there and what are their dimensions)?
- Since rooted hypermonopoles are in bijection with \mathfrak{S}_r , it follows that $\dim(\mathbb{RA}_r(t)) = r!$. How does $\mathbb{RA}_r(t)$ compare with the symmetric group algebra $\mathbb{C}[\mathfrak{S}_r]$? Is there a relation between the two multiplication operators?
- Is the suggested trace (given above) non-degenerate?

5.2. Counting Hypermonopoles According to Genus. In this section we determine the generating series for permutations (or, equivalently, hypermonopoles) according to genus. Notice that permutations of genus 0 (i.e. hypermonopoles of genus 0) are in bijection with partitions of genus 0 and are hence counted by Catalan numbers. This is not true in higher genera. For instance, Figure 10 showed a genus 1 hypermonopole, namely (132), which in fact seems to encode a partition of genus 0.

It is convenient briefly to use the usual terminology of hypermaps. The *hyperedges* of a hypermap H are the faces of one colour, which is to be specified, and the *faces* of H are the faces of the other colour. The *hyperedge partition* is defined in the usual. However, the *degree* of a vertex of H is the number of hyperedges

incident with it. The *edges* of H are the edges of the underlying graph. H is rooted by selecting a mutually incident vertex, edge and face.

A *hypermonopole* is a hypermap with a single vertex. The degree of this vertex in H is therefore precisely half the degree of the vertex in the underlying graph. Let x_s, y_s, z_s be indeterminates marking hyperfaces (shaded faces), faces (unshaded faces) and vertices, respectively, of degree s . Let $\mathbf{x} = (x_1, \dots)$, $\mathbf{y} = (y_1, \dots)$ and $\mathbf{z} = (z_1, \dots)$. The *vertex partition* ν of H is $(\nu_1, \nu_2, \dots) = (1^{k_1}, 2^{k_2}, \dots)$ where k_s is the number of vertices in H of degree s . Let ϕ, η be the hyperedge (shaded face) and face (unshaded face) partitions, respectively. The rooted hypermap generating series for orientable surfaces is

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{\nu, \phi, \eta \in \Phi} h(\phi, \eta, \nu) \mathbf{x}_\phi \mathbf{y}_\eta \mathbf{z}_\nu$$

where $h(\phi, \eta, \nu)$ is the number of rooted hypermaps with vertex partition ν , hyperedge partition ϕ and face partition η .

Let $p_\nu(\mathbf{x}), s_\nu(\mathbf{x})$ denote, respectively, the power sum symmetric function and the Schur function in \mathbf{x} with index ν and let H_θ be the product of the hook lengths of cells in the Ferrers diagram of θ .

Theorem 5.2 (see equation (1) [GJ2]).

$$H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z})) = t \frac{\partial}{\partial t} \log \sum_{\theta \in \Phi} t^{|\theta|} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}) \Big|_{t=1}.$$

The following result gives the generating series for rooted hypermonopoles. The results concerning Schur functions and characters of irreducible representations of the symmetric group that are used here may be found in [Mac]. Let $(x)_k$ denote $x(x-1) \cdots (x-k+1)$, where $k \geq 0$.

Corollary 5.3. *Let $a_r(p, q)$ be the number of rooted hypermonopoles with p hyperedges, q faces and a vertex of degree r (degree $2r$ in the underlying graph). Then*

$$A_r(x, y) = \sum_{p, q \geq 0} a_r(p, q) x^p y^q = \frac{1}{r!} \sum_{k=0}^{r-1} (-1)^{r-k-1} \binom{r-1}{k} (x+k)_r (y+k)_r.$$

Proof. Let $A_r(\mathbf{x}, \mathbf{y})$ be the generating series for rooted hypermonopoles with respect to hyperedge partition and face partition, and with vertex degree r (degree $2r$ as a graph). Then

$$A_r(p(\mathbf{x}), p(\mathbf{y})) = [p_r(\mathbf{z})] H(p_r(\mathbf{x}), p_r(\mathbf{y}), p_r(\mathbf{z}))$$

so, from Theorem 5.2,

$$A_r(p(\mathbf{x}), p(\mathbf{y})) = r \sum_{\theta \vdash r} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) [p_r] s_\theta.$$

Now

$$s_\theta = \sum_{\alpha \vdash r} g(\alpha) \chi_\alpha^\theta p_\alpha$$

where $g(\alpha) = \prod_{i \geq 1} (i^{a_i} a_i!)^{-1}$ with $\alpha = (1^{a_1}, 2^{a_2}, \dots)$, and χ_α^θ is the character of the ordinary irreducible representation of \mathfrak{S}_r indexed by θ and evaluated on the conjugacy class indexed by $\alpha \vdash r$. Then $[p_r] s_\theta = \chi_{(r)}^\theta / r$. But $\chi_{(r)}^\theta = (-1)^k$ if $\theta = (r-k, 1^k)$ from some k such that $0 \leq k < r$, and is 0 otherwise. Thus

$$A_r(p(\mathbf{x}), p(\mathbf{y})) = \sum_{k=0}^{r-1} H_{(r-k, 1^k)} s_{(r-k, 1^k)}(\mathbf{x}) s_{(r-k, 1^k)}(\mathbf{y}).$$

But $\sum_{k=0}^{r-1} \lambda_{\alpha}^{(r-k, 1^k)} u^k = (1+u)^{-1} \prod_{i=1}^r (1 - (-u)^i)^{a_i}$, where not all of the a_i 's are zero. Then

$$s_{(r-k, 1^k)} = [u^k w_1^r] (1+u)^{-1} \left(\exp \left(\sum_{i \geq 1} \frac{1}{i} (1 - (-u)^i) p_i w_1^i \right) - 1 \right).$$

We introduce the homomorphism

$$L: p_i(\mathbf{x}) \mapsto -x \text{ for all } i \geq 1.$$

Then $L s_{(r-k, 1^k)}(\mathbf{x}) = [u^k w_1^r] (X^x - 1)$, where $X = (1 - w_1)(1 + uw_1)^{-1}$. Now rewrite X as

$$X = 1 - w_1(1+u)((1+uw_1)^{-1}).$$

Let ψ_x be the linear operator defined by

$$\psi_x: \binom{x}{k} \mapsto x^k \text{ for } k \geq 0.$$

In particular, if a is independent of x , then $\psi_x(1+a)^x = (1-ax)^{-1}$. Then

$$\psi_x(1+u)^{-1} X^x = (1+u)^{-1} - xw_1(1+xw_1)^{-1}(1+ub)^{-1}$$

where $b = w_1(1+x)(1+xw_1)^{-1}$, whence

$$\psi_x[u^k](1+u)^{-1}(X^x - 1) = (-1)^{k+1} x(1+x)^k w_1^{k+1} (1+xw_1)^{k+1}$$

if $r > 0$. But $\psi_x^{-1} x^{r-k} (1+x)^k = \binom{x+k}{r}$, so

$$[u^k w_1^r] (1+u)^{-1} (X^x - 1) = (-1)^r \binom{r-1}{k} \binom{x+k}{r}.$$

It follows that

$$A_r(-x, -y) = \sum_{k=0}^{r-1} (-1)^k H_{(r-k, 1^k)} \binom{r-1}{k}^2 \binom{x+k}{r} \binom{y+k}{r}.$$

Let f^θ be the degree of the irreducible representation indexed by θ . Then $H_\theta = r!/f^\theta$, so

$$H_{(r-k, 1^k)} = r! \binom{r-1}{k}^{-1}$$

since $f^{(r-k, 1^k)} = [u^k] (1+u)^{r+1} = \binom{r-1}{k}$. Then by routine manipulation

$$A_r(x, y) = r! \sum_{l=0}^{r-1} (-1)^{r-l-1} \binom{r-1}{l} \binom{x+l}{r} \binom{y+l}{r},$$

and the result follows. \square

The series $A_r(x, y)$ appears to be of total degree $2r$. It is, however, of total degree $r+1$, and the following result gives a degree-respecting form for the series. We are indebted to George Andrews for the following proof.

Corollary 5.4.

$$A_r(x, y) = (r-1)! \sum_{k=0}^{\lfloor (r+1)/2 \rfloor} k \binom{x+y+4-2k}{r-2k+1} \binom{x}{k} \binom{y}{k}.$$

Proof. We note first that, from Corollary 5.3, $A_r(x, y)$ can be expressed hypergeometrically as

$$A_r(x, y) = (-1)^{r-1} r! \binom{x}{r} \binom{y}{r} \cdot {}_3F_2 \left[\begin{matrix} -r+1 & x+1 & y+1 \\ - & y-r+1 & x-r+1 \end{matrix} \middle| 1 \right].$$

But, from [Ba] (equ. (1), Sect. 4.5), we have

$${}_4F_3 \left[\begin{matrix} a & b & c & -m \\ - & 1+a-b & 1+a-c & w \end{matrix} \middle| 1 \right] = \frac{(w-a)^{(m)}}{(w)^{(m)}} \cdot G$$

where

$$G = {}_5F_4 \left[\begin{matrix} 1+a-w & \frac{1}{2}a & \frac{1}{2}(1+a) & 1+a-b-c & -m \\ - & 1+a-b & 1+a-c & \frac{1}{2}(1+a-w-m) & 1+\frac{1}{2}(a-w+m) \end{matrix} \middle| 1 \right],$$

which expresses a nearly poised ${}_4F_3$ in terms of a Saalschützian ${}_5F_4$. Letting $w \rightarrow a$, we have

$${}_3F_2 \left[\begin{matrix} -m & b & c \\ - & 1+a-b & 1+a-c \end{matrix} \middle| 1 \right] = \frac{(-1)^m}{(a)^{(m)}} \sum_{m \leq 2j \leq 2m} \frac{(a)^{(2j)} (1+a-b-c)^{(j)} (-m)^{(j)}}{(a+a-b)^{(j)} (a+c-c)^{(j)} (2j-m)!}.$$

Now set $w \mapsto a$, $b \mapsto x+1$, $c \mapsto y+1$, $m \mapsto r-1$, and $a \mapsto x+y-r+1$ so,

$$A_r(x, y) = \sum_{\lceil (r-1)/2 \rceil \leq j \leq r-1} \frac{(x+y)^{(2j-r+1)} (x-r+j+1)^{(r-j)} (y-r+j+1)^{(r-j)} r!}{(r-j)! (r-j-1)! (2j-r+1)!},$$

and the result follows. \square

The terms of top degree $r+1$ in $A_r(x, y)$ are given by by $\frac{1}{r} \sum_{j>0} \binom{r}{2j-1} \binom{2j-1}{j} (x+y)^{r-2j+1} x^j y^j$. In hypergeometric form this is

$$\sum_{s \geq 0} x^{r-s+1} y^s \binom{r-1}{s-1} {}_2F_1 \left[\begin{matrix} -r+s & -s+1 \\ - & 2 \end{matrix} \middle| 1 \right],$$

which is equal to

$$\frac{1}{r+1} \sum_{s=0}^{r-1} x^{r-s} y^{s-1} \binom{r+1}{s+1} \binom{r-1}{s},$$

by the Chu-Vandermonde theorem. Furthermore, at $x = y = 1$, this reduces to $\frac{1}{r+1} \binom{2r}{r}$, again by the Chu-Vandermonde theorem.

An indeterminate z can be introduced to mark genus, by the Euler-Poincaré formula. This series is $z^{(r+1)/2} A_r(xz^{-1/2}, yz^{-1/2})$ which, with a minor abuse of notation, we shall denote by $A_r(x, y, z)$. Clearly, A_r is symmetric in x and y . Table 1 gives A_1, \dots, A_6 compactly in terms of the monomial symmetric functions m_α in x and y .

r	$A_r(x, y, z)$
1	$m_{(1^2)}$
2	$m_{(2,1)}$
3	$(m_{(3,1)} + 3m_{(2^2)}) + m_{(1^2)}z$
4	$(m_{(4,1)} + 6m_{(3,2)}) + 5m_{(1^2)}z$
5	$(m_{(5,1)} + 10m_{(4,2)} + 20m_{(3^2)}) + (15m_{(3,1)} + 40m_{(2^2)})z + 8m_{(1^2)}z^2$
6	$(m_{(6,1)} + 15m_{(5,2)} + 50m_{(4,3)}) + (35m_{(4,1)} + 175m_{(3^2)})z + 84m_{(2,1)}z^2$

TABLE 1. The rooted hypermonopole generating series A_1, \dots, A_6

For example, from Table 1, there are 10 rooted hypermonopoles with 4 faces of one colour and 2 faces of the other colour, and having 5 edges (and therefore genus 0).

Let $G_r(z) = A_r(1, 1, z)$. The coefficients of this polynomial are given in Table 2. We note that, as expected,

r	g		
	0	1	2
1	1		
2	2		
3	5	1	
4	14	10	
5	42	70	8
6	132	420	168

TABLE 2. The number rooted hypermonopole with vertex degree $2r$ and genus g .

the numbers for $g = 0$ are the Catalan numbers.

The coefficients of $A_r(x, y)$ are found by observing that $(x + k)_r = r! \sum_{i=0}^k \binom{k}{i} (x)_{r-k+i} / (r - k + i)!$ and that $(x)_n = \sum_{m=0}^n s_n^{(m)} x^m / m!$ where $n \geq 0$, and $s_n^{(m)}$ is a Stirling number of the first kind defined by $(\log(1 + x))^m / m! = \sum_{n \geq m} s_n^{(m)} x^n / n!$. Thus $[x^p](x + k)_r = r! \sum_{i=0}^k \binom{k}{i} s_{r-k+i}^{(p)} / (r - k + i)!$.

6. THE CONNEXION BETWEEN THE TUTTE INVARIANTS, THE BIRKHOFF-LEWIS EQUATIONS AND THE FOUR-COLOUR THEOREM

The familiar approach to proving the Four-Colour Theorem involves first using a discharging algorithm to find a complete set of unavoidable graphs, that is, a set such that any minimal counterexample to the theorem would have to contain one of these graphs. Kempe chains are then used to obtain relations between (constrained) chromatic chromials of these graphs. If the graphs are large enough then these relations can be used to obtain a contradiction (*i.e.* thus concluding the graphs cannot in fact occur as a subgraph of a minimal counterexample to the Four-Colour Theorem). However, as the graphs grow so does their number (this forces the use of a computer). Recent approaches such as those of Appel and Haken ([AH1],[AHK]) or Robertson *et al.* [RSST] involve reducing the size of the set of unavoidable graphs. The relations obtained from Kempe chains may only comprise a fraction of all the existing relations. Consequently, one characteristic of their approach is that many graphs are required but only some of the relations are used.

In contrast one can try to generate more relations in the hope of being able to use a smaller set of graphs. The Tutte invariants provide a way to obtain all linear relations (see theorem (2.4)) in a systematic manner rather than by using Kempe chains. The tradeoff is having to generate more equations instead of having to check many graphs, so we are somewhat pessimistic that this approach will lead to an algebraic proof of the Four-Colour Theorem which is less dependent on computer assistance than the current one.

We now describe the rôle played by the Tutte invariants in the construction of the Birkhoff-Lewis Equations [BL] and summarise how these relate to the Four Colour Theorem. Let $\text{int}_J G$ and $\text{ext}_J G$ be, respectively, the subgraphs of G in the interior and exterior of J that, by convention, include J .

6.1. Constrained chromials. Let G be a graph and let J be a circuit of G of length r with a non-trivial interior and exterior. The *constrained chromial* $\dot{P}_{(G, \phi)}(t)$ of (G, ϕ) is the number of ways of colouring G with t colours such that the vertices of G in the same block of ϕ receive the same colour and vertices in different blocks receive different colours. The chromial $P_G(t)$ may be expressed immediately in terms of constrained

chromials of graphs with fewer vertices as follows. Let $(t)_k = t(t-1) \cdots (t-k+1)$, where k is a non-negative integer.

Lemma 6.1. *Let G be a graph with a circuit J . Then*

$$P_G(t) = \sum_{\phi \in \Phi(J)} \frac{1}{(t)_{l(\phi)}} \dot{P}_{(\text{int}_J G, \phi)}(t) \dot{P}_{(\text{ext}_J G, \phi)}(t).$$

(Note that $\dot{P}_{(\text{ext}_J G, \phi)}(t)/(t)_{l(\phi)}$ is a polynomial in t .)

Proof. Any colouring of G induces a colouring of J and thence a unique partition $\phi \in \Phi(J)$ whose blocks comprise vertices of J with the same colour. The number of colourings of $(\text{int}_J G, \phi)$ with at most t colours is $\dot{P}_{(\text{int}_J G, \phi)}(t)$. Each such colouring induces a colouring of J with $l(\phi)$ colours selected from $\{1, \dots, t\}$. But this colouring of J induces a unique colouring of $(\text{ext}_J G, \phi)$, which is counted by $\dot{P}_{(\text{ext}_J G, \phi)}(t)/(t)_{l(\phi)}$. The result follows. \square

Lemma 6.2. *Let G be a graph and let J be a circuit of G . Then, for $\phi \in \Phi(J)$,*

$$(6) \quad P_{(\text{ext}_J G, \phi)}(t) = \sum_{\substack{\phi' \in \Phi(J), \\ \phi' \succeq \phi}} \dot{P}_{(\text{ext}_J G, \phi')}(t).$$

Proof. Trivial. \square

6.2. Examples. Substituting equation (6) into Tutte's relations (1) yield what are commonly known as the *Birkhoff-Lewis Equations*. Birkhoff and Lewis [BL] gave such equations explicitly for $r = 4, \dots, 7$, and proposed a somewhat arbitrary method for constructing them in general. Although the function $\vartheta_{\phi, \pi}$ was implicit in their work, there was not an explicit algebraic characterisation of it until the appearance of Tutte's Theorem.

6.2.1. The 4-ring equation and an example. The following example illustrates the use of constrained chromials and Tutte's equations to show that the four-wheel W_4 cannot occur as a subgraph of a minimal counterexample to the Four Colour Theorem. The example serves as the model for the method of excluding a particular graph as a subgraph of a minimum counterexample (refer to the Introduction for the broad picture).

$\dot{P}_{(\text{ext}_J G, \phi)}(t)$	$\phi_i \in \Phi(J)$
\dot{A}_i	$\{[i, i+2], [i+1, i+3], \}$
\dot{B}_i	$\{[i, i+2], [i+1], [i+3]\}$
\dot{C}_i	$\{[i], [i+1], [i+2], [i+3], [i+4]\}$

TABLE 3. Notation for the constrained chromials for the 4-ring

Lemma 6.3. *If G is a minimum counterexample, then W_4 is not a subgraph of G .*

Proof. Suppose G is a minimal counterexample containing W_4 as a subgraph. Let J be such that $\text{int}_J G = W_4$. Let $V(J) = \{1, 2, 3, 4\}$. The constrained chromials of $\text{ext}_J G$ are \dot{A} , \dot{B}_1 , \dot{B}_2 and \dot{C} , as indicated in Table 3. For example, \dot{A} is the constrained chromial corresponding to the only non-planar partition $\{[1, 3], [2, 4]\}$. Let A , B_1 , B_2 and C denote the corresponding free chromials.

There are two sources of relations between these constrained chromials, namely Tutte's relations (Theorem 2.1) and Lemma 6.1, and we consider these in turn.

Theorem 2.1 gives one relation between the free chromials for each non-planar partition. Since the 4-ring has only one such partition, there is precisely one Tutte relation which, by direct calculation, is

$$(t^2 - 5t + 5)A - (t - 1)(t - 3)(B_1 + B_2) + (t - 2)(t - 3)C = 0.$$

From Lemma 6.2, which expresses the free chromials in terms of the constrained chromials, we obtain

$$\begin{aligned} A &= \dot{A}, \\ B_1 &= \dot{B}_1 + \dot{A}, \\ B_2 &= \dot{B}_2 + \dot{A}, \\ C &= \dot{C} + \dot{B}_1 + \dot{B}_2 + \dot{A} \end{aligned}$$

so, substituting these into Tutte's relation gives

$$(7) \quad \dot{A} - (t - 3)(\dot{B}_1 + \dot{B}_2) + (t - 2)(t - 3)\dot{C} = 0.$$

We now use Lemma 6.1 to obtain further relations. Suppose G is not 4-colourable. Then $P_G(4) = 0$ so, from Lemma 6.1, $\dot{P}_{(\text{ext}_J G, \phi)} = 0$ whenever $\dot{P}_{(\text{int}_J G, \phi)} \neq 0$. Thus $\dot{P}_{(W_4, \phi)} \neq 0$ if $\phi = \{[1, 3], [2, 4]\}$ (since the vertices of W_4 on the 4-ring may be coloured 1212 in cyclic order) and hence $\dot{A}(4) = 0$. Similarly, $\dot{B}_1(4) = \dot{B}_2(4) = 0$. This accounts for all of the relations that can be obtained in this way from Lemma 6.1. Combining these with (7) gives $\dot{C}(4) = 0$. But, from Lemma 6.2, we see that $\dot{P}_{(\text{ext}_J G)}(4) = \dot{A}(4) + \dot{B}_1(4) + \dot{B}_2(4) + \dot{C}(4) = 0$. Thus $\text{ext}_J G$ is a subgraph of G that is *not* 4-colourable, so G is not a minimum counterexample, forcing the contradiction. \square

6.2.2. *The 5-ring equations and an example.* As a further example, we use the Tutte relations (Theorem 2.1) to rederive the Birkhoff-Lewis Equations for the 5-ring.

We shall index the vertices of the 5-ring J by $1, \dots, 5$ in cyclic order. In Table 4, it is understood that the

$\dot{P}_{(\text{ext}_J G, \phi)}(t)$	$\phi_i \in \Phi(J)$
\dot{A}_i	$\{[i], [i - 1, i + 2], [i + 1], [i + 3]\}$
\dot{B}_i	$\{[i], [i - 1, i + 2], [i + 1, i + 3]\}$
\dot{G}	$\{[1], [2], [3], [4], [5]\}$

TABLE 4. Notation for the constrained chromials for the 5-ring.

elements in partitions are taken modulo 5, and that 0 is replaced by 5. Suppressing the (routine) details of the matrix computation, we have, from Theorem 2.1, that

$$(t^2 - 3t + 1)B_{i+4} = G + (t - 2)(A_{i+1} + A_{i+2}) + A_{i+4}.$$

These can be expressed wholly in terms of constrained chromials through the relations

$$\begin{aligned} A_2 &= \dot{A}_2 + \dot{B}_5 + \dot{B}_4, \\ A_3 &= \dot{A}_3 + \dot{B}_1 + \dot{B}_5, \\ A_5 &= \dot{A}_5 + \dot{B}_2 + \dot{B}_3, \\ B_5 &= \dot{B}_5, \\ G &= \dot{G} + (\dot{A}_1 + \dots + \dot{A}_5) + (\dot{B}_1 + \dots + \dot{B}_5) \end{aligned}$$

obtained from Lemma 6.2. Substituting these into the above relation, we have

$$\dot{G} + (t-2)(t-3)\dot{B}_{i+4} + \dot{A}_i + \dot{A}_{i+3} - (t-3)(\dot{A}_{i+1} + \dot{A}_{i+2} + \dot{B}_1 + \dot{B}_{i+3}) = 0.$$

Let R_1, \dots, R_5 denote these five relations.

These relations differ from the Birkhoff-Lewis Equations (equ. (8.4) [BL]) which were derived by the method of Kempe chains. We can recover the Birkhoff-Lewis Equations from R_1, \dots, R_5 by taking the linear combination $R_1 + (t-3)R_2 + (t^2 - 5t + 5)R_3 + (t-3)R_4 + R_5$, giving the relation $(t^2 - 3t + 1)(\dot{G} + (t-3)(t-4)\dot{B}_2) = (t^2 - 3t + 1)(t-4)(\dot{A}_4 + \dot{A}_5)$, whence, cancelling the factor $t^2 - 3t + 1$, which has no integer roots, $\dot{G} + (t-3)(t-4)\dot{B}_2 = (t-4)(\dot{A}_4 + \dot{A}_5)$, again with the cyclic permutation of indices modulo 5. This is indeed equation (8.4) of [BL]. The fact that we needed to take linear combinations suggests that this approach may differ from the Kempe chain approach employed by Birkhoff and Lewis.

6.2.3. *The 6-ring equations and an example.* As a final example, we examine the case $\text{int}_J G = T$, where T is given in Figure 14(a). It is considered by Thomas [To] as a candidate for an unavoidable set of configurations, with the hope that this may lead to a set that is smaller than the set used in [RSST]. Its boundary is a circuit of degree 6, so the Birkhoff-Lewis Equations, which are known for the 6-ring, are required. We shall see that the Birkhoff-Lewis Equations for the 6-ring are not enough to prove that T cannot occur as a subgraph of a minimum counterexample (If they were, the present proof of the Four Colour Theorem could be substantially shortened.).

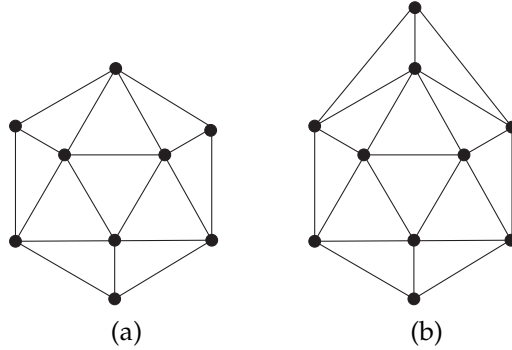


FIGURE 14. Candidates for an unavoidable set.

The names of each of the constrained chromials $\dot{P}_{(\text{ext}_J G, \phi)}(t)$ are given in Table 5 for each partition ϕ of the vertices of J . The index i takes values in $\{1, \dots, 6\}$, and elements of blocks are to be evaluated modulo 6 (with 0 to be replaced with 6.) The unknown constrained chromials, are

$$(8) \quad \dot{B}_i, \dot{F}_i, \dot{G}_i, \dot{J}_i, i = 1, \dots, 6; \quad \dot{C}_i, \dot{H}_i, \dot{I}_i, \dot{K}_i, i = 1, \dots, 3; \quad \dot{E}_i, i = 1, 2; \quad \dot{A}, \dot{D}, \dot{L}.$$

since $\dot{C}_{i+3} = \dot{C}_i, \dot{E}_{i+2} = \dot{E}_i, \dot{H}_{i+3} = \dot{H}_i, \dot{I}_{i+3} = \dot{I}_i$, and $\dot{K}_{i+3} = \dot{K}_i$.

$\dot{P}_{(\text{ext}_J G, \phi)}(t)$	$\phi_i \in \Phi(J)$
\dot{A}	$\{[i, i+2, i+4], [i+1, i+3, i+5]\}$
\dot{B}_i	$\{[i], [i+1, i+3, i+5], [i+2, i+4]\}$
\dot{C}_i	$\{[i, i+3], [i+1, i+5], [i+2, i+4]\}$
\dot{D}	$\{[i, i+3], [i+1, i+4], [i+2, i+5]\}$
\dot{E}_i	$\{[i], [i+1, i+3, i+5], [i+2], [i+4]\}$
\dot{F}_i	$\{[i, i+3], [i+1, i+5], [i+2], [i+4]\}$
\dot{G}_i	$\{[i], [i+1, i+3], [i+2, i+4], [i+5]\}$
\dot{H}_i	$\{[i], [i+1, i+5], [i+2, i+4], [i+3]\}$
\dot{I}_i	$\{[i], [i+1, i+4], [i+2, i+5], [i+3]\}$
\dot{J}_i	$\{[i], [i+1, i+5], [i+2], [i+3], [i+4]\}$
\dot{K}_i	$\{[i], [i+3], [i+1], [i+2], [i+4, i+5]\}$
\dot{L}	$\{[i], [i+1], [i+2], [i+3], [i+4], [i+5]\}$

TABLE 5. Notation for the constrained polynomial for the 6-ring.

The following is the complete set of Birkhoff-Lewis Equations for the 6-ring, and is taken from [BL], p. 442.

$$(9) \quad \left\{ \begin{array}{l} (t-3) \left((t-2)\dot{A} - \dot{B}_i - \dot{B}_{i+1} + \dot{B}_{i+3} - \dot{B}_{i+5} \right) + \dot{G}_i + \dot{G}_{i+1} - \dot{H}_i = 0, \quad (.1) \\ (t-3)(\dot{B}_i - \dot{C}_i) - \dot{E}_i + \dot{F}_i = 0, \quad (.2) \\ \dot{F}_{i+1} + \dot{F}_{i+2} - \dot{G}_{i+5} - \dot{I}_i = 0, \quad (.3) \\ (t-3)(\dot{D} - \dot{C}_i) + \dot{H}_i - \dot{I}_i = 0, \quad (.4) \\ (t-3)(t-4)\dot{B}_i - (t-4)(\dot{E}_i + \dot{H}_i) + \dot{J}_i = 0, \quad (.5) \\ (t-4)(-\dot{F}_i + \dot{G}_{i+3}) - \dot{J}_{i+5} + \dot{K}_i = 0, \quad (.6) \\ (t-3)(t-4)(t-5)\dot{C}_i - (t-4)(t-5)\dot{H}_i - (t-5)\dot{K}_i + \dot{L} = 0, \quad (.7) \end{array} \right.$$

for $i = 1, \dots, 6$, and for all $t \geq 1$.

We now attempt to show that T cannot occur in a minimum counterexample. Suppose that G is a minimum counterexample with $(\text{int}_J G = T)$. It is readily found that the only partitions ϕ that allow a 4-colouring of (T, ϕ) are associated with the constrained chromials $\dot{B}_1, \dot{B}_3, \dot{B}_5, \dot{E}_1, \dot{E}_2, \dot{F}_2, \dot{F}_4, \dot{F}_6, \dot{G}_1, \dots, \dot{G}_6$. Thus

$$(10) \quad \dot{b}_1 = \dot{b}_3 = \dot{b}_5 = \dot{e}_1 = \dot{e}_2 = \dot{f}_2 = \dot{f}_4 = \dot{f}_6 = \dot{g}_1 = \dots = \dot{g}_6 = 0,$$

where, for example, \dot{b}_i denotes $\dot{B}_i(4)$.

If $\ell(\phi) > 4$, then $\dot{P}_{(\text{ext}_J G, \phi)}(4) = 0$, so we set $\dot{P}_{(\text{ext}_J G, \phi)}(t) = (t-4)\ddot{P}_{(\text{ext}_J G, \phi)}(t)$ where $\ddot{P}_{(\text{ext}_J G, \phi)}(t)$ is a polynomial in t . Let $\ddot{J}_i, \ddot{K}_i, \ddot{L}_i$ be, respectively, the corresponding polynomials for \dot{J}_i, \dot{K}_i and \dot{L}_i where these are the constrained polynomials associated with partitions ϕ having $\ell(\phi) > 4$. Now, from (9.5), we have $(t-4)((t-3)\dot{B}_i - (\dot{E}_i + \dot{H}_i) + \dot{J}_i) = 0$, so $(t-3)\dot{B}_i - (\dot{E}_i + \dot{H}_i) + \dot{J}_i = 0$ for all $t \geq 1$. Doing the same with (9.6) and (9.7), we get the equations

$$(11) \quad \left\{ \begin{array}{l} (t-3)\dot{B}_i - \dot{E}_i - \dot{H}_i + \ddot{J}_i = 0, \quad (.5') \\ -\dot{F}_i + \dot{G}_{i+3} - \dot{J}_{i+5} + \ddot{K}_i = 0, \quad (.6') \\ (t-3)(t-5)\dot{C}_i - (t-5)\dot{H}_i - (t-5)\dot{K}_i + \ddot{L} = 0, \quad (.7') \end{array} \right.$$

for all t , to replace equations (9.5), (9.6) and (9.7), respectively. Using (10), the Birkhoff-Lewis Equations therefore become

$$(12) \quad \left\{ \begin{array}{l} 2\dot{a} - \dot{b}_i - \dot{b}_{i+1} + \dot{b}_{i+3} - \dot{h}_i = 0, \quad (.1) \\ \dot{b}_i - \dot{c}_i + \dot{f}_i = 0, \quad (.2) \\ \dot{f}_{i+1} + \dot{f}_{i+2} - \dot{i}_i = 0, \quad (.3) \\ \dot{d} - \dot{c}_i + \dot{h}_i - \dot{i}_i = 0, \quad (.4) \\ \dot{b}_i - \dot{h}_i + \dot{j}_i = 0, \quad (.5') \\ -\dot{f}_i - \dot{j}_{i+5} + \dot{k}_i = 0, \quad (.6') \\ -\dot{c}_i + \dot{h}_i + \dot{k}_i + \dot{\ell} = 0, \quad (.7') \end{array} \right.$$

for $i = 1, \dots, 6$.

From (12.1) with $i = 2$, and (10), we have $\dot{b}_2 = 2\dot{a} - \dot{h}_2$. Again from (12.1), with $i = 1$, and (10), we have $2\dot{a} - \dot{b}_2 + \dot{b}_4 - \dot{b}_6 - \dot{h}_1 = 0$ so elimination of \dot{b}_2 between these gives $\dot{h}_2 + \dot{h}_3 = 2\dot{h}_1$. Similarly, $\dot{h}_1 + \dot{h}_2 = 2\dot{h}_1$ so, on eliminating \dot{h}_2 between the latter, $\dot{h}_1 = \dot{h}_3$, and similarly for \dot{h}_2 . Thus

$$(13) \quad \dot{h}_1 = \dot{h}_2 = \dot{h}_3.$$

Then, using the above expression for \dot{b}_2 we have $\dot{b}_2 = \dot{b}_4 = \dot{b}_6$ and $2\dot{a} = \dot{h}_2 + \dot{b}_2$.

From (12.2), for i odd, we have $\dot{c}_1 = \dot{f}_1, \dot{c}_2 = \dot{f}_3, \dot{c}_3 = \dot{f}_5$ and, again, for i even, $\dot{c}_1 = \dot{b}_4, \dot{c}_2 = \dot{b}_2, \dot{c}_3 = \dot{b}_6$ so combining these with the above observations of \dot{b}_i , we have $\dot{b}_2 = \dot{b}_4 = \dot{b}_6 = \dot{c}_1 = \dot{c}_2 = \dot{c}_3 = \dot{f}_1 = \dot{f}_3 = \dot{f}_5$. But, from (12.3), $\dot{i}_1 = \dot{i}_2 = \dot{i}_3 = \dot{f}_1 = \dot{f}_3 = \dot{f}_5$ so from the previous equation we have

$$(14) \quad \dot{b}_2 = \dot{b}_4 = \dot{b}_6 = \dot{c}_1 = \dot{c}_2 = \dot{c}_3 = \dot{f}_1 = \dot{f}_3 = \dot{f}_5 = \dot{i}_1 = \dot{i}_2 = \dot{i}_3.$$

From (12.4), (13) and (14), $\dot{d} = -\dot{h}_2 + 2\dot{b}_2$. Also, from (12.5'), (13) and the relation among the \dot{b}_i we have $\dot{j}_1 = \dot{j}_3 = \dot{j}_5 = \dot{h}_1 = \dot{h}_2 = \dot{h}_3$ and $\dot{j}_2 = \dot{j}_4 = \dot{j}_6 = \dot{h}_2 - \dot{b}_2$. But, from (9.6'), $\dot{j}_1 = \dot{k}_2, \dot{j}_3 = \dot{k}_1, \dot{j}_5 = \dot{k}_2$ so, combining these results,

$$(15) \quad \dot{h}_1 = \dot{h}_2 = \dot{h}_3 = \dot{j}_1 = \dot{j}_3 = \dot{j}_5 = \dot{k}_1 = \dot{k}_2 = \dot{k}_3,$$

and $\dot{j}_2 = \dot{h}_2 - \dot{b}_2$. The only new relation that (12.7') provides is $-\dot{\ell} = 2\dot{h}_2 - \dot{b}_2$.

We would have liked to have shown that enough of the constrained polynomials are zero at $t = 4$ to be able to obtain a contradiction, as was possible in the case of W_4 . Notice that we are in fact quite close in the sense that we have equations (10), (14) and (15), and we have expressed the remaining unknowns ($\dot{a}, \dot{d}, \dot{j}_2, \dot{\ell}$) only in terms of \dot{b}_2 and \dot{h}_2 . Nevertheless, the Birkhoff-Lewis Equations do not show that G does not contain T as a subgraph.

We remark in passing that Birkhoff and Lewis [BL] proved (using the Birkhoff-Lewis Equations) that the graph in Figure 14(b) cannot occur as a subgraph of a minimum counterexample to the Four Colour Theorem. Interestingly, this is the first graph in the unavoidable set given by [RSST].

Acknowledgments. S.C. thanks the Harvard Mathematics Department. He was supported by a PGSD2-NSERC grant. D.M.J. was supported by an NSERC Discovery Grant

REFERENCES

- [AH1] K. APPEL AND W. HAKEN, Every planar map is four colorable. Part I. Discharging, *Illinois J. Math.* **21** (1977), 429-490.
- [AHK] K. APPEL, W. HAKEN, AND J. KOCH, Every planar map is four colorable. Part II. Reducibility, *Illinois J. Math.* **21** (1977), 491-567.
- [Ba] W.N. BAILEY, Generalized hypergeometric series, Cambridge 1935.
- [BL] G.D. BIRKHOFF AND D.C. LEWIS, Chromatic polynomials, *Trans. AMS* **60** (1946), 355-451.
- [CJ] S. CAUTIS AND D.M. JACKSON, The matrix of chromatic joins and the Temperley-Lieb algebra, *J. Combin. Theory (Ser. B)* **89** (2003), 109-155.
- [GJ1] I.P. GOULDEN AND D.M. JACKSON, "Combinatorial Enumeration," Dover, New York, 2004.

- [GJ2] I.P. GOULDEN AND D.M. JACKSON, Maps in locally orientable surfaces, the double coset algebra, and zonal polynomials, *Can. J. Math.* **48** (1996), 569–584.
- [J] D.M. JACKSON, The lattice of non-crossing partitions, and the Birkhoff-Lewis Equations, *Europ. J. Combinatorics*, **15** (1994), 245–250.
- [Mac] I.G. MACDONALD, “Symmetric functions and Hall polynomials,” Clarendon Press, Oxford, 1979.
- [L] B. LINDSTRÖM, Determinants on semilattices, *Proc. Amer. math. Soc.* **20** (1969), 207–208.
- [R] A. RAM
- [RSST] N. ROBERTSON, D.P. SANDERS, P.D. SEYMOUR AND R. THOMAS, The four colour theorem, *J. Combin. Theory (Ser. B)* **70** (1997), 2–44.
- [S] R. P. STANLEY, *Enumerative Combinatorics*, vol. 1, 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, 1997, 1999.
- [To] Private communication per J. Geelen, December 2004.
- [T1] W.T. TUTTE, On the Birkhoff-Lewis equations, *Discrete Math.* **92** (1991), 417–425.
- [T2] W.T. TUTTE, The matrix of chromatic joins, *J. Combin. Theory (Ser. B)* **57** (1993), 269–288.