THE MATRIX OF CHROMATIC JOINS AND THE TEMPERLEY-LIEB ALGEBRA

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Abstract. We show that the matrix of chromatic joins, that is associated with the revised Birkhoff-Lewis equations, can be expressed completely in terms of functions defined on a generalization of the Temperley-Lieb algebra. We give a self-contained account of the aspects of the Temperley-Lieb algebra that are essential to this context since these are not easily obtainable in this form. Of interest in the theory of these equations are recursions for the inverse of the matrix of chromatic joins. We show that the approach given here is a natural one which provides clear insight into the investigation of the properties of the inverse, and we give an instance of a recursion. It is hoped that these techniques will be of further value in the study of the revised Birkhoff-Lewis equations.

1. Introduction

This paper is concerned with the study of the coefficient matrix $\mathbf{L}_n(q)$, whose elements are monomials in an indeterminate q, for the revised system of Birkhoff-Lewis equations for the n-ring, discovered by Tutte [Tu1] that are associated with the Four Colour Problem. This matrix, whose size is exponential in n, has been termed the *matrix of chromatic joins*. Of interest in the theory of these equations are recursions for the elements of the inverse of this matrix. The value of the determinant has been obtained by Tutte [Tu2], verifying that $\mathbf{L}_n(q)$ is generically invertible, and a recursion for the elements of the inverse has been obtained Dahab [D]. In fact, we shall study a more general matrix $\mathbf{M}_n(x,y)$ which specializes to $\mathbf{L}_n(q)$ through $x=q$ and $y=1$.

The primary purpose of this paper is to show that there is an enrichment $\mathfrak{TL}_n(x,y)$ of the Temperley-Lieb algebra $\mathfrak{TL}_n(q)$ that can be used to study recursions for the elements of the inverse of $\mathbf{M}_n(x,y)$. We have used this algebra to derive such a recursion, and this is the main result (Thm 5.4) of the paper. The recursion involves only the cover relations of the distributive lattice $\mathcal{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$ of Dyck paths in the plane ordered by geometrical inclusion ($\prec_{\mathsf{gi}}$), with scalars that are are square roots of rational functions of generalized Chebyshev polynomials (in the indeterminates x and y). This strongly suggests that $\mathfrak{TL}_n(x,y)$ may be of further value in the examination of the revised Birkhoff-Lewis equations. Generalized Chebyshev polynomials in two indeterminates have also arisen independently in the work of Dahab [D] on the determinant of the chromatic join matrix, but for different reasons. The secondary purpose of this paper is to provide a self-contained and complete account of the aspects of $\mathfrak{TL}_n(x,y)$ that are relevant to this objective. We have used $\mathfrak{TL}_n(q)$ extensively and, where we have been unable to find complete proofs of the results that we require, we have supplied proofs which are extended to $\mathfrak{TL}_n(x,y)$. A number of the more technical proofs appear in the Appendix to avoid distraction from the main argument.

We also remark that there is a close connexion between the present question and the classical problem of counting meanders, that is associated with Hilbert's sixteenth problem, by specializing x

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and y in the matrix $\mathbf{M}_n(x,y)$. For recent developments on meanders see [dF2]. The connexion with Hilbert's sixteenth problem is discussed by Arnol'd [A]. A brief account of the relevant background to the Birkhoff-Lewis equations is now given to motivate the study of the inverse of the matrix of chromatic joins.

We hope that the material on $\mathfrak{TL}_n(x,y)$ given here will lead to further progress on the revised Birkhoff-Lewis equations.

1.1. **Background to the Birkhoff-Lewis equations.** Let M be a planar map bounded by a circuit J, with n vertices, such that each face of M within J is a triangle. J is called an *n-ring*. The chromatic polynomial or *chromial* of M is a polynomial $P(\mathsf{M},\lambda)$ in λ whose value is the number of proper colourings of the vertices of M with λ colours. Details of the theory of chromials of planar triangulations, stated in the dual form, are given by Birkhoff and Lewis [BL]. The *constrained chromial* on a partition π of the vertex set of J is the polynomial $Q(\mathsf{M},\pi,\lambda)$ in λ whose value is defined to be the number of ways of colouring M so that vertices of J have the same colour if and only if they are in the same block of π. If T is a triangulation obtained from M by joining vertices of π across the exterior face of M, then the chromial of T is called a *free chromial* of M. The Birkhoff-Lewis equations express each free chromial as a linear combination of constrained chromials, consistent with the new edges. Birkhoff and Lewis then try to solve these equations, to express the constrained chromials in terms of the free chromials. They solved the equations for the 5-ring and considered the 6-ring [BL]. Further information on the Four Colour Problem is given in [SK].

Tutte [Tu2] has shown that the equations can also be obtained with the more relaxed condition that M be a triangulation and that the neighbouring elements on the n-ring be in distinct colour classes. He derived his equations from the n-ring alone, without edges between neighbouring elements, with the aid of planar partitions. A *non-planar partition* of the vertex set of J is a partition such that two vertices of one block separate two vertices of another on J. The remaining partitions are called *planar*. The *chromatic join* of a pair (π,γ) of planar partitions is the partition with the largest number of blocks that is refined by both of the partitions. The number of blocks in the chromatic join is denoted by $\chi(\pi,\gamma)$. Tutte [Tu2] has shown that the Birkhoff-Lewis equations are recoverable through the inverse of

$$\mathbf{L}_n(q) = \left[q^{\chi(\pi,\gamma)}\right]_{l(n)\times l(n)}, \tag{1}$$

the *matrix of chromatic joins*, where $l(n)$ is the number of planar partitions of J. Planar and non-planar partitions are also called, respectively, *non-crossing* and *crossing* partitions, and have been studied extensively (see [E], for example).

1.2. **Connexion with $\mathfrak{TL}_n(x,y)$.** The idea behind the paper is to use a combinatorial correspondence to express planar partitions of $\{1,\ldots,n\}$ in terms of planar diagrams called strand diagrams, which themselves can be expressed as concatenations of irreducible strand diagrams. The latter correspond to the generators $\mathsf{e}_1,\ldots,\mathsf{e}_n$ of $\mathfrak{TL}_n(x,y)$. The number $\chi(\pi,\gamma)$ of blocks in the chromatic join of π and γ can be obtained from their corresponding $\mathfrak{TL}_n(x,y)$ elements by means of a trace function. With this accomplished, we use $\mathfrak{TL}_n(x,y)$ to construct a particular linear basis to assist in a triangularization argument that expresses $\mathbf{M}_n$ in the form

$$\mathbf{M}_n = (\mathbf{P}_n^{-1})^t\mathbf{D}_n\mathbf{P}_n^{-1}, \tag{2}$$

where $\mathbf{P}_n$ is upper triangular and invertible and $\mathbf{D}_n$ is diagonal. From this the inverse of $\mathbf{M}_n$ is readily determined.

A connexion between the Four Colour Problem and $\mathfrak{TL}_n(q)$ has been observed by others, and has been expressed, for example, by Kauffman and Thomas [KT] as a combinatorial word problem in this algebra. However, the Temperley-Lieb algebra occurs in an entirely different way in this paper, through the use of planar partitions. The authors would like to credit di Francesco [dF1], and di Francesco, Golinelli and Guitter [dFGL] for describing the construction of the basis $\mathcal{B}_2$ of $\mathfrak{TL}_n(q)$ and for indicating the central ideas behind the main results in Appendices B and C.3. We have made extensive use of these papers. For further information on the Temperley-Lieb algebra the reader is directed to [BR, GdlHJ, KL].

1.3. **Organization of the paper.** In Section 2 planar partitions are associated with elements in $\mathfrak{TL}_n(x, y)$. The matrix of chromatic joins is then expressed as the Gram matrix of a bilinear form on $\mathfrak{TL}_n(x, y)$. In Sections 3 we give the construction of another basis for $\mathfrak{TL}_n(x, y)$ that is used in the diagonalization of the bilinear form. In Section 4 we show that there is an ordering of this basis such that the change of basis matrix is triangular. In Section 5 we obtain a recursion for the elements of $\mathbf{P}_n$ (Thm. 5.4) and give an example of its use in obtaining an element of $\mathbf{P}_6$. The determinant of $\mathbf{M}_n(x, y)$ (Thm 5.7) is also given. The Appendices contains proofs of the more technical results that are needed together with examples.

2. Planar partitions and the Temperley-Lieb algebra

To make the connexion between planar partitions and elements of the Temperley-Lieb algebra we use geometrical representations of planar partitions as arch diagrams and strand diagrams. The latter two diagrams are standard ones in the theory of $\mathfrak{TL}_n(q)$. To assist the discussion, we demonstrate the action of the constructions on the two planar partitions $\pi_0 = \{\{1, 2, 6\}, \{3, 4\}, \{5\}\}$ and $\gamma_0 = \{\{1, 4\}, \{2, 3\}, \{5\}, \{6\}\}$. Their chromatic join is $\{\{1, 2, 3, 4, 6\}, \{5\}\}$.

2.1. **Planar partitions.** Let π be a planar partition of $\{1, \ldots, n\}$. Then π can be represented by distributing n points equally spaced on the circumference of a circle oriented in clockwise sense in the plane, with one point on the circle distinguished. The circle is therefore rooted. We adopt the convention that the points are labelled 1 to n starting from the root in clockwise order. If $\{i_1, \ldots, i_j\}_<$ is a block of π, then edges $\{i_1, i_2\}, \ldots, \{i_j, i_1\}$ are drawn in the interior of the circle so that they intersect neither themselves nor the edges corresponding to other blocks. This defines $l(\pi)$ finite and mutually disjoint regions since π is planar. Figure 1 gives such a representation of π_0. The circle may be thought of as J and the points as the vertices in J, labelled clockwise from 1 to n.

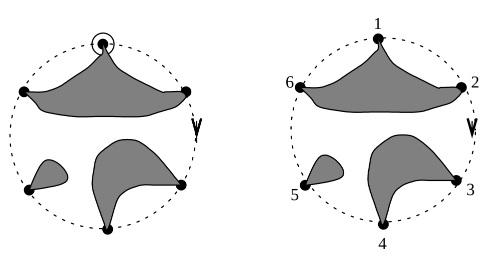
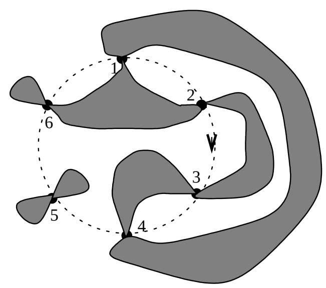


Figure 1. The planar partition π_0, with implicit labelling and shading

The blocks of the chromatic join of π and γ are the vertex sets of the components of the graph constructed by drawing the edges of γ in the exterior of the circle in the planar partition π. This



gives the construction of the chromatic join of π_0 and γ_0 from their diagrams, since the number of shaded components is equal to the number of blocks in the chromatic join.

In the planar diagrams throughout this section the labelling of points and shading of regions are redundant since they are derivable from the rooting, but they are included to assist in the descriptions of the constructions. Also, the number of planar partitions of $\{1, \ldots, n\}$ is $C_n = \binom{2n}{n}/(n+1)$, a Catalan number (see Appendix C.2).

2.2. **A bijection between planar partitions and arch diagrams.** A planar partition π may be represented by $2n$ points $1, 1', \ldots, n, n'$ distributed at unit intervals along an infinite, directed base line, drawn in the plane, together with non-intersecting semicircular arcs

$$\{i'_1, i_2\}, \{i'_2, i_3\}, \ldots, \{i'_{j-1}, i_j\}, \{i'_j, i_1\}$$

centred on the line and drawn on the same side of it. A semicircular arc is called an *arch* and the diagram is called an *arch diagram.* The construction is feasible since π is a planar partition, and is clearly reversible. In Figure 2 the regions corresponding to the blocks of π_0 are shaded.

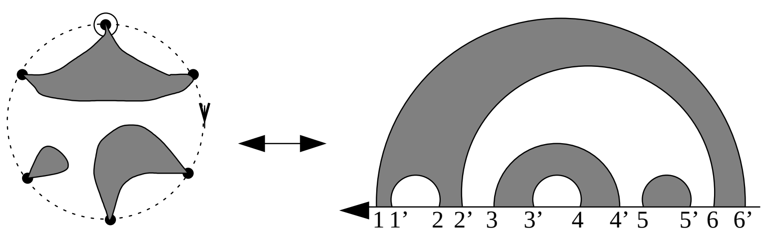


FIGURE 2. The arch diagram of π_0.

Let γ be a planar partition of $\{1, \ldots, n\}$. If the arch diagram of γ is reflected in its base line, and the points on the base line of this diagram are then identified with the points on the base line of the arch diagram of π, then the number of shaded regions in the resulting diagram is immediately identified as $\chi(\pi, \gamma)$. Figure 3 shows that $\chi(\pi_0, \gamma_0) = 2$ since there are two shaded regions in the resulting diagram each of which corresponds naturally to a block in the chromatic join of π and γ.

2.3. **A bijection between arch diagrams and strand diagram.** If n is even, the *strand diagram* of π is obtained by cutting the base line of the arch diagram of π at the mid-point of the segment $[1, n']$, and then rotating one of the semi-infinite line segments so that it is parallel to the other, with j and j' opposite $2n - j$ and $(2n - j)'$, respectively, for $j = 1, \ldots, n$. The arches are

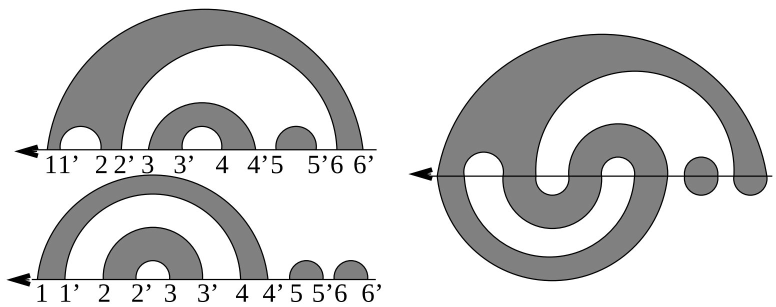
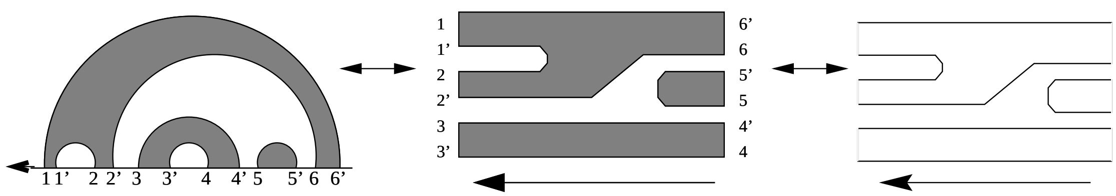


FIGURE 3. Calculation of $\chi(\pi_0, \gamma_0)$ from the arch diagrams of π_0 and γ_0.

smoothly transformed so that they do not intersect each other and are called *strands*. The diagram is given a direction, called its *orientation*, from the line containing 1 (the *head* of the diagram) to the line containing $2n$ (the *tail* of the diagram). The orientation is indicated by an arrow. The strand diagram therefore has a *top* and a *bottom*. Furthermore, two strand diagrams are equivalent if one can be deformed into the other smoothly while keeping each the endpoints i and i' fixed. For example, the arch diagram of π_0 and the corresponding strand diagram, with shading to indicate the blocks of the partition, is



This construction extends naturally to odd n, although attention has to be paid to the shading. For example, the strand diagram for the planar partition $\{\{1, 2, 5\}, \{3\}, \{4\}\}$ is the same as the right hand diagram of the three immediately above, with the sole difference that the final strand in the right hand diagram is removed. Subsequent results hold uniformly across these two cases, and therefore will not be considered separately.

The following operations on strand diagrams will be useful. The strand diagram π^t, obtained from π by reversing its orientation, is called the *transpose* of π. The *concatenation* $\pi\gamma$ of π with γ is the strand diagram obtained by identifying the i-th point from the top in the tail of π with the i-th point from the top in the head of γ, deleting closed loops that may be formed, and assigning to it the orientation of π (or, equivalently, γ). The *closure* $\overline{\pi}$ of π is the planar diagram obtained by joining the i-th point from the top in the tail of π with the i-th point from the top in the head of π, for all such points, with arcs that intersect neither themselves nor the strands of π and that are drawn so that they appear at the bottom of the diagram. We will call such a diagram a *loop diagram*. An example is given in Figure 4.

It is clear from the observation about the construction of the chromatic join that

$$\chi(\pi, \gamma) = \#\{\text{shaded regions in } \overline{\pi\gamma^t}\ \} \tag{3}$$

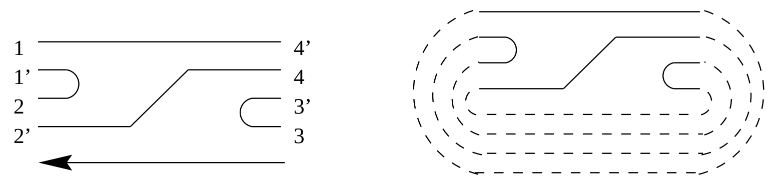
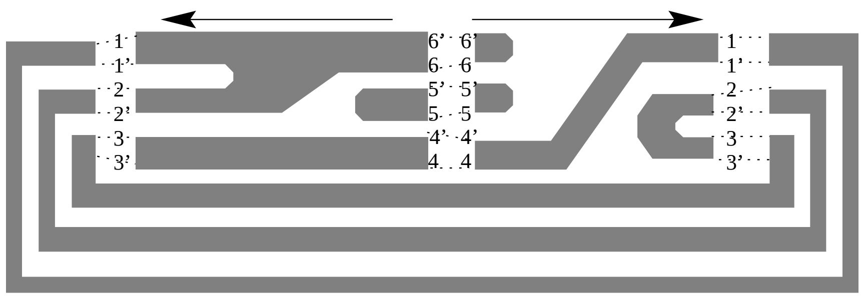


FIGURE 4. A strand diagram and its closure

(or, equivalently, #{shaded regions in $\overline{\gamma\pi^t}$ } since h is symmetric). The construction is illustrated below, where the diagrams on the left and right correspond to the planar partitions π_0 and γ_0, respectively.



These observations lead to the introduction of the following algebra in which $\chi(\pi, \gamma)$ will be seen to be the value of an appropriately defined trace function on an element in the algebra.

2.4. **The Temperley-Lieb algebra.** Throughout, sans serif symbols are used to denote elements of the Temperley-Lieb algebra.

Definition 2.1. *The* Temperley-Lieb algebra *on n elements is a free additive algebra with multiplicative generators* $1, \mathsf{e}_1, \cdots, \mathsf{e}_{n-1}$ *subject to the relations*

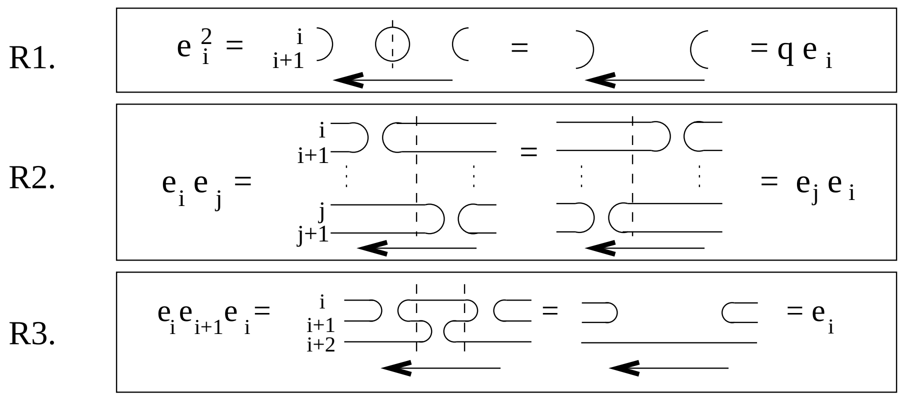
R1 $\mathsf{e}_i^2 = q\mathsf{e}_i$ *for* $i = 1, \ldots, n-1$,
R2 $\mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i$ *if* $|i-j| > 1$,
R3 $\mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i = \mathsf{e}_i$ *for* $i = 1, \ldots, n-1$,

where the indeterminate q commutes with all the elements and 1 *is the multiplicative identity.*

Thus $\mathfrak{TL}_n(q) = \mathbb{C}(q)\langle 1, \mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_{n-1}\rangle/(R1, R2, R3)$.

It is readily seen that every strand diagram can be obtained by concatenating strand diagrams selected from

The product in $\mathfrak{TL}_n(q)$ respects the concatenation of diagrams, with the convention that closed loops formed in the concatenation are marked with the indeterminate q and then deleted. To show this, it is necessary only to verify the assertion for the generators. The sequences



of planar diagrams confirms this. The dimension of this algebra is therefore C_n, the number of planar partitions of $\{1,\ldots,n\}$.

Each planar partition corresponds to a unique strand diagram and thence to a monomial in $\mathsf{e}_1,\ldots,\mathsf{e}_{n-1}$. The set of all such (distinct) monomials, modulo relations R2 and R3, is a linear basis of $\mathfrak{TL}_n(q)$ over $\mathbb{C}(q)$, and is denoted by $\mathcal{B}_1$. For example, $\mathcal{B}_1 = (1,\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_1\mathsf{e}_2,\mathsf{e}_2\mathsf{e}_1)$ is a basis of $\mathfrak{TL}_3(q)$. There is therefore a bijection between the set of all planar partitions of $\{1,\ldots,n\}$ and the elements of the basis $\mathcal{B}_1$.

There is a natural inclusion $\mathfrak{TL}_n(q) \subset \mathfrak{TL}_{n+1}(q)$ via $\mathsf{e}_n = 0$ in $\mathfrak{TL}_{n+1}(q)$, which will be important in the inductive constructions. For strand diagrams, this inclusion corresponds to removing the strand diagram for e_n in $\mathfrak{TL}_{n+1}(q)$ and then deleting the bottom strand from the strand diagram of $\mathsf{e}_i \in \mathfrak{TL}_{n+1}(q)$ for $i = 0,\ldots,n-1$.

Definition 2.2. *A function* $\mathrm{tr}_n\colon \mathfrak{TL}_n(q) \to \mathbb{C}(q)$ *that satisfies conditions (1) and (2) among*

1. tr_n *is a linear functional,*
2. $\mathrm{tr}_n(\mathsf{ab}) = \mathrm{tr}_n(\mathsf{ba})$ *for all* $\mathsf{a},\mathsf{b} \in \mathfrak{TL}_n(q)$,

is called a trace*. If, in addition,* tr_k *satisfies*

3. $\mathrm{tr}_k(\mathsf{a}\mathsf{e}_{k-1}) = \mathrm{tr}_{k-1}(\mathsf{a})$ *whenever* $\mathsf{a} \in \langle 1,\mathsf{e}_1,\ldots,\mathsf{e}_{k-2}\rangle$ *for* $k \geq 1$

then tr_k *is called a* Markov trace*.*

It is shown in Appendix A that if a Markov trace exists then it is unique up to a multiplicative factor.

Let $\#_{\mathrm{loop}}(\mathsf{e})$ be the number of loops in the loop diagram corresponding to the closure of the strand diagram of $\mathsf{e} \in \mathfrak{TL}_n(q)$. Then

$$\mathrm{tr}_n(\mathsf{e}) = q^{\#_{\mathrm{loop}}(\mathsf{e})}$$

extends linearly to a trace function on $\mathfrak{TL}_n(q)$. To check this, it is enough to prove that the trace properties hold for the generators. But this is straightforward. One contribution to the number of loops comes from concatenation, which is recorded through relation R1, and the other contribution comes from the closure of the element. Condition (3) for the Markov trace is shown diagrammatically on Figure 5. Thus tr_n is a Markov trace.

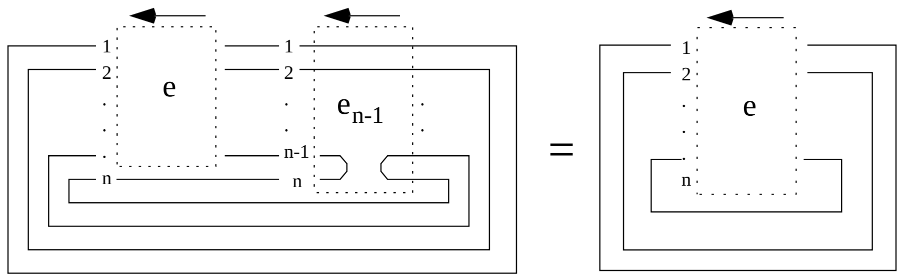


FIGURE 5. The Markov condition for the trace

2.5. **The generalized Temperley-Lieb algebra.** The chromatic join can be incorporated into this setting through a refinement of the above trace. Let e be the Temperley-Lieb element corresponding to π. Let $\#_{\mathrm{sh}}(\mathsf{e})$ and $\#_{\mathrm{ush}}(\mathsf{e})$ denote, respectively, the number of (finite) shaded and unshaded regions in the loop diagram of the closure of e. Then

$$\#_{\mathrm{sh}}(\mathsf{e}) + \#_{\mathrm{ush}}(\mathsf{e}) = \#_{\mathrm{loop}}(\mathsf{e}). \tag{4}$$

We now show that

$$\phi_n(\mathsf{e}) = x^{\#_{\mathrm{sh}}(\mathsf{e})} y^{\#_{\mathrm{ush}}(\mathsf{e})}. \tag{5}$$

is a Markov trace on the following suitable generalization of $\mathfrak{TL}_n(q)$.

Definition 2.3. *The* generalized Temperley-Lieb algebra $\mathfrak{TL}_n(x,y)$ with trace *is a free additive algebra over $\mathbb{C}(x,y)$ with a trace function and with multiplicative generators $1, \mathsf{e}_1, \cdots, \mathsf{e}_{n-1}$ subject to relations*

S1 $\mathsf{e}_i^2 = x_i \mathsf{e}_i$ *for* $i = 1, \ldots n-1$, *where* $x_i = x$ *if* i *is odd and* $x_i = y$ *if* i *is even,*
S2 $\mathsf{e}_i \mathsf{e}_j = \mathsf{e}_j \mathsf{e}_i$ *if* $|i-j| > 1$,
S3 $\mathsf{e}_i \mathsf{e}_{i\pm 1} \mathsf{e}_i = \mathsf{e}_i$ *for* $i = 1, \ldots, n-1$

where the indeterminates x and y commute with all the elements and 1 is the multiplicative identity.

The parity condition in S1 is forced since $\mathrm{tr}(\mathsf{e}_{i-1}\mathsf{e}_{i+1}) = \mathrm{tr}(\mathsf{e}_{i-1}\mathsf{e}_i\mathsf{e}_{i+1}\mathsf{e}_{i-1}) = x_{i-1}\mathrm{tr}(\mathsf{e}_i\mathsf{e}_{i+1}\mathsf{e}_{i-1})$ and, similarly, $\mathrm{tr}(\mathsf{e}_{i-1}\mathsf{e}_{i+1}) = \mathrm{tr}(\mathsf{e}_{i+1}\mathsf{e}_{i-1}\mathsf{e}_i\mathsf{e}_{i+1}) = x_{i+1}\mathrm{tr}(\mathsf{e}_{i-1}\mathsf{e}_i\mathsf{e}_{i+1})$, so equating these gives $x_{i-1} = x_{i+1}$. Note also that the dimension of $\mathfrak{TL}_n(x,y)$ is equal to C_n and that $\mathfrak{TL}_n(q) = \mathfrak{TL}_n(q,q)$. The following result is now easily checked.

Lemma 2.4. *ϕ_n is a Markov trace on $\mathfrak{TL}_n(x,y)$.*

We note that, from Lemma A.1, ϕ_n is unique up to a multiplicative factor. From (5), $\phi_n(\mathsf{e}) = x^{\#_{\mathrm{sh}}(\mathsf{e})}$ When $y = 1$, so this trace function counts the number of parts in the planar partition e. Moreover, from (4), it specializes to $\phi_n(\mathsf{e}) = x^{\#_{\mathrm{loop}}(\mathsf{e})}$ when $x = y$, so this trace function also counts the number of loops in the closure of e.

2.6. **The matrix of chromatic joins.** Since $\chi(\pi,\gamma)$ may be obtained through the construction $\chi(\pi,\gamma) = \#_{\mathsf{sh}}(\overline{\pi\gamma^t})$ on planar diagrams, we now complete the translation of the combinatorial question to the Temperley-Lieb algebra, by defining the transpose of an element of $\mathfrak{TL}_n(x,y)$ so that it is consistent with this construction.

Definition 2.5. *The* transpose $^t : \mathfrak{TL}_n(x,y) \to \mathfrak{TL}_n(x,y)$ *satisfies*

(1) $\mathsf{e}_i^t = \mathsf{e}_i$ *for* $i = 1, 2, \ldots, n-1$,
(2) $(\mathsf{ef})^t = \mathsf{f}^t\mathsf{e}^t$ *for all* $\mathsf{e}, \mathsf{f} \in \mathfrak{TL}_n(x,y)$,
(3) $(\lambda\mathsf{e} + \gamma\mathsf{f})^t = \lambda\mathsf{e}^t + \gamma\mathsf{f}^t$ *for all* $\mathsf{e}, \mathsf{f} \in \mathfrak{TL}_n(x,y)$ *and* $\lambda, \mu \in \mathbb{C}(x_1, \ldots, x_n)$.

Let

$$\langle\cdot,\cdot\rangle_n \colon \mathfrak{TL}_n(x,y) \times \mathfrak{TL}_n(x,y) \to \mathbb{C}(x,y) \colon (\mathsf{e},\mathsf{f}) \mapsto \phi_n(\mathsf{e}\mathsf{f}^t). \tag{6}$$

This is a symmetric bilinear form on $\mathfrak{TL}_n(x,y)$, and the matrix of chromatic joins is the Gram matrix

$$\mathbf{M}_n(x,y) = [\langle \mathsf{a}_i, \mathsf{a}_j\rangle_n]_{C_n \times C_n}\,, \tag{7}$$

of $\langle\cdot,\cdot\rangle_n$ with respect to the standard ordered basis $\mathcal{B}_1 = (\mathsf{a}_1, \ldots \mathsf{a}_{C_n})$ of monomials in $\mathfrak{TL}_n(x,y)$. In particular, from (1), (3) and (5),

$$\mathbf{L}_n = \mathbf{M}_n(q,1). \tag{8}$$

For example, the Gram matrix of $\langle\cdot,\cdot\rangle_n$ with respect to $(1, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1\mathsf{e}_2, \mathsf{e}_2\mathsf{e}_1)$ of $\mathfrak{TL}_3(x,y)$ is

$$\mathbf{M}_3(x,y) = \begin{bmatrix} x^2y & x^2 & xy & x & x \\ x^2 & x^3 & x & x^2 & x^2 \\ xy & x & xy^2 & xy & xy \\ x & x^2 & xy & x^2y & x \\ x & x^2 & xy & x & x^2y \end{bmatrix}.$$

Later, another ordering of $\{1, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1\mathsf{e}_2, \mathsf{e}_2\mathsf{e}_1\}$ will be used, and it will also be denoted by $\mathcal{B}_1$. At this point, however, there is no reason to prefer a particular ordering.

3. The basis $\mathcal{B}_2$

Hereinafter we work within $\mathfrak{TL}_n(x,y)$ to construct an ordered basis $\mathcal{B}_2$ of $\mathfrak{TL}_n(x,y)$ with respect to which $\mathbf{M}_n$ has the form given in (2). The relations S1, S2 and S3 will be used extensively, although explicit reference seldom will be made to them. In addition, it will be more convenient to prove certain of the algebraic results by constructions based on strand diagrams. This is permissible since, as observed earlier, strand diagrams with concatenation, transposition and closure provide a combinatorial presentation of $\mathfrak{TL}_n(x,y)$ with a Markov trace.

3.1. Some special elements in $\mathfrak{TL}_n(x,y)$.

We begin with some observations about the construction of the special elements, E_n, called the Jones-Wenzl projectors, for $\mathfrak{TL}_n(x,y)$ through an analogy with the symmetric group. It is known that $\mathfrak{TL}_n(x,x)$ is a quotient algebra of the Hecke algebra, which in turn is a generalization of the group algebra $\mathbb{C}\mathfrak{S}_n$ of the symmetric group. The latter can be represented as a free additive algebra with multiplicative generators $1, \mathsf{t}_1, \ldots, \mathsf{t}_{n-1}$ subject to the relations

T1 $\mathsf{t}_i^2 = 1$ for $i = 1, \ldots, n-1$,
T2 $\mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i$ if $|i-j| > 1$,
T3 $\mathsf{t}_i\mathsf{t}_{i+1}\mathsf{t}_i = \mathsf{t}_{i+1}\mathsf{t}_i\mathsf{t}_{i+1}$ for $i = 1, \ldots, n-2$,

where t_i is the adjacent transposition $(i, i+1)$. By setting $\mathsf{t}_i = 1 - \mathsf{e}_i$ and taking the quotient modulo the ideal generated by the elements $\mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i - \mathsf{e}_i$ we obtain $\mathfrak{TL}_n(2,2)$. The relation $\mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i - \mathsf{e}_i = 0$ is equivalent to the relation $\mathsf{t}_i\mathsf{t}_{i+1}\mathsf{t}_i = 1 - \mathsf{t}_i - \mathsf{t}_{i+1} + \mathsf{t}_i\mathsf{t}_{i+1} + \mathsf{t}_{i+1}\mathsf{t}_i$ and so $\mathfrak{TL}_n(2,2) \cong \mathbb{C}\mathfrak{S}_n/I$

where I is the ideal generated by the elements $1 - \mathsf{t}_i - \mathsf{t}_{i+1} - t_i t_{i+1} t_i + \mathsf{t}_i \mathsf{t}_{i+1} + \mathsf{t}_{i+1}\mathsf{t}_i$. In the language of classical representation theory this quotient corresponds to reducing $\mathbb{C}\mathfrak{S}_n$ modulo the representations corresponding to Young tableaux with more than two rows. This is because the Young symmetrizers of tableaux containing at least three rows are equal to zero when evaluated modulo the relations $1 - \mathsf{t}_i - \mathsf{t}_{i+1} + \mathsf{t}_i\mathsf{t}_{i+1} + \mathsf{t}_{i+1}\mathsf{t}_i = \mathsf{t}_i\mathsf{t}_{i+1}\mathsf{t}_i$.

Recall also that the Young symmetrizer z corresponding to a tableau with only one column satisfies $\mathsf{t}_i\mathsf{z} = \mathsf{z} = \mathsf{z}\mathsf{t}_i$ for $i = 1, \dots, n-1$ which is equivalent to $\mathsf{e}_i\mathsf{z} = 0 = \mathsf{z}\mathsf{e}_i$. Thus, in the next section we shall construct elements $\mathsf{E}_n \in \mathfrak{TL}_n(x, y)$ such that $\mathsf{e}_i\mathsf{E}_n = 0 = \mathsf{E}_n\mathsf{e}_i$. These elements will play a role in $\mathfrak{TL}_n(x, y)$ analogous to that of the Young symmetrizers in $\mathbb{C}\mathfrak{S}_n$.

With the above comments in mind, we therefore seek $\mathsf{E}_n \in \mathfrak{TL}_n(x, y)$ such that (i) $\mathsf{E}_n\mathsf{e}_i = 0 = \mathsf{e}_i\mathsf{E}_n$ for all $1 \leq i \leq n-1$ and (ii) $\mathsf{E}_n - 1 \in \langle \mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_{n-1}\rangle$. Now $\mathsf{E}_n - 1$ is generated by $\mathsf{e}_1, \dots \mathsf{e}_{n-1}$ so, from (i), $\mathsf{E}_{n+1}(\mathsf{E}_n - 1) = 0 = (\mathsf{E}_n - 1)\mathsf{E}_{n+1}$ so $\mathsf{E}_{n+1}\mathsf{E}_n = \mathsf{E}_{n+1} = \mathsf{E}_n\mathsf{E}_{n+1}$. Then $\mathsf{E}_n\mathsf{E}_{n+1}\mathsf{E}_n = \mathsf{E}_{n+1}\mathsf{E}_n = \mathsf{E}_{n+1}$. Thus $\mathsf{E}_{n+1} = \mathsf{E}_n\mathsf{F}_n\mathsf{E}_n$ for some $\mathsf{F}_n \in \mathfrak{TL}_{n+1}(x, y)$. However, the only monomials that may appear in F_n are 1 and e_n since, by (i), the others give no contribution to $\mathsf{E}_n\mathsf{F}_n\mathsf{E}_n$. Hence $\mathsf{F}_n = \lambda_n 1 - \mu_n\mathsf{e}_n$ for some $\lambda_n, \mu_n \in \mathbb{C}(x, y)$. Clearly, $\lambda_n = 1$ from (ii). Thus if E_n exists then it must satisfy $\mathsf{E}_{n+1} = \mathsf{E}_n(1 - \mu_n\mathsf{e}_n)\mathsf{E}_n = \mathsf{E}_n - \mu_n\mathsf{E}_n\mathsf{e}_n\mathsf{E}_n$. The next result shows that this recursion does yield idempotents satisfying the conditions and characterizes μ_n. Note that $\mathfrak{TL}_n(q, q)$ is the Temperley-Lieb algebra $\mathfrak{TL}_n(q)$, and the E_n are known in this case as the *Jones-Wenzl projectors*.

The above observations provide the motivation for the following lemma, whose proof is an extension of the proof for $\mathfrak{TL}_n(q)$, and is included for completeness. In the statement of the lemma, condition 2 is redundant since $0 = \mathsf{E}_n(\mathsf{E}_n - 1)$ by conditions 1 and 3 so $\mathsf{E}_n^2 = \mathsf{E}_n$. However, condition 2 is retained since it is convenient to have it stated explicitly with the other two.

Lemma 3.1. *There is a unique element $\mathsf{E}_n \in \mathfrak{TL}_n(x, y)$ satisfying the conditions that*

(1) $\mathsf{E}_n - 1$ *belongs to the algebra generated by* $\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_{n-1}$,
(2) $\mathsf{E}_n^2 = \mathsf{E}_n$,
(3) $\mathsf{E}_n\mathsf{e}_i = 0 = \mathsf{e}_i\mathsf{E}_n$ *for* $1 \leq i \leq n-1$.

Moreover,

$$\mathsf{E}_{n+1} = \mathsf{E}_n - \mu_n\mathsf{E}_n\mathsf{e}_n\mathsf{E}_n \tag{9}$$

for $n \geq 1$ *with* $\mathsf{E}_1 = 1$, *where* μ_i *satisfies*

$$\mu_{i+1} = (x_{i+1} - \mu_i)^{-1} \tag{10}$$

for $i \geq 0$ *with* $\mu_0 = 0$.

Proof. We show that E_{n+1} satisfies conditions 1, 2, 3 and $(\mathsf{E}_{n+1}\mathsf{e}_{n+1})^2 = \mu_{n+1}^{-1}\mathsf{E}_{n+1}\mathsf{e}_{n+1}$ under the inductive hypothesis that E_n does.

The result holds trivially for $n = 1$. Now $\mathsf{E}_{n+1} - 1 = (\mathsf{E}_n - 1) - \mu_n\mathsf{E}_n\mathsf{e}_n\mathsf{E}_n$ and is therefore generated by $\mathsf{e}_1, \dots, \mathsf{e}_n$ so condition 1 holds.

Now from equation (9) we have $\mathsf{E}_{n+1}^2 = \mathsf{E}_n - 2\mu_n \mathsf{E}_n\mathsf{e}_n\mathsf{E}_n + \mu_n^2(\mathsf{E}_n\mathsf{e}_n)^2\mathsf{E}_n$. But $\mathsf{E}_n - \mathsf{1}$ is generated by $\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_{n-1}$ so E_n and e_{n+1} commute. Then, from equation (9)

$$\begin{aligned}\mathsf{e}_{n+1}\mathsf{E}_{n+1}\mathsf{e}_{n+1} &= \mathsf{e}_{n+1}\mathsf{E}_n\mathsf{e}_{n+1} - \mu_n\mathsf{e}_{n+1}\mathsf{E}_n\mathsf{e}_n\mathsf{E}_n\mathsf{e}_{n+1} \\ &= x_{n+1}\mathsf{E}_n\mathsf{e}_{n+1} - \mu_n\mathsf{E}_n(\mathsf{e}_{n+1}\mathsf{e}_n\mathsf{e}_{n+1})\mathsf{E}_n \\ &= x_{n+1}\mathsf{E}_n\mathsf{e}_{n+1} - \mu_n\mathsf{E}_n\mathsf{e}_{n+1} = \mu_{n+1}^{-1}\mathsf{E}_n\mathsf{e}_{n+1}.\end{aligned}$$

where we had used the fact that $\mathsf{E}_n\mathsf{e}_{n+1} = \mathsf{e}_{n+1}\mathsf{E}_n$ since E_n is generated by $\langle \mathsf{e}_1, \dots, \mathsf{e}_{n-1}\rangle$. Multiplying on the left by E_{n+1} gives $(\mathsf{E}_{n+1}\mathsf{e}_{n+1})^2 = \mu_{n+1}^{-1}\mathsf{E}_{n+1}\mathsf{E}_n\mathsf{e}_{n+1}$. But, from equation (9), $\mathsf{E}_{n+1}\mathsf{E}_n = \mathsf{E}_n^2 - \mu_n\mathsf{E}_n\mathsf{e}_n\mathsf{E}_n^2 = \mathsf{E}_{n+1}$ so $(\mathsf{E}_{n+1}\mathsf{e}_{n+1})^2 = \mu_{n+1}^{-1}\mathsf{E}_{n+1}\mathsf{e}_{n+1}$. Combining these, we conclude that $\mathsf{E}_{n+1}^2 = \mathsf{E}_n - 2\mu_n\mathsf{E}_n\mathsf{e}_n\mathsf{E}_n + \mu_n\mathsf{E}_n\mathsf{e}_n\mathsf{E}_n = \mathsf{E}_{n+1}$, from equation (9), so condition 2 holds.

From condition 3, by hypothesis and by equation (9), we have $\mathsf{E}_{n+1}\mathsf{e}_i = \mathsf{E}_n\mathsf{e}_i - \mu_n(\mathsf{E}_n\mathsf{e}_n)(\mathsf{E}_n\mathsf{e}_i) = 0$ for $1 \le i \le n-1$. Also $\mathsf{E}_{n+1}\mathsf{e}_n = \mathsf{E}_n\mathsf{e}_n - \mu_n(\mathsf{E}_n\mathsf{e}_n)^2 = 0$, so condition 3 holds.

Finally, from condition 3, $(\mathsf{1} - \mathsf{E}_n)\mathsf{e}_i = \mathsf{e}_i = \mathsf{e}_i(\mathsf{1} - \mathsf{E}_i)$ for $i \le i \le n-1$ in the ring $\langle \mathsf{e}_1, \dots, \mathsf{e}_{n-1}\rangle$, so $\mathsf{1} - \mathsf{E}_n$ is the multiplicative identity, which is necessarily unique, so E_n is unique. □

It follows from Lemma 3.1 that if $V_i \equiv V_i(x, y)$ is written in the form

$$\mu_i = V_{i-1}/V_i \tag{11}$$

for $i \ge 1$ then V_k satisfies the recurrence equation

$$V_{k+1} = x_{k+1}V_k - V_{k-1} \tag{12}$$

for $k \ge 1$ with $V_0(x, y) = 1$ and $V_1(x, y) = x$. Since this identifies $V_k(q, q)$ is the normalized Chebyshev polynomial U_k of the second kind, we regard $V_i(x, y)$ as a generalized Chebyshev polynomial. The inverse relation is

$$V_k = (\mu_1 \dots \mu_k)^{-1}. \tag{13}$$

The appearance of Chebyshev polynomials is not entirely surprising. Trivially, planar partitions are in bijection with planted plane trees through duality, and the generating series for the number of the latter with respect to non-root vertices satisfies the functional equation $T = x(1-T)^{-1}$. This generates a continued fraction whose k-th convergent has the form xQ_k/Q_{k+1} where $Q_{k+1} = Q_k - xQ_{k-1}$ for $k \ge 1$ and $Q_{-1} = 0$, $Q_0 = 1$. This is easily transformed into the generating series for $V_k(x, x)$. The series xQ_k/P_{k+1} counts planted plane trees with height at most k. The corresponding series in x and y counts two-coloured trees, with height at most k, with respect to the number of vertices of each colour.

Corollary 3.2. $\mathsf{E}_n^t = \mathsf{E}_n$.

Proof. Trivially, conditions 1,2 and 3 of Lemma 3.1 hold with E_n replaced with E_n^t. But such an element is unique by Lemma 3.1, and the result follows. □

Corollary 3.3. *For $1 \le k \le n-1$ there is a unique element $\mathsf{E}_n^{(k)} \in \mathfrak{TL}_n(x, y)$ such that*

(1) $\mathsf{E}_n^{(k)} - \mathsf{1}$ *belongs to the algebra generated by* $\mathsf{e}_k, \mathsf{e}_{k+1}, \dots, \mathsf{e}_{n-1}$,
(2) $\mathsf{E}_n^{(k)}\mathsf{E}_n^{(k)} = \mathsf{E}_n^{(k)}$,
(3) $\mathsf{E}_n^{(k)}\mathsf{e}_i = 0 = \mathsf{e}_i\mathsf{E}_n^{(k)}$ *for* $k \le i \le n-1$,

where $\mathsf{E}_n^{(n)} = \mathsf{E}_n^{(n+1)} = \mathsf{1}$ *by convention.*

Proof. This follows directly from above by considering $\mathfrak{TL}_{n-k+1}(x,y)$ and shifting indices $i \mapsto i+k-1$. $\square$

Note that $\mathsf{E}_n^{(1)} = \mathsf{E}_n$.

It follows from Corollary 3.3 that the strand diagram of each monomial in $\mathsf{E}_n^{(n-h+1)}$ is obtained from the strand diagram of the corresponding monomial in $\mathsf{E}_h \in \mathfrak{TL}_h(x,y)$ by attaching $n-h-1$ parallel strands at the top. Thus $\mathsf{E}_n^{(n-h+1)}$ has the same structure as E_h in $\mathfrak{TL}_n(x,y)$. Moreover, $\mathsf{E}_n^{(n-h+1)}$ can therefore be obtained from E_h by the substitutions $\mathsf{e}_i \mapsto \mathsf{e}_{n-h+i}$ for $i=1,\dots,h-1$, where E_h is obtained from the recursion given in (9). Note that this substitution is a bijective ring homomorphism. It is also useful to observe that $\mathsf{E}_n^{(h)}$ is obtained from E_{n-h+1} by

$$\mathsf{e}_i \mapsto \mathsf{e}_{h+i-1} \text{ and } x_i \mapsto x_{h+i-1} \text{ for } i=1,\dots n-h. \tag{14}$$

3.2. **Paths.** A *path* $\mathfrak{p}$ of length $2n$ is a sequence of points $(P_0,\dots,P_{2n})$ in $\mathbb{R}^2$, such that $P_0=(0,0)$ and $P_i - P_{i-1} = (1,\pm 1)$ for $i=1,\dots,2n$. Let $\mathcal{P}_n$ denote the set of all such paths with $P_{2n}=(0,0)$ and such that no point has negative y-coordinate. This is the set of Dyck paths. It is convenient to connect P_{i-1} and P_i by an edge for $i=1,\dots,2n$.

The *height*, $h_i(\mathfrak{p})$, of $\mathfrak{p}$ is the y-coordinate of P_i. Then $\mathfrak{p}$ has a *rise at position* i if $h_{i+1}(\mathfrak{p}) > h_i(\mathfrak{p})$, and a *fall at position* i if $h_{i+1}(\mathfrak{p}) < h_i(\mathfrak{p})$. Also, $\mathfrak{p}$ has a *minimum* in position i if $h_{i-1}(\mathfrak{p}) = h_{i+1}(\mathfrak{p}) = h_i(\mathfrak{p})+1$. Similarly, $\mathfrak{p}$ has a *maximum* in position i if $h_{i-1}(\mathfrak{p}) = h_{i+1}(\mathfrak{p}) = h_i(\mathfrak{p})-1$. In addition, $\mathfrak{p}$ has a *double rise* in position i if $h_{i-1}(\mathfrak{p})+1 = h_i(\mathfrak{p}) = h_{i+1}(\mathfrak{p})-1$, and a *double fall* in position i if $h_{i-1}(\mathfrak{p})-1 = h_i(\mathfrak{p}) = h_{i+1}(\mathfrak{p})+1$. We say that $\mathfrak{p}$ has a *slope* in position i if there is either a double rise or a double fall in position i. Note that if a maximum or minimum occurs in position i and height m then $i \equiv m \bmod 2$.

The steps $(1,1)$ and $(1,-1)$ are called, respectively, a *rise* $\nearrow$ and a *fall* $\searrow$. A path in $\mathcal{P}_n$ can therefore be regarded as sequence of rises and falls, starting at the origin, with no point below the x-axis. Given a path, the corresponding arch diagram is constructed by first associating the step $(1,1)$, with the beginning of an arch and the step $(1,-1)$, with the end of an arch and then completing this uniquely to an arch diagram. The correspondence is clearly bijective. Here and throughout, paths are denoted by Gothic minuscules. Figure 6 shows the path diagram $\nearrow^2\searrow\nearrow^3\searrow^2\nearrow\searrow^3 \in \mathcal{P}_6$ and its corresponding arch diagram.

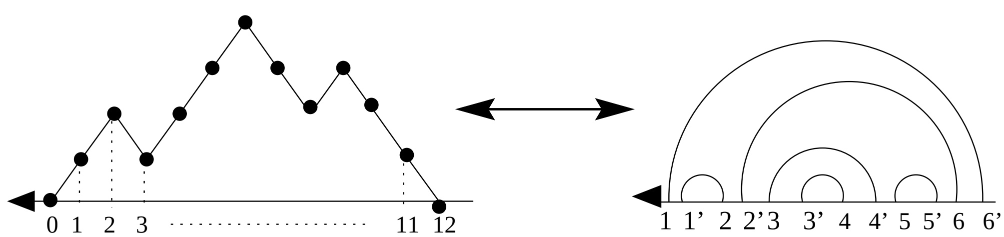


FIGURE 6. A path and the corresponding arch diagram

Combining the above bijection with those from Section 2 gives a bijection

$$(\cdot)_1 \colon \mathcal{P}_n \xrightarrow{\sim} \mathcal{B}_1$$

between $\mathcal{P}_n$ and $\mathcal{B}_1$, which is a basis of both $\mathfrak{TL}_n(q)$ and $\mathfrak{TL}_n(x,y)$, The path corresponding to a monomial $\mathsf{a} \in \mathcal{B}_1$ is denoted by $[\mathsf{a}]$. In this notation therefore $[\mathsf{e}_i^2] = [\mathsf{e}_i]$. Also, if a is any monomial, then $([\mathsf{a}])_1$ is the element in $\mathcal{B}_1$ that is equal to it. For brevity we shall use $[\mathsf{a}]_1$ to denote $([\mathsf{a}])_1$. In particular, if $\mathsf{a} \in \mathcal{B}_1$ then $[\mathsf{a}]_1 = \mathsf{a}$.

Finally, we shall need the transpose of a path. If $\mathfrak{s}$ denotes a step define $\mathfrak{s}^t = \nearrow$ if $\mathfrak{s} = \searrow$ and $\mathfrak{s}^t = \searrow$ if $\mathfrak{s} = \nearrow$ (ie. $\mathfrak{s}^t$ denotes the step complement of $\mathfrak{s}$). If $\mathfrak{p} = \mathfrak{s}_1 \cdots \mathfrak{s}_k$ where $\mathfrak{s}_1, \ldots, \mathfrak{s}_k$ are steps, then $\mathfrak{s}_k^t \cdots \mathfrak{s}_1^t$ is called the *transpose* of $\mathfrak{p}$ and is denoted by $\mathfrak{p}^t$. Thus $\mathfrak{p}^t$ is obtained from $\mathfrak{p}$ by reading the latter backwards and then specifying $(0,0)$ to be its origin. If $\mathfrak{r}$ and $\mathfrak{l}$ are paths, let $\delta_{\mathfrak{r},\mathfrak{l}} = 1$ if $\mathfrak{l} = \mathfrak{r}$ and 0 otherwise.

3.3. **Box addition on paths.** If $\mathfrak{p} \in \mathcal{P}_n$ has a minimum at position i, let $\mathfrak{p} \boxplus \diamond_i$ denote the path obtained combinatorially by complementing the steps incident with position i to make a maximum at position i. We can notionally regard $\mathfrak{p} \boxplus \diamond_i$ as having been obtained from $\mathfrak{p}$ by "adding a box" $\diamond_i$ at the minimum in position i in $\mathfrak{p}$, and we shall refer to this operation as *box addition*. The operation is undefined if $\mathfrak{p}$ does not have a minimum at position i. The reverse operation will be useful. If $\mathfrak{p} \in \mathcal{P}_n$ has a maximum at position i then $\mathfrak{p} \boxminus \diamond_i$ denotes the path obtained by complementing the steps incident with position i to make a minimum in position i. We can regard this operation as *box deletion*. It is easily verified that the basis $\mathcal{B}_1$ of monomials may be obtained through box addition by means of the following algorithm, in which

$$\mathfrak{I}_n^k = (\nearrow\searrow)^j \left(\nearrow^k \searrow^k\right) (\nearrow\searrow)^j \in \mathcal{P}_n$$

where $k + 2j = n$ and $0 \le k \le n$. Such a path is called a *base path*.

Algorithm 3.4. *Let the mapping* $(\cdot)_1$ *act on* $\mathcal{P}_n$ *by* $(\mathfrak{I}_n^n)_1 = 1$ *and, for* $k < n$, *by*

$$(\mathfrak{I}_n^k)_1 = \mathsf{e}_1 \mathsf{e}_3 \ \ldots \mathsf{e}_{n-k-1},$$

where $n \equiv k \bmod 2$. *The rest of the mapping is defined iteratively by*

$$(\mathfrak{p} \boxplus \diamond_i)_1 = \begin{cases} \mathsf{e}_i(\mathfrak{p})_1 & \text{if } i < n \text{ and } \mathfrak{p} \text{ has a minimum at position } i, \\ (\mathfrak{p})_1 \mathsf{e}_{2n-i} & \text{if } i > n, \text{ and } \mathfrak{p} \text{ has a minimum at position } i, \\ \text{undefined} & \text{if } i = n \end{cases} \tag{15}$$

Then the basis $\mathcal{B}_1$ *of* $\mathfrak{TL}_n(x,y)$ *is the set* $(\mathcal{P}_n)_1$ *of images of* $\mathcal{P}_n$ *under* $(\cdot)_1$.

The construction is not defined for $i = n$ since $\mathsf{e}_n \notin \mathfrak{TL}_n(q)$, so a box cannot be added to the midpoint of a path. This observation is a crucial one since it explains the significance of the midpoint of a path in $\mathcal{P}_n$, which plays a key role in the later algebraic and combinatorial constructions.

The height of the midpoint of $\mathfrak{p} \in \mathcal{P}_n$ is called the *mid-height* of $\mathfrak{p}$ and is denoted by $h(\mathfrak{p})$. Figure 7 shows the determination of $[\mathsf{b}] = [\mathsf{e}_2\mathsf{e}_3] \boxplus \diamond_4$ and $[\mathsf{c}] = [\mathsf{e}_2\mathsf{e}_3] \boxplus \diamond_8$ in $\mathfrak{TL}_6(q)$. Every path in $\mathcal{P}_n$ with mid-height k can therefore be obtained, not necessarily uniquely, by means of (15) through box addition from the base path $\mathfrak{I}_n^k$ where $k \equiv n \bmod 2$. The number of base paths in $\mathcal{P}_n$ is $\lfloor \frac{1}{2}(2n+1) \rfloor$. The path $[\mathfrak{I}_9^3]_1 = \mathsf{e}_1\mathsf{e}_3\mathsf{e}_5$ is given in Figure 8.

If $\mathfrak{p} \in \mathcal{P}_n$ and $h(\mathfrak{p}) = h$, then the number of box additions required in the construction of $\mathfrak{p}$ from $\mathfrak{I}_n^h$ using (15) is referred to as the *number of boxes* in $\mathfrak{p}$, and is denoted by $\#_\diamond(\mathfrak{p})$. Thus, $\#_\diamond(\mathfrak{I}_n^h) = 0$. This number will be used in some of the inductive proofs.

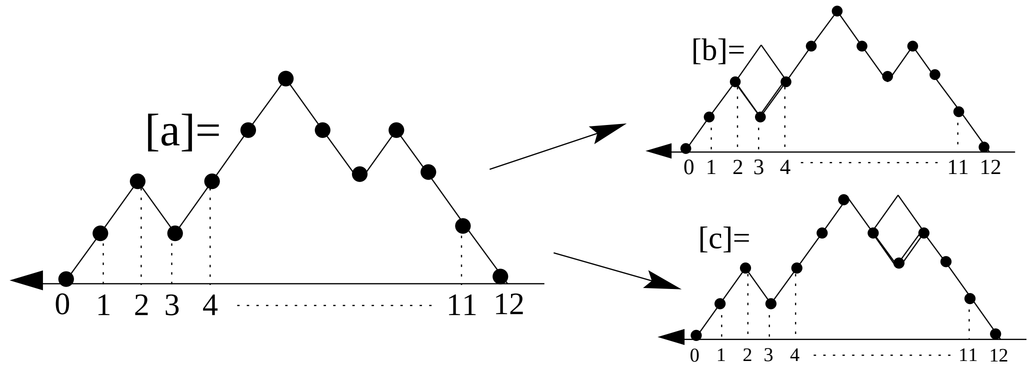


FIGURE 7. Box addition in $\mathfrak{TL}_6(q)$.

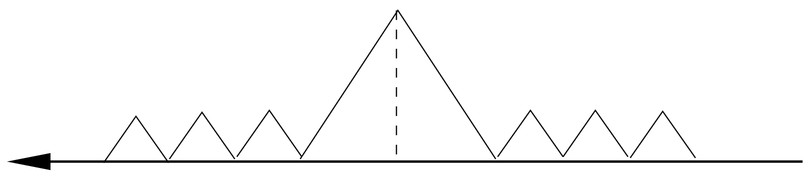


FIGURE 8. The path $\mathfrak{I}_9^3 = [\mathsf{e}_1\mathsf{e}_3\mathsf{e}_5]$.

3.4. **The construction of the basis $\mathcal{B}_2$.** Another basis $\mathcal{B}_2$ is constructed in a similar way to $\mathcal{B}_1$, from the same set of base paths but with a revised action of box addition.

Definition 3.5. *Let the mapping $(\cdot)_2$ act on $\mathcal{P}_n$ by $(\mathfrak{I}_n^n)_2 = 1$ and, for $k < n$, by*

$$(\mathfrak{I}_n^k)_2 = x^{-(n-k)/2}\mathsf{E}_n^{(n-k+1)}\mathsf{e}_1\mathsf{e}_3\ \dots \mathsf{e}_{n-k-1}, \tag{16}$$

where $n \equiv k \mod 2$. The rest of the mapping is defined iteratively by

$$(\mathfrak{p} \boxplus \diamond_i)_2 = \begin{cases} \sqrt{\frac{\mu_{j+1}}{\mu_j}}\,(\mathsf{e}_i - \mu_j 1)(\mathfrak{p})_2 & \text{if } i < n \text{ and } \mathfrak{p} \text{ has a minimum at position } i, \\ \sqrt{\frac{\mu_{j+1}}{\mu_j}}\,(\mathfrak{p})_2(\mathsf{e}_{2n-i} - \mu_j 1) & \text{if } i > n, \text{ and } \mathfrak{p} \text{ has a minimum at position } i, \\ \text{undefined} & \text{if } i = n \end{cases}$$

where $j = h_i(\mathfrak{p}) + 1$. Let $\mathcal{B}_2$ be the set $(\mathcal{P}_n)_2$ of images of $\mathcal{P}_n$ under $(\cdot)_2$.

Note that the e_i's for $1 \leq i \leq n - h - 1$ commute with the $\mathsf{E}_n^{(n-k+1)}$ term in $(\mathfrak{I}_n^k)_2$. This construction is well defined since $(\{[\mathsf{a}] \boxplus \diamond_i\} \boxplus \diamond_j)_2 = (\{[\mathsf{a}] \boxplus \diamond_j\} \boxplus \diamond_i)_2$ whenever $|i - j| > 1$ by S2. If a is a monomial, we denote $([\mathsf{a}])_2$ by $[\mathsf{a}]_2$ for brevity.

For example, $([\mathsf{e}_2\mathsf{e}_3] \boxplus \diamond_3)_2 = \sqrt{\frac{\mu_3}{\mu_2}}(\mathsf{e}_3 - \mu_2)[\mathsf{e}_2\mathsf{e}_3]_2$ and $([\mathsf{e}_2\mathsf{e}_3] \boxplus \diamond_8)_2 = [\mathsf{e}_2\mathsf{e}_3]_2\sqrt{\frac{\mu_4}{\mu_3}}(\mathsf{e}_4 - \mu_3)$. For brevity, here and subsequently, we replace $\mu_j 1$ by μ_j where the context permits.

Any path $\mathfrak{p} \in \mathcal{P}_n$ can be uniquely decomposed into $\mathfrak{p} \mapsto (\mathfrak{l}, \mathfrak{r})$ where $\mathfrak{p} = \mathfrak{l}\mathfrak{r}$, and $\mathfrak{l}$ and $\mathfrak{r}$ are paths of length n. We shall call $\mathfrak{l}\mathfrak{r}$ the *midpoint decomposition* of $\mathfrak{p}$. The main result regarding the elements in $\mathcal{B}_2$ may now be stated.

Theorem 3.6. *If $[\mathsf{a}] = \mathfrak{l}\mathfrak{r}$ and $[\mathsf{a}'] = \mathfrak{l}'\mathfrak{r}'$ are midpoint decompositions, where a and a' are monomials in $\mathfrak{TL}_n(x, y)$, then*

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2.$$

The proof of Theorem 3.6 is technical and so, to avoid interrupting the present development, it is deferred to Appendix B. The theorem shows that $\mathcal{B}_2$ is a semi-orthogonal basis for $\mathfrak{TL}_n(x,y)$ and it provides us with an easy way of multiplying elements of $\mathcal{B}_2$. Moreover, it gives the following structure theorem for $\mathfrak{TL}_n(x,y)$.

Theorem 3.7.

$$\mathfrak{TL}_n(x,y) \cong \mathcal{M}_{k_1}(\mathbb{C}(x,y)) \times \mathcal{M}_{k_2}(\mathbb{C}(x,y)) \times \cdots \times \mathcal{M}_{k_n}(\mathbb{C}(x,y)),$$

where $\mathcal{M}_i(\mathbb{C}(x,y))$ *is the ring of* $i \times i$ *matrices over* $\mathbb{C}(x,y)$ *and* $k_i = \binom{n}{(n-i)/2} - \binom{n}{(n-i-2)/2}$.

Proof. We index the rows and columns of matrices in $\mathcal{M}_{k_i}(\mathbb{C}(x,y))$ by the paths of length n with terminal height i. Consider map ψ given by $(\mathfrak{l}\mathfrak{r}^t)_2 \mapsto \mathsf{e}^{(h)}_{\mathfrak{l},\mathfrak{r}}$ where h is the mid-height of the path $\mathfrak{l}\mathfrak{r}^t \in \mathcal{P}_n$ and $\mathsf{e}^{(h)}_{\mathfrak{l},\mathfrak{r}}$ denotes the matrix in $\mathcal{M}_{k_h}(\mathbb{C}(x,y))$ with a 1 in position $(\mathfrak{l},\mathfrak{r})$ and 0's elsewhere. The matrices from the other constituents are zero. Then, from Theorem 3.6, ψ is a homomorphism and a dimension argument proves that it is also bijective. Now k_i is identified as the number of paths of length n with terminus at height i. The result follows from Lemma C.1, □

An example of the isomorphism is given in Appendix D.4.

Corollary 3.8. $\mathcal{B}_2$ *is a basis of* $\mathfrak{TL}_n(x,y)$.

Proof. Suppose that $\mathcal{B}_2$ is linearly dependent. Then there exist scalars $\alpha_i \in \mathbb{C}(x,y)$, $i = 1,\ldots,C_n$, not all zero, such that $\sum_{i=1}^{C_n} \alpha_i [\mathsf{a}_i]_2 = 0$. Assume that $\alpha_k \neq 0$. Let $[\mathsf{a}_i] = \mathfrak{l}_i\mathfrak{r}_i$ be midpoint decompositions for $i = 1,\ldots,C_n$. Then, from Theorem 3.6,

$$0 = (\mathfrak{l}\mathfrak{l}_k^t)_2 \left(\sum_{i=1}^{C_n} \alpha_i (\mathsf{a}_i)_2\right) (\mathfrak{r}_k^t\mathfrak{r})_2 = \sum_{i=1}^{C_n} \alpha_i (\mathfrak{l}_k\mathfrak{l}_k^t)_2 (\mathfrak{l}_i\mathfrak{r}_i)_2 (\mathfrak{r}_k^t\mathfrak{r}_k)_2 = \sum_{i=1}^{C_n} \alpha_i \delta_{\mathfrak{l}_i,\mathfrak{l}_k}\delta_{\mathfrak{r}_i,\mathfrak{r}_k} (\mathfrak{l}_i\mathfrak{r}_i)_2 = \alpha_k \mathsf{a}_k.$$

Thus $\alpha_k = 0$, which is a contradiction, so $\mathcal{B}_2$ is linearly independent. Finally, since $\dim(\mathfrak{TL}_n(x,y)) = C_n = |\mathcal{B}_2|$ it follows $\mathcal{B}_2$ is a basis. □

It follows immediately that

$$(\cdot)_2 \colon \mathcal{P}_n \xrightarrow{\sim} \mathcal{B}_2$$

is a bijection.

It is now possible to indicate informally the motivation behind the construction of basis $\mathcal{B}_2$ in Definition 3.5. Following the example of the symmetric group algebra, it is natural to obtain construct the elements of $\mathcal{B}_2$ recursively by multiplying left and right by selected factors. It remains to justify why these factors should be $\mathsf{e}_i - \mu_j$ precisely when the corresponding path diagram contains a minimum at position i.

Suppose a factor contains a monomial $\mathsf{e} = \mathsf{e}_{i_1}\mathsf{e}_{i_2}\ldots\mathsf{e}_{i_k}$. Then multiplying by e on the right is the same as first multiplying by e_{i_1} and then by e_{i_2} and so on. So one may as well assume that the monomials are equal to e_i for some i. Thus the factors ought to be polynomials in e_i and since e_i is idempotent each factor ought to have the form $\mathsf{e}_i + c$ for some scalar $c \in \mathbb{C}(x,y)$. Now if the path diagram $[\mathsf{a}]$ has a slope at position i then $[\mathsf{a}]_2(\mathsf{e}_i + c) = c[\mathsf{a}]_2$ since, as is shown in Appendix B.1, $[\mathsf{a}]_2\mathsf{e}_i = 0$ if $[\mathsf{a}]$ has a slope at position i. Similarly if $[\mathsf{a}]$ has a maximum at position i then

$$[\mathsf{a}]_2(\mathsf{e}_i + c) \sim [\mathsf{a}']_2(\mathsf{e}_i - \mu_j)(\mathsf{e}_i + c) \sim [\mathsf{a}']_2(\mathsf{e}_i + c') \sim [\mathsf{a}]_2 + [\mathsf{a}']_2$$

where $c' \in \mathbb{C}(x,y)$ and $\sim$ denotes that we have ignored scalar factors. This shows that multiplying $[\mathsf{a}]_2$ by $\mathsf{e}_i + c$ has no effect if $[\mathsf{a}]$ has a maximum at position i. So to multiply by $\mathsf{e}_i + c$ there ought to be a minimum at position i in $[\mathsf{a}]$. This justifies all the conditions in Definition 3.5 except why $c = \mu_j$. This can be deduced from the proofs of Theorem 3.6 in Appendix B where c must satisfy a recursion enforced by the induction.

4. The matrix of chromatic joins

It is now possible to give a procedure for determining the elements of the inverse of the matrix $\mathbf{L}_n$ of chromatic joins. We now show that the basis $\mathcal{B}_2$ is such that the Gram matrix of $\langle\cdot,\cdot\rangle$ with respect to it is diagonal. Moreover, we show that there is an ordering of the elements of $\mathcal{B}_2$ such that the matrix $\mathbf{P}_n$ that expresses the basis $\mathcal{B}_2$ in terms of $\mathcal{B}_1$ is a triangular matrix.

4.1. **The Gram matrix.** Some preliminary results on the evaluation of the Markov trace ϕ_n are required.

Lemma 4.1. $(\mathfrak{I}_n^h)_2(\mathfrak{I}_n^{h'})_2 = \delta_{h,h'}(\mathfrak{I}_n^h)_2$.

The proof of this is given in Appendix B.1.

Lemma 4.2. $\phi_n(\mathsf{E}_n) = V_n(x,y)$.

Proof. We use induction on n. If $n = 1$ then $\mathsf{E}_1 = 1$ and $V_1 = x = \phi_1(1)$. Now suppose that $\phi_{k-1}(\mathsf{E}_{k-1}) = V_{k-1}$ for $k < n$. The lift of E_{k-1} from $\mathfrak{TL}_{k-1}(x,y)$ to $\mathfrak{TL}_k(x,y)$ appends a strand to each of the strand diagrams, and therefore a region whose (implicit) shading is specified by the parity of k. Then $\phi_k(\mathsf{E}_{k-1}) = x_k\phi_{k-1}(\mathsf{E}_{k-1}) = x_kV_{k-1}$, by hypothesis. From (9) we have $\phi_k(\mathsf{E}_k) = x_kV_{k-1} - \mu_{k-1}\phi_k(\mathsf{E}_{k-1}\mathsf{e}_{k-1}\mathsf{E}_{k-1})$. But ϕ_k is a Markov trace and E_{k-1} is idempotent, so

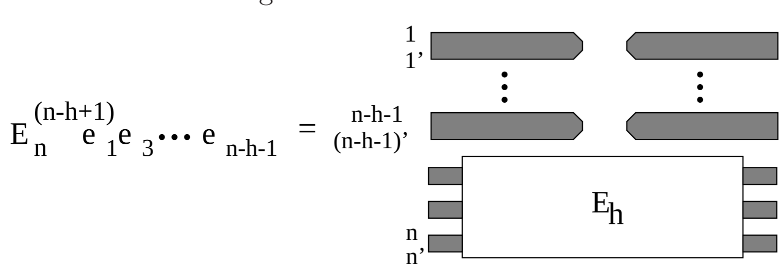
$$\phi_k(\mathsf{E}_k) = x_kV_{k-1} - \mu_{k-1}\phi_k(\mathsf{E}_{k-1}\mathsf{e}_{k-1}) = x_kV_{k-1} - \mu_{k-1}\phi_{k-1}(\mathsf{E}_{k-1}) = x_kV_{k-1} - V_{k-2} = V_k$$

from (12) and the induction hypothesis. The result follows. □

We remark that this is another result in which the parity condition S1 of Definition 2.3 is required.

Lemma 4.3. $\phi_n((\mathfrak{I}_n^h)_2) = V_h$ *whenever* $\mathfrak{I}_n^h$ *exists (*i.e. *when* $n \equiv h \bmod 2$*).*

Proof. Recall that $(\mathfrak{I}_n^h)_2 = x^{-(n-h)/2}\mathsf{E}_n^{(n-h+1)}\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}$. But, from the remark following Corollary 3.3, $\mathsf{E}_n^{(n-h+1)}$ has the same structure as $\mathsf{E}_h \in \mathfrak{TL}_h(x,y)$. Now $n-h+1$ is odd so, by considering the number of regions contributed by E_h, and the fact that each e_{2i+1} contributes a shaded region, it follows from the diagram



that $\phi_n(\mathsf{E}_n^{(n-h+1)}\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}) = x^{(n-h)/2}\phi_h(\mathsf{E}_h)$, so $\phi_n((\mathfrak{I}_n^h)_2) = \phi_h(\mathsf{E}_h) = V_h$, from Lemma 4.2. □

We may now give an explicit expression for the entries of the Gram matrix $\langle\cdot,\cdot\rangle_n$ with respect to the basis $\mathcal{B}_2$.

Corollary 4.4. *If* a,a' *are monomials and* $[\mathsf{a}]$ *has mid-height* h *then*

$$\langle[\mathsf{a}]_2,[\mathsf{a}']_2\rangle_n=\delta_{\mathsf{a},\mathsf{a}'}V_h.$$

Proof. Let $[\mathsf{a}]=\mathfrak{l}\mathfrak{r}$ and $[\mathsf{a}']=\mathfrak{l}'\mathfrak{r}'$ be midpoint decompositions. Then, from (6) and Theorem 3.6, $\langle[\mathsf{a}]_2,[\mathsf{a}']_2\rangle_n=\phi_n((\mathfrak{l}\mathfrak{r})_2(\mathfrak{r}'\mathfrak{l}')_2)=\delta_{\mathfrak{r},\mathfrak{r}'}\phi_n((\mathfrak{l}\mathfrak{l}')_2)$. Since the trace is invariant under a cyclic shift of its argument, then $\langle[\mathsf{a}]_2,[\mathsf{a}']_2\rangle_n=\phi_n((\mathfrak{r}'\mathfrak{l}')_2(\mathfrak{l}\mathfrak{r})_2)=\delta_{\mathfrak{l},\mathfrak{l}'}\phi_n((\mathfrak{r}'\mathfrak{r})_2)$. Thus $\langle[\mathsf{a}]_2,[\mathsf{a}']_2\rangle_n=0$ if $[\mathsf{a}]\neq[\mathsf{a}']$.

In particular, $\langle[\mathsf{a}]_2,[\mathsf{a}]_2\rangle_n=\phi_n((\mathfrak{l}\mathfrak{l})_2)$ and is therefore independent of $\mathfrak{r}$. Similarly, it is independent of $\mathfrak{l}$ so it depends only on h and n. Therefore, $\langle[\mathsf{a}]_2,[\mathsf{a}]_2\rangle_n=\langle(\mathfrak{I}_n^h)_2,(\mathfrak{I}_n^h)_2\rangle_n=\phi_n((\mathfrak{I}_n^h)_2(\mathfrak{I}_n^h)_2^t)$, from (6). But $(\mathfrak{I}_n^h)_2^t=(\mathfrak{I}_n^h)_2$ so, from Lemma 4.1, $\phi_n((\mathfrak{I}_n^h)_2(\mathfrak{I}_n^h)_2^t)=\phi_n((\mathfrak{I}_n^h)_2)$ and the result follows from Lemma 4.3. □

Let $\mathbf{D}_n(x,y)$ be the matrix of $\langle\cdot,\cdot\rangle_n$ with respect to the basis $\mathcal{B}_2=\{[\mathsf{a}]_2\colon i=1,\ldots,C_n\}$ of $\mathfrak{TL}_n(x,y)$. Let $\mathbf{P}_n$ be the matrix that expresses the elements of $\mathcal{B}_2$ in terms of the elements of $\mathcal{B}_1$. Then $\mathbf{P}_n^t\mathbf{M}_n(x,y)\mathbf{P}_n=\mathbf{D}_n(x,y)$. Now $\mathbf{D}_n=[\langle\mathsf{a},\mathsf{a}'\rangle]_{\mathsf{a},\mathsf{a}'\in\mathfrak{TL}_n(x,y)}$ so, from Corollary 4.4, $\mathbf{D}_n$ is generically invertible. Thus $\mathbf{M}_n(x,y)$ is also. Moreover,

$$\mathbf{M}_n^{-1}(x,y)=\mathbf{P}_n\mathbf{D}_n^{-1}(x,y)\mathbf{P}_n^t. \tag{17}$$

4.2. **The change of basis matrix.** We shall be able to select an ordering of $\mathcal{B}_2$ such that the change of basis matrix $\mathbf{P}_n$ is a triangular matrix. This ordering will assist in the explicit determination of elements of $\mathbf{M}_n^{-1}(x,y)$. For this purpose, let $\mathcal{L}(\mathcal{P}_n,\prec_{\mathsf{gi}})$ be the lattice of paths in $\mathcal{P}_n$ where $\mathfrak{p}_1\prec_{\mathsf{gi}}\mathfrak{p}_2$ (geometric inclusion) if $\mathfrak{p}_2$ may be obtained from $\mathfrak{p}_1$ by box additions.

Lemma 4.5. *The basis* $\mathcal{B}_2=\{[\mathsf{a}_i]_2\colon i=1,\ldots,C_n\}$ *of* $\mathfrak{TL}_n(x,y)$ *has the property that*

$$[\mathsf{a}_m]_2\in\operatorname{span}_{\mathbb{C}(x,y)}\{\mathsf{a}\in\mathcal{B}_1\colon[\mathsf{a}]\prec_{\mathsf{gi}}[\mathsf{a}_m]\}$$

for $m=1,\ldots,C_n$.

Proof. We use induction on $\#_\diamond(\mathsf{a}_m)$. The base case is $\mathfrak{I}_n^h$, where h is the mid-height of $[\mathsf{a}_m]$, since $\#_\diamond(\mathfrak{I}_n^h)=0$. Now $(\mathfrak{I}_n^h)_2=x^{-(n-h)/2}\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}\mathsf{E}_n^{(n-h+1)}$ where $\mathsf{E}_n^{(n-h+1)}$ is generated by $\mathsf{e}_{n-h+1},\ldots,\mathsf{e}_{n-1}$. Thus every term in the expansion of $(\mathfrak{I}_n^h)_2$ has the form $\mathsf{a}=\mathsf{e}_1\mathsf{e}_3\cdots\mathsf{e}_{n-h-1}\mathsf{f}$ where $\mathsf{f}\in\langle\mathsf{e}_{n-h+1},\ldots,\mathsf{e}_{n-1}\rangle$. Now consider the concatenation of the strand diagrams of $\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}$ and f to obtain the product a. Since the last h strands of the strand diagram of $\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}$ are horizontal and since $\mathsf{f}\in\langle\mathsf{e}_{n-h+1},\ldots,\mathsf{e}_{n-1}\rangle$, concatenation with the strand diagram of f affects only these last h strands. So

$$[\mathsf{a}]=[\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}\mathsf{f}]=(\nearrow\searrow)^j\,\mathfrak{p}\,(\nearrow\searrow)^j$$

for some $\mathfrak{p}\in\mathcal{P}_h$. But $\nearrow^h\searrow^h$ is the maximal element of the lattice $\mathcal{L}(\mathcal{P}_h,\prec_{\mathsf{gi}})$ so $\mathfrak{p}\prec_{\mathsf{gi}}\nearrow^h\searrow^h$. Moreover,

$$[\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}]=\mathfrak{I}_n^h=(\nearrow\searrow)^j\,(\nearrow^h\searrow^h)\,(\nearrow\searrow)^j,$$

where j is determined from n and h by $n=h-2j$. It follows that $[\mathsf{a}]\prec_{\mathsf{gi}}\mathfrak{I}_n^h$. Thus

$$(\mathfrak{I}_n^h)_2\in\operatorname{span}_{\mathbb{C}(x,y)}\left\{\mathsf{a}\in\mathcal{B}_1\colon[\mathsf{a}]\prec_{\mathsf{gi}}\mathfrak{I}_n^h\right\},$$

establishing the base case.

Now consider $\mathsf{b} \in \mathcal{B}_1$, where $\#_\diamond(\mathsf{b}) > 0$. As the inductive hypothesis we assume that the result is true for all monomials with fewer than $\#_\diamond(\mathsf{b})$ boxes. Then $[\mathsf{b}] = [\mathsf{a}_m] \boxplus \diamond_i$ for some i, so $[\mathsf{a}_m] \prec_{\mathsf{gi}} [\mathsf{b}]$. We assume that $i < n$; the proof for the case $i > n$ is similar and is therefore not included. Now $[\mathsf{b}]_2 = ([\mathsf{a}_m] \boxplus \diamond_i)_2 = \sqrt{\frac{\mu_{j+1}}{\mu_j}}(\mathsf{e}_i - \mu_j)\mathsf{a}_m$ where $j = h_i([\mathsf{a}_m]) + 1$. Then, by hypothesis,

$$\begin{aligned} [\mathsf{b}]_2 &\in \operatorname{span}_{\mathbb{C}(x,y)}\{\mathsf{e}_i\mathsf{a} - \mu_j\mathsf{a} \colon [\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{a}_m]\} \\ &\subset \operatorname{span}_{\mathbb{C}(x,y)}\{\mathsf{e}_i\mathsf{a} \colon [\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{a}_m]\} \cup \operatorname{span}_{\mathbb{C}(x,y)}\{\mathsf{a} \colon [\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{a}_m]\}. \end{aligned}$$

We now consider the expansion of $\mathsf{e}_i\mathsf{a}$. There are three cases, and in each of them we show that $[\mathsf{e}_i\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{b}]$ for each $[\mathsf{e}_i\mathsf{a}]$ in the span.

Case I: Assume that $[\mathsf{a}]$ has a maximum in position i. Then $[\mathsf{a}] = [\mathsf{e}_i\mathsf{f}]$ for some $\mathsf{f} \in \mathfrak{TL}_n(x,y)$. Thus $[\mathsf{e}_i\mathsf{a}] = [\mathsf{e}_i^2\mathsf{f}] = [\mathsf{e}_i\mathsf{f}] = [\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{a}_m] \prec_{\mathsf{gi}} [\mathsf{b}]$, so $[\mathsf{e}_i\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{b}]$ in this case.

Case II: Assume that $[\mathsf{a}]$ has a minimum in position i. Then $[\mathsf{e}_i\mathsf{a}] = [\mathsf{a}] \boxplus \diamond_i \prec_{\mathsf{gi}} [\mathsf{a}_m] \boxplus \diamond_i = \mathsf{b}$, so $[\mathsf{e}_i\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{b}]$ in this case.

Case III: Assume that $[\mathsf{a}]$ has a slope in position i. We assume that the slope is a double rise. The proof for a double fall is similar and is therefore not included.

The arch diagram of a has two arches with heads in consecutive positions. In $[\mathsf{a}]$, these correspond to rises $\overset{i-1}{\nearrow}$ and $\overset{i}{\nearrow}$ in positions $i-1$ and i, together with matching falls $\overset{(i-1)}{\searrow}$ and $\overset{(i)}{\searrow}$, respectively (the bracketted index indicates the matching), corresponding to the ends of the two arches. Thus

$$[\mathsf{a}] = \mathfrak{a} \overset{i-1}{\nearrow} \left(\overset{i}{\nearrow} \mathfrak{b} \overset{(i)}{\searrow} \right) \mathfrak{c} \overset{(i-1)}{\searrow} \mathfrak{d}$$

for some paths $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$. Now consider the concatenation of the strand diagram e_i and the strand diagram a. This affects precisely two strands in the strand diagram a, leaving all the other strands untouched. This is illustrated in Figure 9. It is immediately seen that the corresponding path is

FIGURE 9. The strand diagram a on the left and $\mathsf{e}_i\mathsf{a}$ on the right.

this next expression should be forced to appear before the diagram.

$$[\mathsf{e}_i\mathsf{a}] = \mathfrak{a} \overset{i-1}{\nearrow} \left(\overset{i}{\searrow} \mathfrak{b} \overset{(i)}{\nearrow} \right) \mathfrak{c} \overset{(i-1)}{\searrow} \mathfrak{d}.$$

Thus $[\mathsf{e}_i\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{b}]$ in this case. Combining these cases, we have

$$[\mathsf{b}]_2 \in \operatorname{span}_{\mathbb{C}(x,y)}\{\mathsf{a} \in \mathcal{B}_1 \colon [\mathsf{a}] \prec_{\mathsf{gi}} [\mathsf{b}]\}$$

so the inductive step holds, which completes the proof. □

We may now establish that $\mathbf{P}_n$ is triangular and obtain the elements on its diagonal. The following path statistic is needed. Let $\mathfrak{p} \in \mathcal{P}_n$ with $\mathfrak{p} = \mathfrak{l}\mathfrak{r}$ its midpoint decomposition. Then the

i-th fall in $\mathfrak{l}$ marks a diagonal strip as does the j-th rise in $\mathfrak{r}$. In Figure 10 the shaded regions are the strips of length three. Note that the strips have different orientations depending on whether they occur before or after the midpoint. This means that there are $n-h$ strips in a path $\mathfrak{p} \in \mathcal{P}_n$ with middle height h.

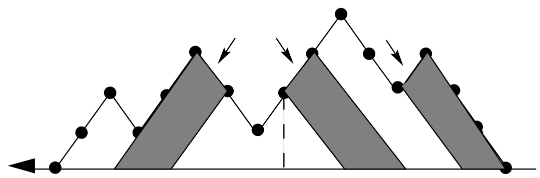


FIGURE 10. Strips of length 3 in path.

Corollary 4.6.

1) $\mathbf{P}_n$ *is triangular,*
2) $[\mathbf{P}_n]_{k,k} = \prod_{i=1}^{n} (\sqrt{\mu_i})^{s_{i,k}}$,

where $s_{i,k}$ *is the number of strips of length* i *in the path* $[\mathsf{a}_k]$.

Proof. 1) A linear extension of the lattice $\mathcal{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$ can be constructed by shelling the lattice from its top element. This shelling order gives a labelling $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_{C_n}$ of the monomials of $\mathfrak{TL}_n(x,y)$ such that $[\mathsf{a}_i] \prec_{\mathsf{gi}} [\mathsf{a}_j] \Rightarrow i < j$. Let $1 \le m \le C_n$. Then there exist $\gamma_{1,m}, \gamma_{2,m}, \ldots, \gamma_{m,m} \in \mathbb{C}(x,y)$ together with a labelling of $\mathsf{a}_1, \ldots, \mathsf{a}_{C_n}$ such that $[\mathsf{a}_m]_2 = \sum_{i=1}^{m} \gamma_{i,m} \mathsf{a}_i$. Thus $\mathbf{P}_n$ is triangular.

2) From Lemma 4.5, $[\mathbf{P}_n]_{k,k}$ is equal to the coefficient of a_k in the expansion of $[\mathsf{a}_k]_2$ in the basis $\mathcal{B}_1$. Suppose $[\mathsf{a}_k]$ has mid-height h. We determine $[\mathsf{a}_k]_2$ by box addition to $\mathfrak{I}_n^h$ to build $[\mathsf{a}_k]$. From Definition 3.5, each box addition at height $j-1$ contributes a multiplicative factor $\sqrt{\frac{\mu_j}{\mu_{j-1}}}$ to $[\mathbf{P}_n]_{k,k}$. A strip of length i therefore contributes a multiplicative factor

$$\sqrt{\frac{\mu_i}{\mu_{i-1}}}\sqrt{\frac{\mu_{i-1}}{\mu_{i-2}}} \cdots \sqrt{\frac{\mu_2}{\mu_1}} = \sqrt{\frac{\mu_i}{\mu_1}}.$$

From (16), $(\mathfrak{I}_n^h)_2 = x^{-(n-h)/2} \mathsf{E}_n^{(n-h+1)} \mathsf{e}_1 \mathsf{e}_3 \ldots \mathsf{e}_{n-h+1}$ contributes the multiplicative factor $x^{-(n-h)/2}$ and, from Corollary 3.3 (1), the coefficient of $\mathsf{1}$ in $\mathsf{E}_n^{(n-h+1)}$ is 1. Since there are $\sum_{i=1}^{n} s_{i,k} = n-h$ strips in a_k and $\mu_1 = 1/x$ we have $[\mathbf{P}_n]_{k,k} = x^{-(n-h)/2} \prod_{i=1}^{n} \sqrt{\frac{\mu_i}{\mu_1}}^{s_{i,k}} = \prod_{i=1}^{n} \sqrt{\mu_i}^{s_{i,k}}$. □

As an example of orderings of $\mathcal{B}_1$, Figure 11 gives a possible ordering for $\mathcal{B}_1$ in $\mathcal{L}(\mathcal{P}_4, \prec_{\mathsf{gi}})$ as the

$$(\mathsf{e}_1\mathsf{e}_3, \mathsf{e}_1\mathsf{e}_3\mathsf{e}_2, \mathsf{e}_1, \mathsf{e}_2\mathsf{e}_1\mathsf{e}_3, \mathsf{e}_1\mathsf{e}_2, \mathsf{e}_2\mathsf{e}_1\mathsf{e}_3\mathsf{e}_2, \mathsf{e}_2\mathsf{e}_1, \mathsf{e}_3\mathsf{e}_2\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1\mathsf{e}_2\mathsf{e}_3, \mathsf{e}_3\mathsf{e}_2, \mathsf{e}_2\mathsf{e}_3, \mathsf{e}_3, \mathsf{1}).$$

Similarly, a possible ordering for $\mathcal{B}_1$ in $\mathcal{L}(\mathcal{P}_3, \prec_{\mathsf{gi}})$ is $(\mathsf{e}_1, \mathsf{e}_2\mathsf{e}_1, \mathsf{e}_1\mathsf{e}_2, \mathsf{e}_2, \mathsf{1})$. Then, by explicit computation, $[\mathsf{e}_2\mathsf{e}_1]_2 = \sqrt{\frac{\mu_2}{\mu_1}}(\mathsf{e}_2 - \mu_1)\mu_1\mathsf{e}_1 = \sqrt{\mu_1\mu_2}(-\mu_1\mathsf{e}_1 + \mathsf{e}_2\mathsf{e}_1)$ and it is noted that this is independent of the elements $\mathsf{e}_1\mathsf{e}_2$, e_2 and $\mathsf{1}$ in $\mathcal{B}_1$.

5. A recursion for the elements of $\mathbf{M}_n^{-1}(x,y)$

We may now address the matter of constructing a recursion for the elements of $\mathbf{M}_n^{-1}(x,y)$, and therefore, by specialization, a recursion for the elements of $\mathbf{L}_n(q)$. We propose to do this by first obtaining a recursion for the elements of $\mathbf{P}_n$, for then the elements of $\mathbf{M}_n^{-1}(x,y)$ may be obtained through (17).

The determination of $\mathbf{M}_n^{-1}(x,y)$ has been reduced to the problem of expressing the basis $\mathcal{B}_2$ in terms of the elements of $\mathcal{B}_1$. This, in principle, can be done through the recursive of Definition 3.5 by means of box additions. However, the construction terminates on the base path $\mathfrak{I}_n^k$, which contains no boxes and, to complete the construction of $\mathcal{B}_2$, it would be necessary to use the difficult non-linear recursion given in (9) for $\mathsf{E}_n^{(k)}$. Recall that box additions cannot be made at the midpoint of a path. We now show that it is possible to avoid this difficulty and to give a recursion for the elements of $\mathcal{B}_2$ that operates uniformly down to the base path $\mathfrak{I}_n^0$ or $\mathfrak{I}_n^{(1)}$. To do this, we append a particular fixed path to the paths $\mathfrak{p}$ in $\mathcal{P}_n$ so the midpoint of $\mathfrak{p}$, at which box addition cannot be carried out, is no longer the midpoint. It is on the augmented path that the desired box addition can be made. This means that we work in a left ideal of $\mathfrak{TL}_{2n}(x,y)$.

5.1. **The ideal $\mathfrak{K}_{2n}(x,y)$.** We begin by defining a natural map from $\mathfrak{TL}_n(x,y)$ to $\mathfrak{K}_{2n}(x,y) = \mathfrak{TL}_{2n}(x,y)(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{2n-1})$. This map is defined on monomials by $\mathsf{e}\mapsto([\mathsf{e}]\,(\nearrow\searrow)^n)_1$, and extended linearly to $\mathfrak{TL}_n(x,y)$. Thus $\Gamma(\mathsf{e})$ is the element of $\mathfrak{K}_{2n}(x,y)$ whose path is obtained from $[\mathsf{a}]$ by concatenation on the right by $(\nearrow\searrow)^n$, the path corresponding to $\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{2n-1}$. Clearly Γ is bijective. To simplify subsequent expressions we shall abuse notation slightly and denoted by $\Gamma((\mathfrak{p})_1)$ by $\Gamma(\mathfrak{p})$ where $\mathfrak{p}$ is a path.

Lemma 5.1. (1) $\Gamma([\mathsf{a}]+\diamond_i)=\Gamma[\mathsf{a}]+\diamond_i$ *for* $[\mathsf{a}]\in\mathcal{P}_n$ *and* $1\le i\le 2n-1$.
(2) $\Gamma(\mathsf{e}_i\mathsf{e})=\mathsf{e}_i\Gamma(\mathsf{e})$ *for* $\mathsf{e}\in\mathfrak{TL}_n(x,y)$ *and* $1\le i\le n-1$.
(3) $\Gamma(\mathsf{e}\mathsf{e}_i)=\mathsf{e}_{2n-i}\Gamma(\mathsf{e})$ *for* $\mathsf{e}\in\mathfrak{TL}_n(x,y)$ *and* $1\le i\le n-1$.

Proof. Part 1 follows immediately by considering box additions on paths. Parts 2 and 3 follow immediately by considering concatenation of strand diagrams. □

The next result shows that the basis $\mathcal{B}_2$ for $\mathfrak{TL}_n(x,y)$ and $\mathfrak{K}_{2n}(x,y)$ are directly related. This correspondence and the fact that box additions at position n in paths associated with $\mathfrak{K}_{2n}(x,y)$ are allowed will enable us to obtain a general recursion for all elements in the basis $\mathcal{B}_2$ of $\mathfrak{TL}_n(x,y)$ without resorting to the E_i's.

Lemma 5.2. *Let* $[\mathsf{a}]\in\mathcal{P}_n$. *Then*

$$\Gamma([\mathsf{a}]_2)=\sqrt{x^nV_k}[\Gamma([\mathsf{a}])]_2,$$

where $k=h(\mathsf{a})$.

Proof. First note that $[\Gamma(\mathsf{a})]$ is a path corresponding to a monomial in $\mathfrak{K}_{2n}(x,y)$. The corresponding $\mathcal{B}_2$ basis element $[\Gamma([\mathsf{a}])]_2$ is obtained by regarding $\mathfrak{K}_{2n}$ as a subset of $\mathfrak{TL}_{2n}(x,y)$. Then, by Theorem 4.5, the element $[\Gamma([\mathsf{a}])]_2$ belongs to $\mathfrak{K}_{2n}(x,y)$.

We will prove the result by induction on the number of boxes in $[\mathsf{a}]$. The base case is $[\mathsf{a}]=\mathfrak{I}_n^k$ since $\#_\diamond(\mathfrak{I}_n^k)=0$. Recall from (16) that $(\mathfrak{I}_n^k)_2=x^{-(n-k)/2}\mathsf{E}_n^{(n-k+1)}\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k-1}$ where, from Corollary 3.3, $\mathsf{E}_n^{(n-k+1)}$ is generated by $\mathsf{e}_{n-k+1},\ldots,\mathsf{e}_{n-1}$. Let

$$\mathsf{F}=\Gamma^{-1}\left((\Gamma\mathfrak{I}_n^k)_2\right), \tag{18}$$

so $\mathsf{F} \in \mathfrak{TL}_n(x,y)$. Our intention is to show that F is equal to $(\mathfrak{I}_n^k)_2$, up to a multiplicative scalar in $\mathbb{C}(x,y)$. Now if $n-k+1 \le i \le n-1$ then $\mathsf{e}_i\mathsf{F} = \Gamma^{-1}(\mathsf{e}_i(\mathfrak{I}_n^k)_2) = \Gamma^{-1}(0) = 0$ since, by Lemma B.1, $\mathfrak{I}_n^k$ has a slope in position i. Similarly $\mathsf{F}\mathsf{e}_i = 0$ and thus

$$\mathsf{e}_i\mathsf{F} = 0 = \mathsf{F}\mathsf{e}_i \qquad \text{for } i = n-k+1,\ldots,n-1. \tag{19}$$

Moreover, from Lemma 4.5,

$$(\Gamma\mathfrak{I}_n^k)_2 = \sum_{\substack{\mathsf{e}\in\mathfrak{K}_{2n}(x,y)\\ [\mathsf{e}]\prec_{\mathsf{gi}}[\Gamma\mathfrak{I}_n^k]}} \beta_{\mathsf{e}}\mathsf{e}$$

where $\beta_{\mathsf{e}} \in \mathbb{C}(x,y)$, so

$$\mathsf{F} = \sum_{\substack{\mathsf{e}\in\mathfrak{K}_{2n}(x,y)\\ [\mathsf{e}]\prec_{\mathsf{gi}}[\Gamma\mathfrak{I}_n^k]}} \beta_{\mathsf{e}}\Gamma^{-1}\mathsf{e} = \sum_{\substack{\mathsf{e}\in\mathfrak{TL}_n(x,y)\\ [\mathsf{f}]\prec_{\mathsf{gi}}\mathfrak{I}_n^k}} \beta_{\mathsf{f}}\mathsf{f}.$$

But $\mathfrak{I}_n^k = (\nearrow\searrow)^j(\nearrow^k\searrow^k)(\nearrow\searrow)^j$ where $2j = n-k$, so $[\mathsf{f}] = (\nearrow\searrow)^j\mathfrak{p}(\nearrow\searrow)^j$ where $\mathfrak{p} \prec_{\mathsf{gi}} \nearrow^k\searrow^k$. Then F has the form $\mathsf{F} = \beta(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k-1})\mathsf{F}'$, for some $\beta \in \mathbb{C}(x,y)$, where $\mathsf{F}'-\mathsf{1}$ is generated by $\mathsf{e}_{n-k+1},\ldots,\mathsf{e}_{n-1}$, and where the $\mathsf{1}$ arises from the contribution to the sum from $[\mathsf{f}] = \mathfrak{I}_n^k$. Then, from (19), $0 = \mathsf{e}_i\mathsf{F} = \beta(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k+1})(\mathsf{e}_i\mathsf{F}')$ so $\mathsf{e}_i\mathsf{F}' = 0$ and, similarly, $\mathsf{F}'\mathsf{e}_i = 0$, for $i = n-k+1,\ldots,n-1$.

Then, by Corollary 3.3, F' is unique so $\mathsf{F}' = \mathsf{E}_n^{(k)}$. Thus $\mathsf{F} = \beta(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k-1})\mathsf{E}_n^{(k)} = \gamma(\mathfrak{I}_n^k)_2$ where $\gamma \in \mathbb{C}(x,y)$. Thus $(\Gamma\mathfrak{I}_n^k)_2 = \gamma\Gamma\left((\mathfrak{I}_n^k)_2\right)$. We determine γ by comparing the coefficients of $\Gamma(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k-1})$ in $(\Gamma\mathfrak{I}_n^k)_2$ and $\Gamma\left((\mathfrak{I}_n^k)_2\right)$.

In the construction of $\left(\Gamma(\mathfrak{I}_n^k)\right)_2$, each box addition at height j contributes a factor of $\sqrt{\mu_{j+1}/\mu_1}$ to the coefficient of $\Gamma(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k-1})$. This coefficient is equal to

$$\mu_1^n\sqrt{\frac{\mu_k}{\mu_1}}\sqrt{\frac{\mu_{k-1}}{\mu_1}}\cdots\sqrt{\frac{\mu_1}{\mu_1}} = \frac{\mu_1^{n-k/2}}{\sqrt{V_k}},$$

by counting strips, using the argument in the proof of Corollary 4.6. Moreover, the coefficient of $\Gamma(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k-1})$ in $\Gamma\left((\mathfrak{I}_n^k)_2\right)$ is equal to the coefficient of $\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-k-1}$ in $(\mathfrak{I}_n^k)_2$, which is $\mu_1^{(n-k)/2}$. Thus $\gamma^{-1} = \sqrt{x^nV_k}$, and the base case of the induction is established.

For the inductive step, we assume that the result holds for $[\mathsf{a}]$, and we consider the addition of a box at position i and height j in $[\mathsf{a}]$. It will be assumed that $i < n$. The case $i > n$ is similar and will not be given. From Definition 3.5,

$$\begin{aligned}\Gamma\left(([\mathsf{a}]\boxplus\diamond_i)_2\right) &= \sqrt{\frac{\mu_{j+1}}{\mu_j}}\Gamma\left((\mathsf{e}_i-\mu_j)[\mathsf{a}]_2\right)\\ &= \sqrt{\frac{\mu_{j+1}}{\mu_j}}(\mathsf{e}_i-\mu_j)\Gamma([\mathsf{a}]_2) \qquad \text{(by Lemma 5.1)}\\ &= \sqrt{\frac{\mu_{j+1}}{\mu_j}}\sqrt{x^nV_k}\,(\mathsf{e}_i-\mu_j)[\Gamma(\mathsf{a})]_2\end{aligned}$$

by the induction hypothesis. Thus, by Definition 3.5,

$$\Gamma\left(([\mathsf{a}]\boxplus\diamond_i)_2\right) = \sqrt{x^nV_k}\ \left([\Gamma(\mathsf{a})]\boxplus\diamond_i\right)_2 = \sqrt{x^nV_k}\ [\Gamma([\mathsf{a}]\boxplus\diamond_i)]_2$$

from Lemma 5.1. This completes the induction, and the result follows. □

The next example shows the use of Γ in constructing E_2 in $\mathfrak{TL}_2(x,y)$ by box addition rather than by the non-linear recursion given in (9).

Example 5.3. From Definition 3.5 we note that $\mathsf{E}_2 = (\mathfrak{I}_2^2)_2$. Then, by considering the concatenation of paths, we have $[\Gamma\mathfrak{I}_2^2] = \mathfrak{I}_4^0 \boxplus \diamond_2$, so $(\Gamma\mathfrak{I}_2^2)_2 = (\mathfrak{I}_4^0 \boxplus \diamond_2)_2$. Thus

$$\mathsf{E}_2 = x\sqrt{V_2}\,\Gamma^{-1}\left((\mathfrak{I}_4^0 \boxplus \diamond_2)_2\right)$$

from Lemma 5.2. To evaluate this expression we work in $\mathfrak{TL}_4(x,y)$. From Definition 3.5, it follows that $(\mathfrak{I}_4^0 \boxplus \diamond_2)_2 = \sqrt{\mu_2/\mu_1}\,(\mathsf{e}_2 - \mu_1)(\mathfrak{I}_4^0)_2$ where $(\mathfrak{I}_4^0)_2 = x^{-2}\mathsf{e}_1\mathsf{e}_3$ from (16) and the convention of Corollary 3.3. Thus

$$(\mathfrak{I}_4^0 \boxplus \diamond_2)_2 = x^{-2}\sqrt{\frac{\mu_2}{\mu_1}}(\mathsf{e}_2\mathsf{e}_1\mathsf{e}_3 - \mu_1\mathsf{e}_1\mathsf{e}_3).$$

To apply Γ^{-1}, note that $[\mathsf{e}_1\mathsf{e}_3] = (\nearrow\searrow)^2(\nearrow\searrow)^2$ so $[\Gamma^{-1}(\mathsf{e}_1\mathsf{e}_3)] = (\nearrow\searrow)^2 = [\mathsf{e}_1]$ whence $\mathsf{e}_1\mathsf{e}_3 = \Gamma\mathsf{e}_1$. Similarly, by path concatenation, $[\mathsf{e}_2\mathsf{e}_1\mathsf{e}_3] = (\nearrow^2\searrow^2)(\nearrow\searrow)^2$ so $[\Gamma^{-1}(e_2\mathsf{e}_1\mathsf{e}_3)] = \nearrow^2\searrow^2 = [1]$ whence $\mathsf{e}_2\mathsf{e}_1\mathsf{e}_3 = \Gamma 1$. Thus $\mathsf{e}_2\mathsf{e}_1\mathsf{e}_3 - \mu_1\mathsf{e}_1\mathsf{e}_3 = \Gamma(1-\mu_1\mathsf{e}_2)$, so $\mathsf{E}_2 = x\sqrt{V_2}x^{-2}\sqrt{\mu_2/\mu_1}(1-\mu_1\mathsf{e}_1)$. But $V_0 = 1, V_1 = x$ and $V_2 = xy-1$ from (11) and (12), so $\mu_1 = x^{-1}$ and $\mu_2 = x(xy-1)^{-1}$. Therefore,

$$\mathsf{E}_2 = 1 - x^{-1}\mathsf{e}_1.$$

This is, of course, in agreement with the recursion for the Jones-Wenzl projectors given in (9).

5.2. **The recursion.** We may now give a linear recursion for the elements of $\mathbf{P}_n^{-1}$ that operates on paths by box addition in $\mathbf{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$ using scalars from $\mathbb{C}(x,y)$. The following notation will be used for this purpose. Let $\mathfrak{p}$ be a path in $\mathcal{P}_n$ with a double rise at position i. Then $\mathfrak{p}$ has the form

$$\mathfrak{p} = \mathfrak{a} \overset{i-1}{\nearrow} \left(\overset{i}{\nearrow} \mathfrak{b} \overset{(i)}{\searrow} \right) \mathfrak{c} \overset{(i-1)}{\searrow} \mathfrak{d}$$

where $\overset{i}{\nearrow}$ denotes a rise at position i and $\overset{(i)}{\searrow}$ denotes the matching step, so $\overset{i}{\nearrow}$ and $\overset{(i)}{\searrow}$ are associated with the same arch (the bracketted i indicates this pairing and *not* the position of the step). There is a similar form for $\mathfrak{p}$ if it has a double fall at position i. So, if $\mathfrak{p}$ has a slope at position i, let

$$\kappa_i(\mathfrak{p}) = \begin{cases} \mathfrak{a} \overset{i-1}{\nearrow} \left(\overset{i}{\searrow} \mathfrak{b} \overset{(i)}{\nearrow} \right) \mathfrak{c} \overset{(i-1)}{\searrow} \mathfrak{d} & \text{if} \quad \mathfrak{p} = \mathfrak{a} \overset{i-1}{\nearrow} \left(\overset{i}{\nearrow} \mathfrak{b} \overset{(i)}{\searrow} \right) \mathfrak{c} \overset{(i-1)}{\searrow} \mathfrak{d} \\ \mathfrak{a} \overset{(i)}{\nearrow} \mathfrak{b} \left(\overset{(i-1)}{\searrow} \mathfrak{c} \overset{i-1}{\nearrow} \right) \overset{i}{\searrow} \mathfrak{d} & \text{if} \quad \mathfrak{p} = \mathfrak{a} \overset{(i)}{\nearrow} \mathfrak{b} \left(\overset{(i-1)}{\nearrow} \mathfrak{c} \overset{i-1}{\searrow} \right) \overset{i}{\searrow} \mathfrak{d}. \end{cases}$$

Thus $\kappa_i(\mathfrak{p})$ is obtained by collapsing $\mathfrak{p}$ according to the type of step at position i. In the first case the subpath $\mathfrak{b}$ is 'lowered' by the complementation of the rise at position i and the matching step; in the second case the subpath $\mathfrak{c}$ is 'lowered' by complementation of the fall at position i and the matching step. We shall call $\kappa_i(\mathfrak{p})$ the *collapse* of $\mathfrak{p}$ at the slope at position i.

If $\mathfrak{q}$ is a path in $\mathcal{P}_n$ with a slope at position i, let $\kappa^{-1}(\mathfrak{q})$ denote the set of all paths $\mathfrak{p}$ such that $\kappa_i(\mathfrak{p}) = \mathfrak{q}$. For example, $\kappa_3^{-1}((\nearrow\searrow)^3) = \{\nearrow^2\searrow^2\nearrow\searrow, \nearrow\searrow\nearrow^2\searrow^2\}$.

It is a straightforward matter to show, by carrying out the product through concatenation of arch diagrams, that

$$[\mathsf{e}_i(\mathfrak{p})_1] = \kappa_i(\mathfrak{p}) \tag{20}$$

where $\mathfrak{p}$ has a slope at position i. We may now state the recursion for elements of $\mathbf{P}_n$.

Theorem 5.4. *Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{P}_n$ and let $\mathfrak{b}$ have a minimum at position i. Then*

$$[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}\boxplus\diamond_i} = -\Delta_{i,\mathfrak{b}}\sqrt{\mu_j\mu_{j+1}}\,[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}} + \delta_{i,\mathfrak{a}}\Delta_{i,\mathfrak{b}}\sqrt{\frac{\mu_{j+1}}{\mu_j}}\left(x_i[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}} + [\mathbf{P}_n]_{\mathfrak{a}\boxminus\diamond_i,\mathfrak{b}} + \sum_{\mathfrak{e}\in\kappa_i^{-1}(\mathfrak{a})}[\mathbf{P}_n]_{\mathfrak{e},\mathfrak{b}}\right)$$

where

$$\delta_{i,\mathfrak{a}} = \begin{cases} 1 & \text{if } \mathfrak{a} \text{ has a maximum at position } i, \\ 0 & \text{otherwise} \end{cases}, \qquad \Delta_{i,\mathfrak{b}} = \begin{cases} \sqrt{\frac{V_{j+1}}{V_{j-1}}} & \text{if } \; i = n, \\ 1 & \text{if } \; i \neq n \end{cases},$$

and $j = h_i(\mathfrak{b}) + 1$. The initial condition is

$$[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}} = \begin{cases} 0 & \text{if } \; \mathsf{a} \not\prec_{\mathsf{gi}} \mathfrak{b}, \\ \prod_{i=1}^n \sqrt{\mu_i}^{s_{i,\mathfrak{a}}} & \text{if } \; \mathfrak{a} = \mathfrak{b}, \end{cases}$$

where $s_{i,\mathfrak{a}}$ is the number of strips of length i in $\mathfrak{a}$.

Proof. If $\mathfrak{a}, \mathfrak{b} \in \mathcal{P}_n$, and let $[\![(\mathfrak{a})_1]\!](\mathfrak{b})_2$ denote the coefficient of $(\mathfrak{a})_1$ in $(\mathfrak{b})_2$. Therefore, in this notation, $[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}\boxplus\diamond_i} = [\![(\mathfrak{a})_1]\!]\,(\mathfrak{b}\boxplus\diamond_i)_2$. From Lemma 5.2, for $1 \le i \le 2n-1$, we have $\Gamma((\mathfrak{b}\boxplus\diamond_i)_2) = \sqrt{x^nV_k}(\Gamma(\mathfrak{b}\boxplus\diamond_i))_2$ where $k = h(\mathfrak{b}\boxplus\diamond_i)$. But, from Definition3.5, $(\Gamma(\mathfrak{b}\boxplus\diamond_i))_2 = \sqrt{\mu_{j+1}/\mu_j}(\mathsf{e}_i - \mu_j)(\Gamma(\mathfrak{b}))_2$ where $j = h_i(\mathfrak{b}) + 1$ and, from Lemma 5.2, $(\Gamma(\mathfrak{b}))_2 = \sqrt{x^nV_{k'}}\Gamma((\mathfrak{b})_2)$ where $k' = h(\mathfrak{b})$. Then

$$\Gamma((\mathfrak{b}\boxplus\diamond_i)_2) = \sqrt{\frac{V_k\mu_{j+1}}{V_{k'}\mu_j}}(\mathsf{e}_i - \mu_j)\Gamma((\mathfrak{b})_2).$$

By considering the effect of box addition at position $i = n$, we have $k = k' + 2$ if $i = n$ and $k = k'$ otherwise so

$$(\mathfrak{b}\boxplus\diamond_i)_2 = \Delta_{i,\mathfrak{b}}\sqrt{\frac{\mu_{j+1}}{\mu_j}}(\mathsf{e}_i - \mu_j)(\mathfrak{b})_2.$$

Thus

$$[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}\boxplus\diamond_i} = \Delta_{i,\mathfrak{b}}\sqrt{\frac{\mu_{j+1}}{\mu_j}}\,[\![(\mathfrak{a})_1]\!]\left((\mathsf{e}_i - \mu_j)(\mathfrak{b})_2\right). \tag{21}$$

There are two cases to be considered.

Case I: Suppose $\mathfrak{a}$ does not have a maximum at position i. Since $[\mathsf{e}_i(\mathfrak{b})_2]$ has a maximum at position i then $[\![(\mathfrak{a})_1]\!][\mathsf{e}_i(\mathfrak{b})_2] = 0$ so the contribution to $[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}\boxplus\diamond_i}$ from this case is $-\Delta_{i,\mathfrak{b}}\sqrt{\mu_{j+1}\mu_j}\,[\![(\mathfrak{a})_1]\!](\mathfrak{b})_2$, which is equal to

$$-\Delta_{i,\mathfrak{b}}\sqrt{\mu_{j+1}\mu_j}\,[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}}.$$

Case II: Suppose $\mathfrak{a}$ has a maximum at position i. Then, using part of the argument from case I, we see that the contribution to $[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}\boxplus\diamond_i}$ is

$$-\Delta_{i,\mathfrak{b}}\sqrt{\mu_{j+1}\mu_j}\,[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}} + \Delta_{i,\mathfrak{b}}\sqrt{\frac{\mu_{j+1}}{\mu_j}}\,[\![(\mathfrak{a})_1]\!][\mathsf{e}_i(\mathfrak{b})_2].$$

We now consider $[\![(\mathfrak{a})_1]\!][\mathsf{e}_i(\mathfrak{b})_2]$. From Lemma 4.5, $(\mathfrak{b})_2 = \sum_{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b}}\alpha_{\mathsf{e}}\mathsf{e}$ where $\alpha_{\mathsf{e}} \in \mathbb{C}(x,y)$. Then

$$[\![(\mathfrak{a})_1]\!]\,\mathsf{e}_i(\mathfrak{b})_2 = \sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}] \text{ has max. at } i}}\alpha_{\mathsf{e}}[\![(\mathfrak{a})_1]\!]\,\mathsf{e}_i\mathsf{e} + \sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}] \text{ has min. at } i}}\alpha_{\mathsf{e}}[\![(\mathfrak{a})_1]\!]\,\mathsf{e}_i\mathsf{e} + \sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}] \text{ has slope at } i}}\alpha_{\mathsf{e}}[\![(\mathfrak{a})_1]\!]\,\mathsf{e}_i\mathsf{e}.$$

There are therefore three cases to be considered. First, suppose that $[\mathsf{e}]$ has a maximum at position i. Then $\mathsf{e} = \mathsf{e}_i\mathsf{e}'$ for some monomial $\mathsf{e}' \in \mathfrak{TL}_n(x,y)$. Thus $\mathsf{e}_i\mathsf{e} = x_i\mathsf{e}_i\mathsf{e}' = x_i\mathsf{e}$. For a contribution to $[\![(\mathfrak{a})_1]\!]\, \mathsf{e}_i(\mathfrak{b})_2$ it is necessary that $(\mathfrak{a})_1 = \mathsf{e}_i\mathsf{e} = x_i\mathsf{e}$ so $\mathfrak{a} = [\mathsf{e}]$. Then $[\![(\mathfrak{a})_1]\!]\, \mathsf{e}_i\mathsf{e} = [\![\mathsf{e}]\!]x_i\mathsf{e} = x_i$. Thus

$$\sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}] \text{ has max. at } i}} \alpha_{\mathsf{e}}[\![(\mathfrak{a})_1]\!]\, \mathsf{e}_i\mathsf{e} = x_i\alpha_{\mathsf{a}} = x_i[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}}.$$

Next, suppose that $[\mathsf{e}]$ has a minimum at position i. For a contribution to $[\![(\mathfrak{a})_1]\!]\, \mathsf{e}_i(\mathfrak{b})_2$ it is necessary that $[\mathsf{e}] = \mathfrak{a} \boxminus \diamond_i$. Then

$$\sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}] \text{ has min. at } i}} \alpha_{\mathsf{e}}[\![(\mathfrak{a})_1]\!]\, \mathsf{e}_i\mathsf{e} = \alpha_{\mathfrak{a}\boxminus\diamond_i} = [\mathbf{P}_n]_{\mathfrak{a}\boxminus\diamond_i,\mathfrak{b}}.$$

Finally, suppose that $[\mathsf{e}]$ has a slope at position i. Then, from (20),

$$\sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}] \text{ has slope at } i}} \alpha_{\mathsf{e}}[\![(\mathfrak{a})_1]\!]\, \mathsf{e}_i\mathsf{e} \quad = \quad \sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}] \text{ has slope at } i}} \alpha_{\mathsf{e}}[\![(\mathfrak{a})_1]\!]\, (\kappa_i([\mathsf{e}]))_1 = \sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}]\in\kappa_i^{-1}(\mathfrak{a})}} \alpha_{\mathsf{e}} = \sum_{\substack{[\mathsf{e}]\prec_{\mathsf{gi}}\mathfrak{b} \\ [\mathsf{e}]\in\kappa_i^{-1}(\mathfrak{a})}} [\mathbf{P}_n]_{[\mathsf{e}],\mathfrak{b}}.$$

The contribution to $[\mathbf{P}]_{\mathfrak{a},\mathfrak{b}\boxplus\diamond_i}$ from case II is therefore

$$\Delta_{i,\mathfrak{b}}\sqrt{\frac{\mu_{j+1}}{\mu_j}}\left(x_i[\mathbf{P}_n]_{\mathfrak{a},\mathfrak{b}} + [\mathbf{P}_n]_{\mathfrak{a}\boxminus\diamond_i,\mathfrak{b}} + \sum_{\mathfrak{e}\in\kappa_i^{-1}(\mathfrak{a})}[\mathbf{P}_n]_{\mathfrak{e},\mathfrak{b}}\right).$$

The recursion now follows by combining cases I and II by means of $\delta_{i,\mathfrak{a}}$. The initial condition follows from Corollary 4.6. ☐

This is a linear recurrence equation with at most $\lfloor n/2\rfloor + 2$ terms and with coefficients that are square roots of rational functions of generalized Chebyshev polynomials. The $k+1$-st column of $\mathbf{P}_n$ involves therefore at most $\lfloor n/2\rfloor + 2$ elements from the k-th column of $\mathbf{P}_n$. Often there will be a position i at which $\mathfrak{b}$ has a maximum and $\mathfrak{a}$ does not. This will yield a recurrence equation with only two terms.

5.3. **Example: Computation of an element of $\mathbf{P}_6$.** We conclude this discussion of the recursion with a specific example of the computation of an element of $\mathbf{P}_6$, namely $[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}}$, where

$$\mathfrak{a} = \nearrow\nearrow\searrow\nearrow\searrow\nearrow\searrow\nearrow\searrow\searrow\nearrow\searrow, \qquad \mathfrak{b} = \nearrow\nearrow\nearrow\searrow\nearrow\nearrow\searrow\searrow\nearrow\searrow\searrow\searrow.$$

This will require several invocations of the recursion given in Theorem 5.4. To simplify the discussion, $\stackrel{i}{=}$ will be used to indicate an equality between two expressions that follows immediately from the use of the recursion at position i. In addition, $\mathfrak{p}\boxminus\diamond_{k_1k_2\ldots k_r}$ denotes $\mathfrak{p}\boxminus\diamond_{k_1}\boxminus\cdots\boxminus\diamond_{k_r}$, where the k_i's are not necessarily distinct since there may be more than one box above position i in $\mathfrak{p}$. In the interests of brevity, the determination of $\Delta_{i,\mathfrak{b}}$ and $\delta_{i,\mathfrak{a}}$ at each stage are suppressed, but these are readily checked by hand from the information that is provided. The strategy is to remove maxima from $\mathfrak{b}$ and to drive the recursion towards the boundary conditions. It is clear that the boxes in $\mathfrak{b}\backslash\mathfrak{a}$, where $\mathfrak{a}\prec_{\mathsf{gi}}\mathfrak{b}$, play an important part in this calculation.

The path $\mathfrak{b}$ has maxima in positions $i = 3$ and $i = 9$, and $\mathfrak{a}$ does not have a maximum at these points, so

$$[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}} \stackrel{3}{=} -\sqrt{\mu_2\mu_3}\,[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_3} \stackrel{9}{=} \mu_2\mu_3\,[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_{3\,9}}.$$

Since $\mathfrak{b} \boxminus \diamond_{3\,9}$ has a maximum at position $i = 10$, and $\mathfrak{a}$ does not have a maximum at this point,

$$[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}} \stackrel{10}{=} -\mu_2\mu_3\sqrt{\mu_1\mu_2}\,[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_{3\,9\,10}}$$

Now $\mathfrak{b} \boxminus \diamond_{3\,9\,10}$ has a maximum at $i = 6$, the midpoint of the path, and $\mathfrak{a}$ also has a maximum at this point. Moreover, $\mathfrak{a} \boxplus \diamond_5$ and $\mathfrak{a} \boxplus \diamond_7$ have a slope at $i = 6$, so $\kappa_6^{-1}(\mathfrak{a}) = \{\mathfrak{a} \boxplus \diamond_5, \mathfrak{a} \boxplus \diamond_7\}$. Then

$$[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_{3\,9\,10}} \stackrel{6}{=} -\sqrt{\frac{V_4}{V_2}}\sqrt{\mu_3\mu_4}\,[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} + \sqrt{\frac{V_4}{V_2}}\sqrt{\frac{\mu_4}{\mu_3}}\,f$$

where

$$f = y[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} + [\mathbf{P}_6]_{\mathfrak{a}\boxminus\diamond_6,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} + [\mathbf{P}_6]_{\mathfrak{a}\boxplus\diamond_5,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} + [\mathbf{P}_6]_{\mathfrak{a}\boxplus\diamond_7,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}}.$$

We now determine each of the four matrix elements in f. The path $\mathfrak{b} \boxminus \diamond_{3\,9\,10\,6}$, that appears in each case, does not have maxima at $i = 5$ or $i = 7$. First, $\mathfrak{a}$ does not have maxima at $i = 5$ or $i = 7$, so

$$[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} \stackrel{5}{=} -\sqrt{\mu_2\mu_3}\,[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6\,5}} \stackrel{7}{=} \mu_2\mu_3\,[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{a}} = \mu_2\mu_3(\mu_1^{1/2}\mu_2^{3/2}),$$

since $\mathfrak{b} \boxminus \diamond_{3\,9\,10\,6\,5\,7} = \mathfrak{a}$. The last equality is by the boundary condition. Second, $\mathfrak{a} \boxminus \diamond_6$ does not have a maximum at $i = 6$, so

$$\begin{aligned}[\mathbf{P}_6]_{\mathfrak{a}\boxminus\diamond_6,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} &\stackrel{5}{=} -\sqrt{\mu_2\mu_3}\,[\mathbf{P}_6]_{\mathfrak{a}\boxminus\diamond_6,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6\,5}} \stackrel{7}{=} \mu_2\mu_3\,[\mathbf{P}_6]_{\mathfrak{a}\boxminus\diamond_6,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6\,5\,7}}\\ &\stackrel{6}{=} -\mu_2\mu_3\sqrt{\frac{V_2}{V_0}}\sqrt{\mu_1\mu_2}\,[\mathbf{P}_6]_{\mathfrak{a}\boxminus\diamond_6,\mathfrak{a}\boxminus\diamond_6} = -\mu_2\mu_3\sqrt{\frac{V_2}{V_0}}\sqrt{\mu_1\mu_2}\,(\mu_1^{3/2}\mu_2^{3/2}),\end{aligned}$$

since $\mathfrak{b} \boxminus \diamond_{3\,9\,10\,6\,5\,7\,6} = \mathfrak{a} \boxminus \diamond_6$, and the last equality is by the boundary condition. Thirdly, $\mathfrak{a} \boxplus \diamond_5$ does not have a maximum at $i = 7$ so

$$[\mathbf{P}_6]_{\mathfrak{a}\boxplus\diamond_5,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} \stackrel{7}{=} -\sqrt{\mu_2\mu_3}[\mathbf{P}_6]_{\mathfrak{a}\boxplus\diamond_5,\mathfrak{a}\boxplus\diamond_5} = -\sqrt{\mu_2\mu_3}(\mu_1^{1/2}\mu_2\mu_3^{1/2}),$$

since $\mathfrak{b} \boxminus \diamond_{3\,9\,10\,6\,7} = \mathfrak{a} \boxplus \diamond_5$. The last equality is by the boundary condition. Finally, $\mathfrak{a} \boxplus \diamond_7$ does not have a maximum at $i = 5$ so

$$[\mathbf{P}_6]_{\mathfrak{a}\boxplus\diamond_7,\mathfrak{b}\boxminus\diamond_{3\,9\,10\,6}} \stackrel{5}{=} -\sqrt{\mu_2\mu_3}\,[\mathbf{P}_6]_{\mathfrak{a}\boxplus\diamond_7,\mathfrak{a}\boxplus\diamond_7} = -\sqrt{\mu_2\mu_3}(\mu_1^{1/2}\mu_2\mu_3^{1/2}),$$

since $\mathfrak{b} \boxminus \diamond_{3\,9\,10\,6\,5} = \mathfrak{a} \boxplus \diamond_7$. The last equality is by the boundary condition.

Combining these evaluations and using (10), we have

$$[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}} = \mu_1\mu_2^3\mu_3\,(\mu_2\mu_3 - y\mu_2 + \mu_1\mu_2 + 2) = \mu_1\mu_2^3\mu_3\,(1 + \mu_2\mu_3)\,.$$

5.4. **Direct determination of $[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}}$.** To confirm the above calculation independently, we determine $[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}}$ directly through the explicit computation of $[\![(\mathfrak{a})_1]\!](\mathfrak{b})_2$ in $\mathfrak{TL}_6(x, y)$. Note that $(\mathfrak{a})_1 = \mathsf{e}_2\mathsf{e}_1\mathsf{e}_4$, by decomposing the strand diagram corresponding to $\mathfrak{a}$ in terms of the strand diagrams of the generators of $\mathfrak{TL}_6(x, y)$. From (16) we have $(\mathfrak{I}_6^4)_2 = x^{-1}\mathsf{E}_6^3\mathsf{e}_1$. Now $\mathsf{e}_3(\mathfrak{I}_6^4)_2 = (\mathfrak{I}_6^4)_2\mathsf{e}_3 = 0$ from Lemma B.1, since $\mathfrak{I}_6^4$ has a slope at position 3 so, from Definition 3.5,

$$\begin{aligned}(\mathfrak{b})_2 &= x^{-1}\frac{\mu_3}{\mu_1}(\mathsf{e}_3 - \mu_2)(\mathsf{e}_2 - \mu_1)\mathsf{E}_6^3\mathsf{e}_1(\mathsf{e}_2 - \mu_1)(\mathsf{e}_3 - \mu_2)\\ &= \mu_3(\mathsf{e}_3\mathsf{e}_2 - \mu_2\mathsf{e}_2 + \mu_1\mu_2)\mathsf{E}_6^3\mathsf{e}_1(\mathsf{e}_2\mathsf{e}_3 - \mu_2\mathsf{e}_3 + \mu_1\mu_2).\end{aligned}$$

From (14), E_6^3 is obtained from E_4 by the substitutions $\mathsf{e}_i \mapsto e_{i+2}$ and $x_i \mapsto x_{i+2}$ for $i = 1, 2, 3$. Under these transformations μ_i is unchanged since the displacement in i is even. Thus, from recursion (9) for E_i, we have $\mathsf{E}_4 = \mathsf{E}_3 - \mu_3\mathsf{E}_3\mathsf{e}_3\mathsf{E}_3$ where $\mathsf{E}_3 = 1 - \mu_1(1 + \mu_1\mu_2)\mathsf{e}_1 - \mu_2\mathsf{e}_2 + \mu_1\mu_2\mathsf{e}_1\mathsf{e}_2 + \mu_1\mu_2\mathsf{e}_2\mathsf{e}_1$. Then $\mathsf{E}_6^3 = \mathsf{F}_3 - \mu_2\mathsf{F}_3\mathsf{e}_5\mathsf{F}_3$ where $\mathsf{F}_3 = 1 - \mu_1(1 + \mu_1\mu_2)\mathsf{e}_3 - \mu_2\mathsf{e}_4 + \mu_1\mu_2\mathsf{e}_3\mathsf{e}_4 + \mu_1\mu_2\mathsf{e}_4\mathsf{e}_3$. We shall write $\mathsf{p} \rightsquigarrow \mathsf{q}$, where $\mathsf{p}, \mathsf{q} \in \mathfrak{TL}_6(x, y)$, if $[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}}$ is unaffected by the replacement of p by q. We need retain in F only terms having an occurrence of e_4 and no occurrence of e_5, since $(\mathfrak{a})_1 = \mathsf{e}_2\mathsf{e}_1\mathsf{e}_4$. This is because, in the formation of $(\mathfrak{b})_2$, E_6^3 is multiplied on the left and the right by terms involving only $\mathsf{e}_1, \mathsf{e}_2$ and e_3, and such terms do not lower the suffix of e_5 and cannot introduce an e_4. Then $\mathsf{F}_3\mathsf{e}_5\mathsf{F}_3 \rightsquigarrow \mu_2^2\mathsf{e}_4\mathsf{e}_5\mathsf{e}_4 = \mu_2^2\mathsf{e}_4$, so $\mathsf{E}_6^3 \rightsquigarrow -\mu_2(1+\mu_2\mu_3)\mathsf{e}_4 + \mu_1\mu_2\mathsf{e}_3\mathsf{e}_4 + \mu_1\mu_2\mathsf{e}_4\mathsf{e}_3$. Then

$$(\mathfrak{b})_2 \rightsquigarrow -\mu_1\mu_2^2\mu_3\mathsf{e}_2\mathsf{E}_6^3\mathsf{e}_1 \rightsquigarrow \mu_1\mu_2^3\mu_3(1+\mu_2\mu_3)\mathsf{e}_2\mathsf{e}_1\mathsf{e}_4,$$

so $[\mathbf{P}_6]_{\mathfrak{a},\mathfrak{b}} = \mu_1\mu_2^3\mu_3(1+\mu_2\mu_3)$, in agreement with the result of the recursion.

5.5. **The determinant of the Gram matrix.** We conclude by confirming that $\det \mathbf{L}_n$ can now be determined. A preliminary result about path enumeration is needed for this purpose.

Proposition 5.5. *Let b_i be the total number of strips of length i in all the set of all paths in $\mathcal{P}_n$. Let c_i is the number of paths in $\mathcal{P}_n$ with mid-height at least i. Then*

$$b_i + c_i = \binom{2n}{n-i} - \binom{2n}{n-i-1}.$$

The proof of this is given in Appendix C.

Theorem 5.6.

$$\det(\mathbf{M}_n(x,y)) = \prod_{i=1}^{n} V_i^{a_{n,i}} \quad \textit{where } a_{n,i} = \binom{2n}{n-i} - 2\binom{2n}{n-i-1} + \binom{2n}{n-i-2}.$$

Proof. Let $\mathcal{B}_2$ now denote the ordered basis $(\mathsf{a}_1,\ldots,\mathsf{a}_{C_n})$ of $\mathfrak{TL}_n(x,y)$ such that $[\mathsf{a}_i] \prec_{\mathsf{gi}} [\mathsf{a}_j] \Rightarrow i<j$ for $1 \le i,j \le C_n$. From (7), $\mathbf{M}_n(x,y) = [\langle \mathsf{a}_i, \mathsf{a}_j\rangle_n]_{C_n,C_n}$, the Gram matrix of $\langle\cdot,\cdot\rangle_n$ with respect to $\mathcal{B}_1$. Let $\mathbf{D}_n = [\langle [\mathsf{a}_i]_2, [\mathsf{a}_j]_2\rangle_n]_{C_n,C_n}$, the Gram matrix of $\langle\cdot,\cdot\rangle_n$ with respect to $\mathcal{B}_2$ for $\mathfrak{TL}_n(x,y)$. Then $\mathbf{P}_n^t\mathbf{M}_n(x,y)\mathbf{P}_n = \mathbf{D}_n$. Now from Corollary 4.4, $\langle [\mathsf{a}_i]_2, [\mathsf{a}_j]_2\rangle_n = \delta_{i,j}V_h$ where $h = h(\mathsf{a}_i)$, the mid-height of $[\mathsf{a}_i]$. Then $\det \mathbf{D}_n = \prod_{h=0}^n V_h^{\alpha_h}$ where α_h is the number of paths in $\mathcal{P}_n$ with mid-height exactly h. Thus, from (13), so $\det \mathbf{D}_n(x,y) = \prod_{h=1}^n \mu_h^{-c_h}$ where c_h is the number of paths in $\mathcal{P}_n$ with height at least h. Now from Corollary 4.6, $\det \mathbf{P}_n = \prod_{h=1}^{C_n}\prod_{i=1}^n \sqrt{\mu_i}^{s_{i,k}}$ so $(\det \mathbf{P}_n)^2 = \prod_{i=1}^n \mu_i^{b_i}$ where b_i is the number of strips of length i in the set of all paths in $\mathcal{P}_n$. Then $\det(\mathbf{M}_n(x,y)) = \det(\mathbf{P}_n)^{-2}\det(\mathbf{D}_n) = \prod_{i=1}^n (\mu_i)^{-b_i-c_i}$. The result follows from (11) and Proposition 5.5. ☐

We may now obtain the expression given by Tutte [Tu2] for the determinant of the matrix of chromatic joins.

Theorem 5.7. *The determinant of the chromatic join matrix is given by:*

$$\det(\mathbf{L}_n) = \prod_{i=1}^{n} V_i(q,1)^{\binom{2n}{n-i} - 2\binom{2n}{n-i-1} + \binom{2n}{n-i-2}}.$$

Proof. In view of (8), the result follows from Theorem 5.6 by setting $y=1$. ☐

The lattice of planar partitions is not closed under chromatic join so the evaluation of $\det \mathbf{L}_n$ does not appear to be feasible through the lattice theoretic approach discussed by Jackson [J] for determinants of matrices of meets and joins in the lattice of partitions or the lattice of planar partitions.

We note in passing that the number of occurrences of x in $\mathbf{M}_n(x,x)$ is the number of configurations of the type that appears in the right hand diagram in Figure 3, with the condition that there is a single loop (the indicated diagram has three loops). Such configurations are called closed *meanders*, and they are the topic of extensive study. The number of closed meanders is therefore the number of occurrences of the monomial x in $\mathbf{M}_n(x,x)$.

5.6. **Closing remark.** An important aspect of the recursion given in Theorem 5.4 is that, in principle, it gives a starting point for the further study of the matrix $\mathbf{L}_n^{-1}(q)$. Although we have been unsuccessful in solving the recursion, it is clearly natural to ask whether the induced recursion for the elements of $\mathbf{L}_n^{-1}(q) = \mathbf{P}_n\mathbf{D}_n^{-1}(q)\mathbf{P}_n^t$ might have a solution. It appears that this is a question for which the incidence algebra of $\mathcal{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$ is natural setting and that the properties of this lattice will be important to the investigation of this question.

References

[A] V.Arnol'd, *The branched covering of* $\mathbb{CP}_2 \to \mathbb{S}_4$, *hyperbolicity and projective topology*, Siberian Math. J., **29** (1988), 717–726.

[BR] H.Barcelo and A.Ram, *Combinatorial representation theory.* In "New Perspectives in Algebraic Combinatorics", (eds. L.J.Billera, A, Björner, C.Greene, R.E.Simion and R.P.Stanley), Mathematical Sciences Research Institute Publications **38**, Cambridge University Press, Cambridge, England, 1999, pp. 23–90.

[BL] G.D.Birkhoff and D.C.Lewis, *Chromatic polynomials*, Trans. Amer. Math. Soc., **60** (1946), 355–451.

[D] R.Dahab, "The Birkhoff-Lewis equations," Ph.D. Thesis, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada, 1993.

[E] P.Edelman, *Chain enumeration and non-crossing partitions*, Discrete Math., **31** (1980), 171–180.

[dF1] P. di Francesco *Meander determinants.* hep-th/9612026.

[dF2] P. di Francesco *Folding and coloring problems in Mathematics and Physics*, Bull. A.M.S. **37** (2000), 251–307.

[dFGL] P. di Francesco, O.Golinelli and E.Guitter, *Meanders and the Temperley-Lieb algebra*, Commun. Math. Phys., **186** (1997), 1–59.

[GdlHJ] F.M.Goodman, P. de la Harpe and V.F.R.Jones, "Coxeter graphs and towers of algebra", Mathematical Sciences Research Institute Publications **14**, Springer, New York, 1989.

[J] D.M.Jackson, *The lattice of non-crossing partitions and the Birkhoff-Lewis equations*, Europ. J. Combinatorics, **15** (1994), 245–250.

[KL] L.H.Kauffman and S.Lins, "Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds", Annals of Mathematical Studies **134**, Princeton University Press, New Jersey, 1994.

[KT] L.Kauffman and R.Thomas, *Temperley-Lieb algebras and the Four Colour Theorem.* Available at http://math.uic.edu/ kauffman/TLFCT.pdf.

[SK] T.L.Saaty and P.C. Kainan, "The four colour problem," Dover, New York, 1986.

[Tu1] W.T.Tutte, *On the Birkhoff-Lewis equations*, Discrete Math., **92** (1991), 417–425.

[Tu2] W.T.Tutte, *The matrix of chromatic joins*, J. Combinatorial Theory (B), **57** (1993), 269–288.

Appendix A. Uniqueness of the Markov trace

The following result shows that the Markov trace in $\mathfrak{TL}_k(x,y)$ is unique up to a multiplicative constant.

Lemma A.1. *A Markov trace* Tr_k *in* $\mathfrak{TL}_k(x,y)$ *is uniquely determined by any one of its evaluations.*

Proof. It is sufficient to prove the result for a monomial g. We may fix the evaluation of any monomial so, for convenience, we fix $\mathrm{Tr}_1(1)$. We show, using the relations (i) $\mathrm{Tr}_k(\mathsf{ef}) = \mathrm{Tr}_k(\mathsf{fe})$ and (ii) $\mathrm{Tr}_k(\mathsf{e}\mathsf{e}_{k-1}) = \mathrm{Tr}_{k-1}(\mathsf{e})$ for $\mathsf{e} \in \langle 1, \mathsf{e}_1, \ldots, \mathsf{e}_{k-2}\rangle$ from Definition 2.2, that $\mathrm{Tr}_k(\mathsf{g})$ can be reduced to $\mathrm{Tr}_1(1)$. Suppose that $\mathsf{g} \neq 1$. Then $\mathsf{g} = \mathsf{e}_{j_1} \ldots \mathsf{e}_{j_l}$ for some integers $j_1, \ldots, j_l \in [1, k-1]$. The strategy is to use S3 of Definition 2.3 (*viz.* $\mathsf{e}_i = \mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i$), to force the appearance of precisely one e_{k-1} and then, by (i), to move e_{k-1} to the end of g. We then use (ii) and repeat this procedure, terminating at the evaluation of $\mathrm{Tr}_1(1)$. To do this, let $M = \max\{j_1, \ldots, j_l\}$. Then $\mathsf{g} = \mathsf{g}_1\mathsf{e}_M\mathsf{g}_2$ for some $\mathsf{g}_1, \mathsf{g}_2 \in \mathfrak{TL}_k(x,y)$. If $M < k-1$ then

$$\begin{aligned}
\mathrm{Tr}_k(\mathsf{g}) &= \mathrm{Tr}_k(\mathsf{g}_1\mathsf{e}_M\mathsf{g}_2) = \mathrm{Tr}_k(\mathsf{g}_2\mathsf{g}_1\mathsf{e}_M) \qquad \text{(by (i))}\\
&= \mathrm{Tr}_k(\mathsf{g}_2\mathsf{g}_1\mathsf{e}_M\mathsf{e}_{M+1}\ldots\mathsf{e}_{k-2}\mathsf{e}_{k-1}\mathsf{e}_{k-2}\ldots\mathsf{e}_{M+1}\mathsf{e}_M) \qquad \text{(by (iii))}\\
&= \mathrm{Tr}_k(\mathsf{e}_{k-2}\mathsf{e}_{k-3}\ldots\mathsf{e}_M\mathsf{g}_2\mathsf{g}_1\mathsf{e}_M\mathsf{e}_{M+1}\ldots\mathsf{e}_{k-2}\mathsf{e}_{k-1}) \qquad \text{(by (i))}.
\end{aligned}$$

Since e_{k-1} has been forced to appear as the rightmost generator in g, we may suppose without loss of generality that $M = k-1$.

If g now contains two e_{k-1}'s, it is rewritten with them as close together as possible. If they are adjacent, e_{k-1}^2 is replaced by $x_{k-1}\mathsf{e}_{k-1}$, removing an e_{k-1}. If not, $\mathsf{g} = \cdots\mathsf{e}_{k-1}\mathsf{e}_{l_1}\mathsf{e}_{l_2}\cdots\mathsf{e}_{l_s}\mathsf{e}_{k-1}\cdots$ where $l_1 = k-2$ for otherwise we could use $\mathsf{e}_{k-1}\mathsf{e}_{l_1} = \mathsf{e}_{l_1}\mathsf{e}_{k-1}$ to make the e_{k-1}'s closer. Similarly $l_2 = k-3, \ldots, l_s = k-1-s$. If $s > 1$ we commute e_{l_s} and e_{k-1}, making the e_{k-1}'s closer. Therefore, $s = 1$ and $\mathsf{g} = \cdots\mathsf{e}_{k-1}\mathsf{e}_{k-2}\mathsf{e}_{k-1}\cdots = \cdots\mathsf{e}_{k-1}\cdots$, removing an e_{k-1}. Repeating this, we may assume that g contains exactly one e_{k-1}. But then $\mathsf{g} = \mathsf{g}'\mathsf{e}_{k-1}$ where $\mathsf{g}' \in \langle 1, \mathsf{e}_1, \ldots, \mathsf{e}_{k-2}\rangle$ so $\mathrm{Tr}_k(\mathsf{g}) = \mathrm{Tr}_{k-1}(\mathsf{g}')$ by (ii). Repeating this gives $\mathrm{Tr}_k(\mathsf{g}) = \mathrm{Tr}_{k'}(1)$ for some integer $k' \geq 1$. Finally, to determine $\mathrm{Tr}_{k'}(1)$, we have

$$\begin{aligned}
\mathrm{Tr}_{k'}(1) &= \mathrm{Tr}_{k'+1}(\mathsf{e}_{k'-1}) = \mathrm{Tr}_{k'+1}(\mathsf{e}_{k'-1}\mathsf{e}_{k'}\mathsf{e}_{k'-1}) = \mathrm{Tr}_{k'+1}(\mathsf{e}_{k'-1}\mathsf{e}_{k'-1}\mathsf{e}_{k'}) \quad \text{(from (ii), S3, (i))}\\
&= x_{k'-1}\mathrm{Tr}_{k'+1}(\mathsf{e}_{k'-1}\mathsf{e}_{k'}) = x_{k'-1}\mathrm{Tr}_{k'}(\mathsf{e}_{k'-1}) = x_{k'-1}\mathrm{Tr}_{k'-1}(1) \quad \text{(from S1, (ii), (ii))}
\end{aligned}$$

Iterating this gives $\mathrm{Tr}_{k'}(1) = x_{k'-1}x_{k'-2}\ldots x_1\mathrm{Tr}_1(1)$, so $\mathrm{Tr}_k(\mathsf{g}) = x_{k'-1}x_{k'-2}\ldots x_1\mathrm{Tr}_1(1)$. □

Appendix B. Inductive proof of Theorem 3.6

The main idea behind the determination of $[\mathsf{a}]_2[\mathsf{a}']_2$ is to obtain a recurrence equation for it by moving a box from $\mathfrak{r}$ to $\mathfrak{l}'$, where $\mathfrak{lr}$ and $\mathfrak{l}'\mathfrak{r}'$ are the midpoint decompositions of $[\mathsf{a}]$ and $[\mathsf{a}']$, respectively. This idea leads to the statement of the theorem, which is now proved by an induction that is based on the recursion.

The base case of the induction is given in Appendix B.1 and the inductive step is given in Appendix B.2. Throughout, let

$$\theta_j = \sqrt{\frac{\mu_{j+1}}{\mu_j}}.$$

B.1. **The base case of the induction.**

Proof. (**Lemma 4.1**): For $(\mathfrak{I}_n^h)_2$ and $(\mathfrak{I}_n^{h'})_2$ to exist we require that $n \equiv h \bmod 2$ and $n \equiv h' \bmod 2$, so $h' \equiv h \bmod 2$. Now

$$(\mathfrak{I}_n^h)_2(\mathfrak{I}_n^{h'})_2 = x^{-(n-h)/2-(n-h')/2}\left(\mathsf{E}_n^{(n-h+1)}\mathsf{e}_1\mathsf{e}_3\dots\mathsf{e}_{n-h-1}\right)\left(\mathsf{E}_n^{(n-h'+1)}\mathsf{e}_1\mathsf{e}_3\dots\mathsf{e}_{n-h'-1}\right) = 0$$

since $\mathsf{e}_{n-h-1}\mathsf{E}_n^{(n-h'+1)} = 0$ for $h < h'$ from Corollary 3.3. Similarly, by reorganizing the products,

$$(\mathfrak{I}_n^h)_2(\mathfrak{I}_n^{h'})_2 = x^{-(n-h)/2-(n-h')/2}\left(\mathsf{e}_1\mathsf{e}_3\dots\mathsf{e}_{n-h-1}\mathsf{E}_n^{(n-h+1)}\right)\left(\mathsf{E}_n^{(n-h'+1)}\mathsf{e}_{n-h'-1}\dots\mathsf{e}_3\mathsf{e}_1\right) = 0$$

since $\mathsf{E}_n^{(n-h+1)}\mathsf{e}_{n-h'-1} = 0$ for $h > h'$ from Corollary 3.3. Finally,

$$(\mathfrak{I}_n^h)_2^2 = x^{-(n-h)}\left(\mathsf{E}_n^{(n-h+1)}\right)^2\mathsf{e}_1^2\mathsf{e}_3^2\dots\mathsf{e}_{n-h-1}^2 = x^{-(n-h)/2}\mathsf{E}_n^{(n-h+1)}\mathsf{e}_1\mathsf{e}_3\dots\mathsf{e}_{n-h-1} = (\mathfrak{I}_n^h)_2,$$

since $\mathsf{e}_i^2 = x\mathsf{e}_i$ when i is odd and $\mathsf{E}_n^{(n-h+1)}$ is idempotent. This gives the result. □

The base case of the induction corresponds to the case $\#_\diamond(\mathfrak{r}) = 0$, and is established through the following lemmas.

Lemma B.1. *Let* a *be a monomial such that* $[\mathsf{a}]$ *has a slope at position* i. *Then*

$$\begin{aligned} \mathsf{e}_i[\mathsf{a}]_2 &= 0 \qquad \text{if } i < n, \\ [\mathsf{a}]_2\mathsf{e}_{2n-i} &= 0 \qquad \text{if } i > n. \end{aligned}$$

Proof. We consider the case in which $i < n$ and there is a double rise at position i in $[\mathsf{a}]$. Since the three other cases ($i < n$ and double fall; $i > n$ and double rise or double fall) are similar they are not included.

We use induction on $\#_\diamond([\mathsf{a}])$. The base case of the induction is therefore $\mathfrak{I}_n^h$ where h is the mid-height of $[\mathsf{a}]$. Now $\mathfrak{I}_n^h$ has a double rise at position i, for $n-h+1 \le i \le n-1$. But

$$\mathsf{e}_i(\mathfrak{I}_n^h)_2 = x^{-(n-h)/2}\mathsf{e}_i\mathsf{E}_n^{(n-h+1)}\mathsf{e}_1\mathsf{e}_3\dots\mathsf{e}_{n-h-1} = 0$$

by Corollary 3.3, so the result holds for $[\mathsf{a}] = \mathfrak{I}_n^h$.

We may now assume that $\mathfrak{I}_n^h \prec_{\mathsf{gi}} [\mathsf{a}]$. Let $m-1$ be the height of the double rise at position i in $[\mathsf{a}]$. Then $m-1 \equiv i \bmod 2$. Now $[\mathsf{a}]$ is constructed by box additions $\diamond_i$ and $\diamond_{i+1}$ either i) to a double slope in the portion of $[\mathsf{a}]$ associated with $\mathsf{E}_n^{(n-h+1)}$ or ii) it is not. In case (i), the product is zero by the inductive hypothesis. In case (ii), $[\mathsf{a}]$ is constructed by box additions $\diamond_i$ and $\diamond_{i+1}$ to a path $\mathfrak{g}$ that therefore must have a double rise at position $i+1$. Thus, by box addition,

$$[\mathsf{a}]_2 = \theta_{m+1}\theta_m\mathsf{f}(\mathsf{e}_{i+1} - \mu_{m+1})(\mathsf{e}_i - \mu_m)(\mathfrak{g})_2,$$

where $\mathfrak{g} \prec_{\mathsf{gi}} [\mathsf{a}]$. Moreover, no paths corresponding to monomials in f have boxes in positions $i-1$ or i since $[\mathsf{a}]$ has a double rise in position i. Nor do these have a box in position $i+1$ since position $i+1$ in $[\mathsf{a}]$ is not a minimum. Thus f is generated by $\{\mathsf{e}_j : |j-i| > 1\}$. Then $\mathsf{e}_i\mathsf{f} = \mathsf{f}\mathsf{e}_i$. Thus

$$\begin{aligned} \mathsf{e}_i[\mathsf{a}]_2 &= \theta_{m+1}\theta_m\mathsf{f}\mathsf{e}_i(\mathsf{e}_{i+1} - \mu_{m+1})(\mathsf{e}_i - \mu_m)(\mathfrak{g})_2 \\ &= \theta_{m+1}\theta_m\mathsf{f}\left((1 - x_i\mu_{m+1} + \mu_m\mu_{m+1}) - \mu_{m+1}\mathsf{e}_i\mathsf{e}_{i+1}\right)(\mathfrak{g})_2. \end{aligned}$$

But $x_i = x_{m+1}$ since $m-1 \equiv i \bmod 2$, so $\mathsf{e}_i[\mathsf{a}]_2 = -\mu_{m+1}\theta_{m+1}\theta_m\mathsf{f}\mathsf{e}_i\mathsf{e}_{i+1}(\mathfrak{g})_2$ from the recurrence equation for μ_m. But $\mathsf{e}_{i+1}(\mathfrak{g})_2 = 0$ by the inductive hypothesis, and the result follows. □

Lemma B.2. *Let* a *be a monomial, and let* $\mathfrak{I}_n^h = \mathfrak{l}\mathfrak{r}$ *and* $[\mathsf{a}] = \mathfrak{l}'\mathfrak{r}'$ *be midpoint decompositions. Then*

$$(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2.$$

Proof. We use induction on $\#_\diamond(\mathfrak{l}')$. Then the base case for the induction is $[\mathsf{a}] = \mathfrak{I}_n^{h'}$, where h' is the mid-height of $[\mathsf{a}]$, since $\#_\diamond(\mathfrak{l}') = 0$. From Lemma 4.1, $(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = (\mathfrak{I}_n^h)_2(\mathfrak{I}_n^{h'})_2\mathsf{f} = \delta_{h,h'}(\mathfrak{I}_n^h)_2\mathsf{f} = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$, where f arises from the factors associated with box addition on the right of $[\mathsf{a}]$. Thus the result holds in this case.

We may now assume that $\#_\diamond(\mathfrak{l}') > 0$. Let i be a position in which a box appears and let $j-1$ be the height in this position at which the box addition takes place. Then $j-1 \equiv i \mod 2$, and $i < n$. There are two cases to consider.

Case I: Assume that i is odd. Now $[\mathsf{a}]_2 = (([\mathsf{a}] \boxminus \diamond_i) \boxplus \diamond_i)_2 = \theta_j(\mathsf{e}_i - \mu_j)([\mathsf{a}] \boxminus \diamond_i)_2$. Then

$$\begin{aligned}(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 &= x^{-(n-h)/2}\theta_j\mathsf{E}_n^{(n-h+1)}(\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_i\cdots\mathsf{e}_{n-n-1})(\mathsf{e}_i-\mu_j)([\mathsf{a}]\boxminus\diamond_i)_2\\ &= \theta_j(x_i-\mu_j)(\mathfrak{I}_n^h)_2([\mathsf{a}]\boxminus\diamond_i)_2.\end{aligned}$$

But i is odd, so $[\mathsf{a}]\boxminus\diamond_i$ contains at least two boxes (at positions i and $i+1$), so $(\mathfrak{I}_n^h)_2([\mathsf{a}]\boxminus\diamond_i)_2 = 0$ by the inductive hypothesis, whence $(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = 0$. Since $\#_\diamond(\mathfrak{l}') > 0$, then $\mathfrak{r}^t \neq \mathfrak{l}$ so $\delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2 = 0$, whence $(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$ so the inductive step holds in this case.

Case II: Assume that there is no such odd i. Then select the smallest even i such that the last box addition was at position i, to a point at height $j-1$ where $j \geq 1$. Then $i \equiv j \mod 2$. Then $[\mathsf{a}] = ([\mathsf{a}] \boxminus \diamond_i \boxminus \diamond_{i-1}) \boxplus \diamond_{i-1} \boxplus \diamond_i$ so

$$[\mathsf{a}]_2 = \sqrt{\frac{\mu_{j+1}}{\mu_{j-1}}}(\mathsf{e}_i-\mu_j)(\mathsf{e}_{i-1}-\mu_{j-1})(\mathfrak{p})_2$$

where $\mathfrak{p} = [\mathsf{a}] \boxminus \diamond_i \boxminus \diamond_{i-1}$. Now

$$(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = \sqrt{\frac{\mu_{j+1}}{\mu_{j-1}}}x^{-(n-h)/2}\left(\mathsf{E}_n^{(n-h+1)}\mathsf{e}_1\mathsf{e}_3\ldots\mathsf{e}_{n-h-1}\right)(\mathsf{e}_i-\mu_j)(\mathsf{e}_{i-1}-\mu_{j-1})(\mathfrak{p})_2.$$

But $\mathsf{e}_{i-1}(\mathsf{e}_i-\mu_j)(\mathsf{e}_{i-1}-\mu_{j-1}) = (1-\mu_jx_{i-1}+\mu_j\mu_{j-1})\mathsf{e}_{i-1} - \mu_{j-1}\mathsf{e}_{i-1}\mathsf{e}_i = -\mu_{j-1}\mathsf{e}_{i-1}\mathsf{e}_i$ from the recurrence (10) for μ_j, since $x_{i-1} = x_{j-1}$ for $j \equiv i \mod 2$. Then $(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = -\sqrt{\mu_{j+1}\mu_{j-1}}\,(\mathfrak{I}_n^h)_2\mathsf{e}_i(\mathfrak{p})_2$. But $\mathfrak{p}$ has a double rise at position i so $\mathsf{e}_i(\mathfrak{p})_2 = 0$ by Lemma B.1. Thus $(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = 0 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$. On the other hand, if $j=1$ then $\mu_{j-1} = 0$ so $(\mathfrak{I}_n^h)_2[\mathsf{a}]_2 = 0 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$. This establishes the inductive step in this case. The result follows. □

The following is the base of the induction for the proof of Theorem 3.6.

Corollary B.3. *Let* $\mathfrak{r}$ *be the right factor of the midpoint decomposition of* $\mathfrak{I}_n^h$. *Let* $[\mathsf{a}] = \mathfrak{l}\mathfrak{r}$ *and* $[\mathsf{a}'] = \mathfrak{l}'\mathfrak{r}'$ *be midpoint decompositions. Then*

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2.$$

Proof. $[\mathsf{a}]$ can be constructed from $\mathfrak{I}_n^h$ by box additions $\diamond_i$ with $i < n$. By Definition 3.5 this implies that $[\mathsf{a}]_2 = \mathsf{b}(\mathfrak{I}_n^h)_2$ for some $\mathsf{b} \in \mathfrak{T}\mathfrak{L}_n(x,y)$. Then $[\mathsf{a}]_2[\mathsf{a}']_2 = \mathsf{b}(\mathfrak{I}_n^h)_2[\mathsf{a}']_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$ by Lemma B.2. □

B.2. **The inductive step.** We are now in a position to complete the proof of Theorem 3.6. We consider midpoint decompositions of paths $\mathfrak{p} \in \mathcal{P}_n$ and some further notation will be useful. Let $\mathfrak{p} = \lambda(\mathfrak{p})\rho(\mathfrak{p})$ be the midpoint decomposition of $\mathfrak{p}$, and let $\lambda^t(\mathfrak{p})$ denote $(\lambda(\mathfrak{p}))^t$. Thus $\mathfrak{p}^t = \rho^t(\mathfrak{p})\lambda^t(\mathfrak{p})$. Note that if $\mathfrak{p}$ has a maximum, minimum or slope in position $i > n$, then $\rho(\mathfrak{p})$ has the same in position $i - n$.

Proof. We begin by attaching the notation of the enunciation of the theorem: $[\mathsf{a}]$ and $[\mathsf{a}']$ are monomials in $\mathfrak{TL}_n(x, y)$, and $[\mathsf{a}] = \mathfrak{l}\mathfrak{r}$ and $[\mathsf{a}'] = \mathfrak{l}'\mathfrak{r}'$ are midpoint decompositions. We use induction on $\#_\diamond(\mathfrak{r})$. The strategy for the inductive step is to consider the effect of moving a box from $\mathfrak{r}$ to $\mathfrak{l}'$. The base case of the induction is for $\#_\diamond(\mathfrak{r}) = 0$, and this is established immediately by Corollary B.3.

We may now assume that $\#_\diamond(\mathfrak{r}) > 0$. Then there is a box in $[\mathsf{a}]$ at some position $i > n$, so $[\mathsf{a}] = [\mathsf{b}] \boxplus \diamond_i$ where $[\mathsf{b}]$ has a minimum in position i, at height $j - 1$, say. Then

$$[\mathsf{a}]_2 = \theta_j [\mathsf{b}]_2(\mathsf{e}_{2n-i} - \mu_j) \tag{22}$$

where $i \equiv j - 1 \bmod 2$. Then

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \theta_j [\mathsf{b}]_2(\mathsf{e}_{2n-i} - \mu_j)[\mathsf{a}']_2.$$

There are three cases.

Case I: Assume that $[\mathsf{a}']$ has a minimum in position $2n - i < n$, and at height $k - 1$, say. Then $([\mathsf{a}'] \boxplus \diamond_{2n-i})_2 = \theta_k(\mathsf{e}_{2n-i} - \mu_k)[\mathsf{a}']_2$ so

$$[\mathsf{b}]_2([\mathsf{a}'] \boxplus \diamond_{2n-i})_2 = \theta_k [\mathsf{b}]_2(\mathsf{e}_{2n-i} - \mu_k)[\mathsf{a}']_2.$$

But $[\mathsf{b}]$ has a minimum in position i and $[\mathsf{a}'] \boxplus \diamond_{2n-i}$ has a maximum in position $2n-i$. Thus, by the induction hypothesis, $[\mathsf{b}]_2([\mathsf{a}'] + \diamond_{2n-i})_2 = 0$ since $\#_\diamond([\mathsf{b}]) < \#_\diamond([\mathsf{a}])$. Then $[\mathsf{b}]_2\mathsf{e}_{2n-i}[\mathsf{a}'] = \mu_k[\mathsf{b}]_2[\mathsf{a}']_2$. Eliminating $[\mathsf{b}]_2\mathsf{e}_{2n-i}[\mathsf{a}']$ with this from the expression for $[\mathsf{a}]_2[\mathsf{a}']_2$, we have

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \theta_j(\mu_k - \mu_j)[\mathsf{b}]_2[\mathsf{a}']_2.$$

If $\rho(\mathsf{b}) = \lambda^t([\mathsf{a}'])$, then $j = k$ since $[b]$ and $[\mathsf{a}']$ have minima at height $j-1$ and $k-1$, respectively. Thus $\mu_j = \mu_k$ whence $[\mathsf{a}]_2[\mathsf{a}']_2 = 0$. On the other hand, if $\rho([\mathsf{b}]) \neq \lambda^t([\mathsf{a}'])$ then, by the induction hypothesis, $[\mathsf{b}]_2[\mathsf{a}']_2 = 0$ so, again, $[\mathsf{a}]_2[\mathsf{a}']_2 = 0$. But $\mathfrak{r}^t \neq \mathfrak{l}'$ since $[\mathsf{a}]$ has a maximum in position i and $[\mathsf{a}']$ has a minimum in position $2n - i$. Thus $\delta_{\mathfrak{r}^t,\mathfrak{l}'} = 0$ whence $[\mathsf{a}]_2[\mathsf{a}']_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$, so the inductive step holds in this case.

Case II: Assume that $[\mathsf{a}']$ has a slope in position $2n - i < n$. Then

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \theta_j [\mathsf{b}]_2(\mathsf{e}_{2n-i} - \mu_j)[\mathsf{a}']_2.$$

But $\mathsf{e}_{2n-i}[\mathsf{a}']_2 = 0$ by Lemma B.1, and $[\mathsf{b}]_2[\mathsf{a}']_2 = 0$ by the induction hypothesis, so $[\mathsf{a}]_2[\mathsf{a}']_2 = 0$. Moreover, $\delta_{\mathfrak{r}^t,\mathfrak{l}'} = 0$, since $\mathfrak{r}^t$ has a maximum in position $n - i$, where $\mathfrak{l}'$ has a slope. Thus $[\mathsf{a}]_2[\mathsf{a}']_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$, so the inductive step holds in this case.

Case III: Assume that $[\mathsf{a}']$ has a maximum in position $2n - i < n$, at height $k + 1$, say. Then $[\mathsf{a}'] \boxminus \diamond_{2n-i}$ has a minimum in position $2n - i$ at height $k - 1$. Now $[\mathsf{a}'] = [\mathsf{a}'] \boxminus \diamond_{2n-i} \boxplus \diamond_{2n-i}$ so $[\mathsf{a}']_2 = \theta_j\theta_k(\mathsf{e}_{2n-i} - \mu_k)([\mathsf{a}'] \boxminus \diamond_{2n-i})_2$ so, from (22),

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \theta_j\theta_k[\mathsf{b}]_2(\mathsf{e}_{2n-i} - \mu_j)(\mathsf{e}_{2n-i} - \mu_k)([\mathsf{a}'] \boxminus \diamond_{2n-i})_2.$$

But $(\mathsf{e}_{2n-i} - \mu_j)(\mathsf{e}_{2n-i} - \mu_k) = (x_{2n-i} - \mu_j - \mu_k)(\mathsf{e}_{2n-i} - \mu_k) + \mu_k(x_{2n-i} - \mu_k)$. It follows from these three expressions that

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \theta_j(x_{2n-i} - \mu_j - \mu_k)[\mathsf{b}]_2[\mathsf{a}']_2 + \theta_j\theta_k\,\mu_k(x_{2n-i} - \mu_k)[\mathsf{b}]_2([\mathsf{a}'] \boxminus \diamond_{2n-i})_2$$

so

$$[\mathsf{a}]_2[\mathsf{a}']_2 = \theta_j\theta_k\, \mu_k(x_{2n-i} - \mu_k)[\mathsf{b}]_2([\mathsf{a}'] \boxminus \diamond_{2n-i})_2$$

since $[\mathsf{b}]_2[\mathsf{a}']_2 = 0$ by the induction hypothesis.

Now $[\mathsf{b}]$ has a minimum in position i and $[\mathsf{a}'] \boxminus \diamond_{2n-i}$ has a minimum in position $2n-i$. Thus $\rho([\mathsf{b}])$ and $\lambda^t([\mathsf{a}'] \boxminus \diamond_{2n-i})$ both have minima in position $i-n$. We now examine this situation. There are two cases.

i) If $\rho^t([\mathsf{b}]) \neq \lambda([\mathsf{a}'] \boxminus \diamond_{2n-i})$ then $[\mathsf{b}]_2([\mathsf{a}'] \boxminus \diamond_{2n-i})_2 = 0$ by the induction hypothesis. But in this case, $\mathfrak{r}^t \neq \mathfrak{l}$ so $\delta_{\mathfrak{r}^t,\mathfrak{l}'} = 0$. Thus $[\mathsf{a}]_2[\mathsf{a}']_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$, so the inductive step holds in case (i).

ii) On the other hand, if $\rho([\mathsf{b}]) = \lambda^t([\mathsf{a}'] \boxminus \diamond_{2n-i})$ then $[\mathsf{b}]_2([\mathsf{a}'] \boxminus \diamond_{2n-i})_2 = (\mathfrak{l}\mathfrak{r}')_2$ by the induction hypothesis. Moreover, in this case $j = k$ so, from (10),

$$[\mathsf{a}]_2[\mathsf{a}']_2 \quad = \quad \mu_{j+1}(x_{2n-i} - \mu_j)(\mathfrak{l}\mathfrak{r}')_2 = \frac{x_{2n-i} - \mu_j}{x_{j+1} - \mu_j}\,(\mathfrak{l}\mathfrak{r}')_2 = (\mathfrak{l}\mathfrak{r}')_2$$

since $i \equiv j-1 \bmod 2$. But $\rho^t([\mathsf{b}]) = \lambda([\mathsf{a}'] \boxminus \diamond_{2n-i})$ so $\mathfrak{r}^t = \rho^t([\mathsf{a}]) = \lambda([\mathsf{a}']) = \mathfrak{l}'$ so $[\mathsf{a}]_2[\mathsf{a}']_2 = \delta_{\mathfrak{r}^t,\mathfrak{l}'}(\mathfrak{l}\mathfrak{r}')_2$, so the inductive step holds in case (ii).

Thus the inductive step holds in case III. The result now follows. □

We note that this proof requires the parity condition in S1.

Appendix C. Paths

C.1. **The lattice $\mathcal{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$.** Included in this Appendix are further facts about $\mathcal{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$. The lattice is graded and distributive. Its rank function $\mathrm{rank}(\mathfrak{p})$ is given by the area (number of unit squares) of the lattice polygon bounded by $\mathfrak{p} \in \mathcal{P}_n$ and the minimal element $(\nearrow\searrow)^n$. The number $r_{k,n}$ of elements in $\mathcal{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$ with rank equal to k is $[u^k x^n]F_0$ where F_i satisfies the recursion $F_k = (1 - u^k x F_{k+1})^{-1}$ for $k \geq 0$. This generates the continued fraction of Ramanujan and it can be shown that

$$F_0 = \frac{1 + \sum_{n\geq 1}(-1)^n x^n u^{n^2} \prod_{i=1}^n (1-u^i)^{-1}}{1 + \sum_{n\geq 1}(-1)^n x^n u^{n(n-1)} \prod_{i=1}^n (1-u^i)^{-1}}.$$

The initial terms are

$$F_0 = 1 + x + (u+1)x^2 + (u^3 + u^2 + 2u + 1)x^3 + (u^6 + u^5 + 2u^4 + 3u^3 + 3u^2 + 3u + 1)x^4 + \cdots.$$

The rank generating series for the lattice $\mathcal{L}(\mathcal{P}_4, \prec_{\mathsf{gi}})$ whose Hasse diagram is given in Figure 11 is therefore

$$\sum_{k=0}^{8} r_{k,4} u^k = u^6 + u^5 + 2u^4 + 3u^3 + 3u^2 + 3u + 1.$$

This polynomial lists the number of elements at each rank.

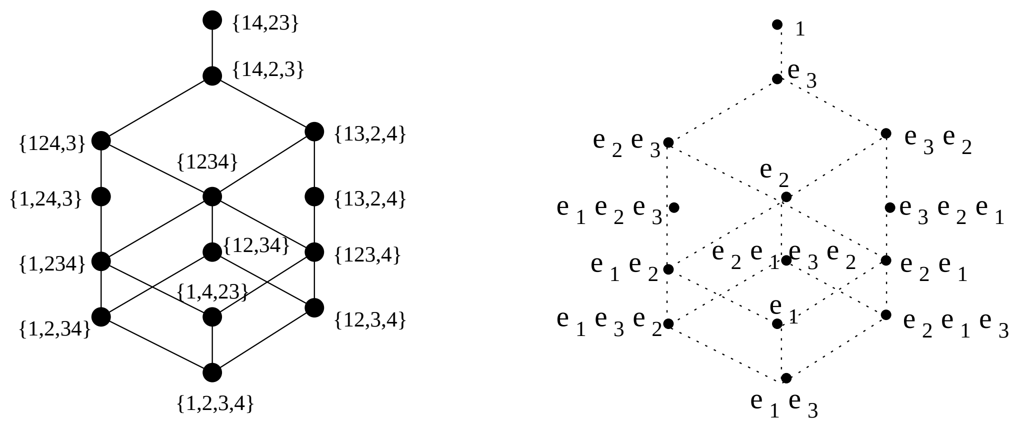


FIGURE 11. The lattice $\mathcal{L}(\mathcal{P}_4, \prec_{\mathsf{gi}})$

The paths corresponding to the elements in $\mathcal{L}(\mathcal{P}_4, \prec_{\mathsf{gi}})$ are given below, but without the cover relations.

$$[1] = \nearrow^4 \searrow^4$$

$$[\mathsf{e}_3] = \nearrow^3 \searrow \nearrow \searrow^3$$

$$[\mathsf{e}_2\mathsf{e}_3] = \nearrow^2 \searrow \nearrow^2 \searrow^3 \qquad [\mathsf{e}_3\mathsf{e}_2] = \nearrow^3 \searrow^2 \nearrow \searrow^2$$

$$[\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3] = \nearrow \searrow \nearrow^3 \searrow^3 \qquad [\mathsf{e}_2] = \nearrow^2 (\searrow \nearrow)^2 \searrow^2 \qquad [\mathsf{e}_3\mathsf{e}_2\mathsf{e}_1] = \nearrow^3 \searrow^3 \nearrow \searrow$$

$$[\mathsf{e}_1\mathsf{e}_2] = \nearrow \searrow \nearrow^2 \searrow \nearrow \searrow^2 \qquad [\mathsf{e}_2\mathsf{e}_1\mathsf{e}_3\mathsf{e}_2] = (\nearrow^2 \searrow^2)^2 \qquad [\mathsf{e}_2\mathsf{e}_1] = \nearrow^2 \searrow \nearrow \searrow^2 \nearrow \searrow$$

$$[\mathsf{e}_1\mathsf{e}_3\mathsf{e}_2] = (\nearrow \searrow)^2 \nearrow^2 \searrow^2 \qquad [\mathsf{e}_1] = \nearrow \searrow \nearrow^2 \searrow^2 \nearrow \searrow \qquad [\mathsf{e}_2\mathsf{e}_1\mathsf{e}_3] = \nearrow^2 \searrow^2 (\nearrow \searrow)^2$$

$$[\mathsf{e}_1\mathsf{e}_3] = (\nearrow \searrow)^4$$

C.2. **Enumeration of paths.** The number of planar partitions of $\{1, \ldots, n\}$ is equal to the number of arch diagrams on $2n$ points, and this is equal to the number of well formed brackets with n of each type. The set $\mathcal{S}$ of well formed brackets with the null bracket ε adjoined can be decomposed as $\mathcal{S} - \{\varepsilon\} \xrightarrow{\sim} (\mathcal{S})\mathcal{S}$. The number of planar partitions of $\{1, \ldots, n\}$ is therefore the coefficient of x^n in S where S is the solution of $S - 1 = xS^2$ with non-negative coefficients. This is $C_n = \binom{2n}{n}/(n+1)$, a Catalan number.

Lemma C.1. *The number of paths of length n with no point at negative height and with terminus at height i is*

$$k_i = \binom{n}{\frac{1}{2}(n-i)} - \binom{n}{\frac{1}{2}(n-i-2)}.$$

Proof. We use the the Reflexion Principle as follows. Consider the set of all paths from $(0, 0)$ to (n, i). From this we wish to remove paths having a point at negative height, which are characterized as those that cross the line $y = -1$. Such a path $\mathfrak{p}$ may be written uniquely as $\mathfrak{p} = \mathfrak{a}\mathfrak{b}$ where the terminus of $\mathfrak{a}$ is the first point (from the left) in $\mathfrak{p}$ with negative height. Let $\mathfrak{a}^r$ be the reflexion of $\mathfrak{a}$ in the line $y = -1$. Thus $\mathfrak{q} = \mathfrak{a}^r\mathfrak{b}$ is a path from $(0, -2)$ to (n, i). Then the mapping $\mathfrak{p} \mapsto \mathfrak{q}$ defines a bijection between the set of all paths from $(0,0)$ to (n,i) with at least one point at negative height and the set of all paths from $(0,-2)$ to (n,i). Then $k_i = N((0,0),(n,i)) - N((0,-2),(n,i))$ where $N((a,b),(r,s))$ be the number of paths from (a,b) to (r,s) where $r \geq a$. But, trivially,

$$N((a,b),(r,s)) = \binom{r-a}{\frac{1}{2}(r+s-a-b)}.$$

The result follows. □

C.3. **Proof of Proposition 5.5.** The proof given here is essentially a combination of proofs given by di Francesco [dF1]. Let $\#^j_{\nearrow}(\mathfrak{p},[r,s])$ denote the number of rises at height j in $\mathfrak{p} \in \mathcal{P}_n$ in positions $r,\dots,s$ where $r \leq s$. The corresponding number of falls is $\#^j_{\searrow}(\mathfrak{p},[r,s])$.

Proof. A strip of length j in $\mathfrak{p} \in \mathcal{P}_n$ corresponds to a step in position i in $\mathfrak{p}$ at height j, which is a fall if $0 \leq i \leq n-i$ or a rise if $n \leq i \leq 2n-1$. But $\#^j_{\searrow}(\mathfrak{p},[0,n-1]) + \#^j_{\nearrow}(\mathfrak{p},[n,2n-1]) = \#^j_{\searrow}(\mathfrak{p},[0,2n-1]) - \zeta(j,\mathfrak{p})$. where $\zeta(j,\mathfrak{p}) = 1$ if $j < h(\mathfrak{p})$ and 0 otherwise. Now b_j is the number of strips of length j in the set of all paths in $\mathcal{P}_n$, and is therefore equal to the number of paths in $\mathcal{P}_n$ with a distinguished strip of length j. Then $b_j = \#\{\text{paths in } \mathcal{P}_n \text{ with a distinguished fall at height } j\} - c_j$, so $b_j + c_j = \#\{\text{paths in } \mathcal{P}_n \text{ with a distinguished fall at height } j\}$.

Let $\mathfrak{q}$ be a path in $\mathcal{P}_n$ with a distinguished fall at height j. Then $\mathfrak{q}$ can be factorized uniquely into $\mathfrak{q} = \mathfrak{l}\mathfrak{r}$ where the distinguished fall is the last step of $\mathfrak{l}$. Then $\mathfrak{q}' = \mathfrak{l}\mathfrak{r}^t$ is a path of length $2n$ with terminal height $2j$ and with no point at negative height. This construction is reversible since the distinguished fall is the first fall from the right in $\mathfrak{q}'$ at height j, and this step is the terminal step of $\mathfrak{l}$, thereby recovering the factorization. Thus $b_j + c_j$ is the number of paths of length $2n$ with terminal height $2j$ and with no point at negative height. The result follows from Lemma C.1. □

Appendix D. The construction of the change of basis matrix $\mathcal{B}_2$ for $\mathfrak{TL}_3(x,y)$

This appendix gives the construction of the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathfrak{TL}_3(x,y)$, the change of basis matrix $\mathbf{P}_3$ and $\mathbf{D}_3$ directly from the set of paths $\mathcal{P}_3$ using box addition.

D.1. **Determining $\mathbf{P}_3$ and the box additions.** The base paths are $\mathfrak{I}_3^1$ and $\mathfrak{I}_3^3$, that have mid-height 1 and 3, respectively. They are given by $\mathfrak{I}_3^1 = (\nearrow\searrow)^3$ and $\mathfrak{I}_3^3 = \nearrow^3\searrow^3$. The elements of $\mathcal{P}_3$ are constructed from these by box additions in increasing order of mid-height h and, within paths of the same mid-height, in increasing order of $\#_\diamond$. The box addition are to be used to determine $\mathcal{B}_2$. There are $C_3 = 5$ paths to be constructed.

Below is a listing of the paths of $\mathcal{P}_3$ constructed in this order, each presented as a midpoint decomposition.

h	$\#_\diamond$	(posn., height)	box additions	path $\mathfrak{p}$ in $\mathcal{P}_3$	$[\mathfrak{p}]$
1	0	$(2,0),(4,0)$	$\mathfrak{I}_3^1$	$(\nearrow\searrow\nearrow)(\searrow\nearrow\searrow)$	e_1
1	1	$(4,0)$	$\mathfrak{I}_3^1 \boxplus \diamond_2$	$(\nearrow^2\searrow)(\searrow\nearrow\searrow)$	$\mathsf{e}_2\mathsf{e}_1$
1	1	$(2,0)$	$\mathfrak{I}_3^1 \boxplus \diamond_4$	$(\nearrow\searrow\nearrow)(\nearrow\searrow^2)$	$\mathsf{e}_1\mathsf{e}_2$
1	2	-	$\mathfrak{I}_3^1 \boxplus \diamond_2 \boxplus \diamond_4$	$(\nearrow^2\searrow)(\nearrow\searrow^2)$	e_2
3	0	-	$\mathfrak{I}_3^3$	$(\nearrow^3)(\searrow^3)$	1

The first column gives the mid-height h of the path $\mathfrak{p}$ given in the fifth column. The second column gives the number of boxes added to a base path to form the path $\mathfrak{p}$. The base paths are therefore those corresponding to a 0 in the second column. The third column gives the position and height of the minima in $\mathfrak{p}$, with the exception of a minimum in the middle position, which can never receive a box. Box additions are specified in the fourth column. The fifth column gives the midpoint decomposition $\mathfrak{lr}$ of $\mathfrak{p}$. The sixth column gives the result $[\mathfrak{p}]$ of applying Algorithm 3.4 with these box additions. The paths are listed in the order they are constructed.

The corresponding element of $\mathfrak{TL}_3(x,y)$ can also be determined combinatorially by first transforming a path into an arch diagram, and then transforming this into a strand diagram. The strand diagram is then rewritten as a concatenation of strand diagrams for the generators e_1 and e_2 in a straightforward way.

The lattice $\mathcal{L}(\mathcal{P}_3, \prec_{\mathsf{gi}})$ of these path in $\mathcal{P}_3$ ordered by geometric inclusion is

$$\mathcal{L}(\mathcal{P}_3, \prec_{\mathsf{gi}}) = \begin{array}{ccccc} & & \mathsf{1} & & \\ & & \downarrow & & \\ & & \mathsf{e}_2 & & \\ & \swarrow & & \searrow & \\ \mathsf{e}_1\mathsf{e}_2 & & & & \mathsf{e}_2\mathsf{e}_1 \\ & \searrow & & \swarrow & \\ & & \mathsf{e}_1 & & \end{array}$$

A linear extension of this lattice gives the ordered basis

$$\mathcal{B}_1 = (\mathsf{e}_1, \mathsf{e}_2\mathsf{e}_1, \mathsf{e}_1\mathsf{e}_2, \mathsf{e}_2, \mathsf{1})$$

of $\mathfrak{TL}_3(x,y)$.

D.2. **Construction of the Jones-Wenzl projectors.** From (10), $\mu_1 = x^{-1}$ and $\mu_2 = x/(xy-1)$. Moreover, $V_1(x,y) = x$ and $V_2(x,y) = xy-1$. Then, from Lemma 3.1,

$$\begin{aligned} \mathsf{E}_1 &= 1, \\ \mathsf{E}_2 &= \mathsf{E}_1 - \mu_1 \mathsf{E}_1\mathsf{e}_1\mathsf{E}_1 = \mathsf{1} - x^{-1}\mathsf{e}_1, \\ \mathsf{E}_3 &= \mathsf{E}_2 - \mu_2 \mathsf{E}_2\mathsf{e}_2\mathsf{E}_2 = \mathsf{1} - \frac{1}{xy-1}\left(y\mathsf{e}_1 + x\mathsf{e}_2 - \mathsf{e}_1\mathsf{e}_2 - \mathsf{e}_2\mathsf{e}_1\right). \end{aligned}$$

Also, from Corollary 3.3,

$$\mathsf{E}_3^{(1)} = \mathsf{E}_3 = \mathsf{1} - \frac{1}{xy-1}\left(y\mathsf{e}_1 + x\mathsf{e}_2 - \mathsf{e}_1\mathsf{e}_2 - \mathsf{e}_2\mathsf{e}_1\right).$$

$\mathsf{E}_3^{(2)}$ is obtained from E_2 by $\mathsf{e}_1 \mapsto \mathsf{e}_2$ and $x_1 \mapsto x_2$, so $\mathsf{E}_3^{(2)} = \mathsf{1} = y^{-1}\mathsf{e}_2$, and $\mathsf{E}_3^{(3)} = \mathsf{1}$.

By direct computation with S1, S2 and S3, it is readily seen, for example, that $\mathsf{e}_1\mathsf{E}_3 = 0$ and $\mathsf{e}_2\mathsf{E}_3 = 0$ (*c.f.* Lemma 3.1).

D.3. **Construction of $\mathcal{B}_2$ and $\mathbf{P}_3$.** The box addition data given in Appendix D.1 is used in the construction of $\mathcal{B}_2$. From Definition 3.5

$$\begin{aligned} (\mathfrak{I}_3^1)_2 &= x^{-1}\mathsf{E}_3^{(3)}\mathsf{e}_1 = x^{-1}\mathsf{e}_1, \\ (\mathfrak{I}_3^3)_2 &= \mathsf{E}_3^{(1)} = \mathsf{1} - \frac{1}{xy-1}\left(y\mathsf{e}_1 + x\mathsf{e}_2 - \mathsf{e}_1\mathsf{e}_2 - \mathsf{e}_2\mathsf{e}_1\right). \end{aligned}$$

Again, from Definition 3.5,

$$\mathcal{B}_2 = ([\mathsf{e}_1]_2, [\mathsf{e}_2\mathsf{e}_1]_2, [\mathsf{e}_1\mathsf{e}_2]_2, [\mathsf{e}_2]_2, [1]_2),$$

where, by carrying out the explicit computations using S1, S2 and S3,

$$\begin{aligned}
[\mathsf{e}_1]_2 &= (\mathfrak{I}_3^1)_2 = x^{-1}\mathsf{e}_1 \\
[\mathsf{e}_2\mathsf{e}_1]_2 &= (\mathfrak{I}_3^1 \boxplus \diamond_2)_2 = \sqrt{\frac{\mu_2}{\mu_1}}(\mathsf{e}_2 - \mu_1)(\mathfrak{I}_3^1)_2 = \frac{1}{\sqrt{xy-1}}(-x^{-1}\mathsf{e}_1 + \mathsf{e}_2\mathsf{e}_1), \\
[\mathsf{e}_1\mathsf{e}_2]_2 &= (\mathfrak{I}_3^1 \boxplus \diamond_4)_2 = \frac{1}{\sqrt{xy-1}}(-x^{-1}\mathsf{e}_1 + \mathsf{e}_1\mathsf{e}_2), \\
[\mathsf{e}_2]_2 &= (\mathfrak{I}_3^1 \boxplus \diamond_2 \boxplus \diamond_4)_2 = \sqrt{\mu_2/\mu_1}\ (\mathfrak{I}_3^1 \boxplus \diamond_2)_2(\mathsf{e}_2 - \mu_1) = \mu_2(\mathsf{e}_2 - \mu_1)\mathsf{e}_1(\mathsf{e}_2 - \mu_1) \\
&= \frac{x}{xy-1}(x^{-2}\mathsf{e}_1 - x^{-1}\mathsf{e}_2\mathsf{e}_1 - x^{-1}\mathsf{e}_1\mathsf{e}_2 + \mathsf{e}_2) \\
&\quad \text{since the minimum in position 2 in } [\mathsf{e}_2\mathsf{e}_1]_2 \text{ is at height 0} \\
[1]_2 &= (\mathfrak{I}_3^3)_2 = \mathsf{E}_3^{(1)} = \frac{1}{xy-1}\left(-y\mathsf{e}_1 + \mathsf{e}_2\mathsf{e}_1 + \mathsf{e}_1\mathsf{e}_2 - x\mathsf{e}_2\right) + 1.
\end{aligned}$$

By explicit computation with these expressions we have $[\mathsf{e}_1\mathsf{e}_2]_2[\mathsf{e}_2\mathsf{e}_1]_2 = -x^{-1}\mathsf{e}_1 = [\mathsf{e}_1]_2$. The same result is obtained from Theorem 3.6 as follows. For the midpoint decompositions $[\mathsf{e}_1\mathsf{e}_2] = (\nearrow\searrow\nearrow)(\nearrow\searrow^2) = \mathfrak{l}\mathfrak{r}$ and $[\mathsf{e}_2\mathsf{e}_1] = (\nearrow^2\searrow)(\searrow\nearrow\searrow) = \mathfrak{l}'\mathfrak{r}'$ we note that $\mathfrak{r}^t = \mathfrak{l}$ so $[\mathsf{e}_1\mathsf{e}_2]_2[\mathsf{e}_2\mathsf{e}_1]_2 = (\mathfrak{l}\mathfrak{r}')_2 = (\nearrow\searrow\nearrow)(\searrow\nearrow\searrow) = [\mathsf{e}_1]_2$. Similarly, $[\mathsf{e}_2\mathsf{e}_1]_2[\mathsf{e}_1\mathsf{e}_2]_2 = [\mathsf{e}_2]_2$.

The matrix that expresses $\mathcal{B}_2$ in terms of $\mathcal{B}_1$ is obtained immediately from the above expressions and is

$$\mathbf{P}_3 = \begin{pmatrix}
\frac{1}{x} & -\frac{1}{x\sqrt{xy-1}} & -\frac{1}{x\sqrt{xy-1}} & \frac{1}{x(xy-1)} & -\frac{y}{xy-1} \\
0 & \frac{1}{\sqrt{xy-1}} & 0 & -\frac{1}{xy-1} & \frac{1}{xy-1} \\
0 & 0 & \frac{1}{\sqrt{xy-1}} & -\frac{1}{xy-1} & \frac{1}{xy-1} \\
0 & 0 & 0 & \frac{x}{xy-1} & -\frac{x}{xy-1} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

D.4. **A matrix algebra isomorphic to $\mathfrak{TL}_3(x,y)$.** The following is an example of the isomorphism given in Theorem 3.7. The terminal height of paths of length 3 with no points at negative height are 1 and 3. The paths with terminal height 3 are $\mathfrak{p}_1^{(1)} = \nearrow\searrow\nearrow$ and $\mathfrak{p}_2^{(1)} = \nearrow^2\searrow$, so $k_1 = 2$. The path with terminal height 3 is $\mathfrak{p}_1^{(3)} = \nearrow^3$, so $k_3 = 1$. Thus

$$\mathfrak{TL}_n(x,y) \cong \mathcal{M}_2(\mathbb{C}(x,y)) \times \mathcal{M}_1(\mathbb{C}(x,y)).$$

To determine the image of $[\mathsf{e}_1\mathsf{e}_2]_2$, note that $[\mathsf{e}_1\mathsf{e}_2] = \mathfrak{p}_1^{(1)}(\mathfrak{p}_2^{(1)})^t$, so

$$[\mathsf{e}_1\mathsf{e}_2]_2 \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus [0],$$

where the rows and columns of this matrix are indexed by members of the ordered set $(\mathfrak{p}_1^{(2)}, \mathfrak{p}_2^{(2)}, \mathfrak{p}_1^{(3)})$. Similarly,

$$[\mathsf{e}_1]_2 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus [0], [\mathsf{e}_2\mathsf{e}_1]_2 \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus [0], [\mathsf{e}_2]_2 \leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \oplus [0], [1]_2 \leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \oplus [1].$$

For example, using the expressions from Appendix D.3, we have

$$[\mathsf{e}_1\mathsf{e}_2]_2[\mathsf{e}_2\mathsf{e}_1]_2 \leftrightarrow \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus [0]\right)\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus [0]\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus [0] \leftrightarrow [\mathsf{e}_1]_2,$$

so $[\mathsf{e}_1\mathsf{e}_2]_2[\mathsf{e}_2\mathsf{e}_1]_2 = [\mathsf{e}_1]_2$. This is in agreement with the direct computation

$$[\mathsf{e}_1\mathsf{e}_2]_2[\mathsf{e}_2\mathsf{e}_1]_2 = \frac{1}{xy-1}(-x^{-1}\mathsf{e}_1 + \mathsf{e}_1\mathsf{e}_2)(-x^{-1}\mathsf{e}_1 + \mathsf{e}_2\mathsf{e}_1) = x^{-1}\mathsf{e}_1 = [\mathsf{e}_1]_2.$$

Notation

$\mathsf{a}, \mathsf{b}, \ldots$	elements of $\mathfrak{TL}_n(x,y)$ (sans serif)
$\mathfrak{a}, \mathfrak{b}, \ldots$	paths in $\mathcal{P}_n$ (Gothic)
$\mathcal{B}_1$	monomial basis of $\mathfrak{TL}_n(x,y)$
$\mathcal{B}_2$	semi-orthogonal basis of $\mathfrak{TL}_n(x,y)$
C_n	Catalan number
$\mathbb{C}(x,y)$	rational functions in x, y over $\mathbb{C}$
$\mathbf{D}_n(x,y)$	matrix of $\langle\cdot,\cdot\rangle_n$ wrt $\mathcal{B}_2$
e_i	a generator of $\mathfrak{TL}_n(x,y)$
$\mathsf{e}^t, \mathfrak{e}^t$	transposes
E_n	Jones-Wenzl projector
$\mathsf{E}_n^{(k)}$	shifted Jones-Wenzl projector
$h(\mathfrak{a})$	mid-height of $\mathfrak{a}$
$h_i(\mathfrak{a})$	height of $\mathfrak{a}$ at position i
$\mathfrak{I}_n^k$	a base path
$\mathfrak{K}_{2n}(x,y)$	a left ideal of $\mathfrak{TL}_{2n}(x,y)$
$\mathbf{L}_n(q)$	matrix of chromatic joins
$\mathcal{L}(\mathcal{P}_n, \prec_{\mathsf{gi}})$	lattice of paths ordered by $\prec_{\mathsf{gi}}$
$\mathbf{M}_n(x,y)$	generalization of $\mathbf{L}_n(q)$
$\mathcal{M}_n$	set of all $n \times n$ matrices
$\mathbf{P}_n$	matrix for $\mathcal{B}_2$ in terms of $\mathcal{B}_1$
$\mathcal{P}_n$	set of all Dyck paths of length $2n$
q	an indeterminate
$R1, R2, R3$	defining relations for $\mathfrak{TL}_n(q)$
$S1, S2, S3$	defining relations for $\mathfrak{TL}_n(x,y)$
$s_{i,\mathfrak{a}}$	number of strips of length i in $\mathfrak{a}$
$\mathfrak{S}_\mathfrak{n}$	symmetric group on n symbols
$\mathfrak{TL}_n(q)$	Temperley-Lieb algebra
$\mathfrak{TL}_n(x,y)$	generalized Temperley-Lieb algebra
tr_n	a Markov trace on $\mathfrak{TL}_n(x,y)$
$V_i(x,y)$	generalized Chebyshev polynomial
x, y	indeterminates

Γ	bijective map $\mathfrak{TL}_n(x,y) \to \mathfrak{K}_{2n}(x,y)$
θ_i	$\sqrt{\mu_{i+1}/\mu_i}$
$\kappa_i(\mathfrak{p})$	collapse of $\mathfrak{p}$ at slope in position i of $\mathfrak{p}$
μ_n	scalar for E_{n+1}
π, γ	planar partitions
$\pi\gamma$	concatenation of strand diagrams of π, γ
ϕ_n	weight function for planar partition
$\chi(\pi,\gamma)$	chromatic join of γ, π
$\mathsf{1}$	multiplicative identity in $\mathfrak{TL}_n(x,y)$
$\overline{\mathsf{e}}$	closure of strand diagram of e
$\#_{\mathrm{loop}}(\mathsf{e})$	no. loops in loop diagram of $\overline{\mathsf{e}}$
$\#_{\mathrm{sh}}(\mathsf{e})$	no. of shaded regions in $\overline{\mathsf{e}}$
$\#_{\mathrm{ush}}(\mathsf{e})$	no. of unshaded regions in $\overline{\mathsf{e}}$
$\#_{\diamond}(\mathsf{e})$	no. of box additions to construct e
$[\mathsf{e}]$	path corresponding to e
$[\mathsf{e}]_1$	abbreviation for $([\mathsf{e}])_1$
$[\mathsf{e}]_2$	abbreviation for $([\mathsf{e}])_2$
$[\![\mathsf{a}]\!]\mathsf{f}$	coefficient of $\mathsf{a} \in \mathcal{B}_1$ in $\mathsf{f} \in \mathfrak{TL}_n(x,y)$
$\langle\cdot,\cdot\rangle_n$	bilinear form on $\mathfrak{TL}_n(x,y)$
$\nearrow, \searrow$	rise and fall, respectively, in a path
$\nearrow^k$	sequence of k rises
$\overset{i}{\nearrow}$	rise in position i in a path
$\overset{(i)}{\searrow}$	unique fall in a path to match $\overset{i}{\nearrow}$
$\boxplus$	box addition
$\boxminus$	box deletion
$\diamond_i$	box at position i in a path
$\prec_{\mathsf{gi}}$	geometrical inclusion of paths
$(\mathfrak{a})_1$	$\mathcal{B}_1$ element corresponding to $\mathfrak{a}$
$(\mathfrak{a})_2$	$\mathcal{B}_2$ element corresponding to $\mathfrak{a}$

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