A LIOUVILLE THEOREM FOR SOLUTIONS TO THE LINEARIZED MONGE-AMPERE EQUATION

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ABSTRACT. We prove that global Lipschitz solutions to the linearized Monge-Ampere equation

\[ L_\varphi u := \sum \varphi^{ij} u_{ij} = 0 \]

must be linear in 2D. The function \( \varphi \) is assumed to have the Monge-Ampere measure \( \det D^2 \varphi \) bounded away from 0 and 1.

1. Introduction. In this paper we consider global \( C^2 \) solutions \( u : \mathbb{R}^2 \to \mathbb{R} \) that satisfy certain types of degenerate elliptic equations

\[ \sum a_{ij}(x)u_{ij} = 0 \quad \text{in } \mathbb{R}^2. \tag{1} \]

We are interested in equations (1) that appear as the linearized operator for the Monge-Ampere equation. We show that the only global Lipschitz solutions i.e

\[ \|\nabla u(x)\|_{L^\infty(\mathbb{R}^2)} \leq C. \tag{2} \]

must be linear.

For simplicity we assume throughout the paper that the coefficients \( a_{ij} \) are smooth and satisfy the ellipticity condition:

\[ A(x) := (a_{ij}(x))_{ij} > 0. \]

We start by recalling two classical Liouville type theorems concerning global solutions of (1). The first is due to Bernstein (see [6], [1]) and asserts:

A global \( C^2 \) solution (in \( \mathbb{R}^2 \)) which is bounded must be constant.

The result fails if one allows linear growth for \( u \) at \( \infty \) as it can be seen from the following simple example

\[ u(x) = \sqrt{1+x_1^2} - \sqrt{1+x_2^2}, \]

for appropriate \( A(x) \).

The second theorem states that global solutions satisfying (1), (2) must be linear if the coefficients are uniformly elliptic i.e

\[ \lambda I \leq A(x) \leq \Lambda I, \quad x \in \mathbb{R}^2. \]

This follows from the classical \( C^{1,\alpha} \) interior estimates in 2D due to Morrey [7] and Nirenberg [8].

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In this short paper we prove a similar Liouville theorem for solutions to the linearized operator of the Monge-Ampere equation
\[ \det D^2 \varphi = f, \quad \lambda \leq f \leq \Lambda. \] (3)

**Theorem 1.1.** Assume \( \varphi \) is a smooth convex function in \( \mathbb{R}^2 \) satisfying
\[ \lambda \leq \det D^2 \varphi \leq \Lambda, \]
and denote by \((\varphi^{ij})\) the inverse matrix of \( D^2 \varphi \). If \( u \in C^2 \) is globally Lipschitz (i.e. satisfies (2)) and solves
\[ L_{\varphi} u := \sum \varphi^{ij} (x) u_{ij} = 0 \quad \text{in} \ \mathbb{R}^2, \] (4)
then \( u \) is linear.

Equation (4) was studied by Caffarelli and Gutierrez in [3]. It appears for example in fluid mechanics (see [2], [4]), or in the affine maximal graph equation (see [10]) etc. The main result in [3] states that solutions of (4) satisfy the Harnack inequality in the sections of \( \varphi \) (see Section 2 for the precise statement). When dealing with the degenerate equation \( L_{\varphi} u = 0 \), the sections of \( \varphi \) play the same role as the euclidean balls do in the theory of uniformly elliptic equations.

Theorem 1.1 suggests that in \( \mathbb{R}^2 \), solutions of (4) satisfy stronger estimates than those obtained from Harnack inequality. In a forthcoming paper we intend to obtain interior \( C^{1,\alpha} \) estimates for the equation \( L_{\varphi} u = 0 \) in \( 2D \).

Theorem 1.1 can be proved in fact in a more general form, where the coefficient matrix \( A(x) \) is “uniformly elliptic” with respect to the inverse of \( D^2 \varphi \) i.e.
\[ c(D^2 \varphi)^{-1} \leq A(x) \leq C(D^2 \varphi)^{-1}, \quad 0 < c < C, \]
and the Monge-Ampere measure
\[ \det D^2 \varphi = \mu \]
satisfies a standard doubling condition (see conditions 0.3-0.4 of [3]). In this setting, the Liouville theorem for uniformly elliptic equations mentioned above appears as a consequence of Theorem 1.1 by taking \( \varphi(x) := |x|^2 \).

The proof of Theorem 1.1 follows the same strategy as the proof of Bernstein theorem for elliptic equations in \( 2D \). If \( u \) is a solution to (1) and is not linear, then any tangent plane splits the graph of \( u \) into at least 4 unbounded connected components. Then we apply the Harnack inequality of Caffarelli and Gutierrez in certain nondegenerate directions and obtain a contradiction. Similar ideas have been used in [9], [5] for other degenerate equations.

**2. Geometry of sections and Harnack inequality.** Let \( S_h(x) \), the section of \( \varphi \) at the point \( x \) and of height \( h > 0 \), be defined as
\[ S_h(x) := \{ y \in \mathbb{R}^2 \mid \varphi(y) < \varphi(x) + \nabla \varphi(x) \cdot (y - x) + h \}. \]

We list below the key properties (see for example [3]) for the sections of a convex function \( \varphi \) that satisfies
\[ \det D^2 \varphi = f, \quad \lambda \leq f \leq \Lambda. \]

a) \( S_h(x) \) is convex, and if \( h \leq t \) then \( S_h(x) \subset S_t(x) \).
b) To each section $S_h(x)$ we can associate an ellipse

$$E_h(x) = A_h(x)B_1,$$

with $A_h(x)$ a symmetric matrix

such that

$$E_h(x) \subset S_h(x) - x \subset C E_h(x),$$

with the constant $C$ depending only on $\lambda, \Lambda$. In what follows we denote by

$|E_h(x)|$ – the ratio between the longest and the shortest axis of $E_h(x)$

$\xi_h(x)$ – the direction of the longest axis of $E_h(x)$.

\[ \text{(5)} \]

\[ \text{c) If } x_1 \in S_h(x_0) \text{ then } \]

$$S_h(x_0) \subset S_{C_1h}(x_1) \subset S_{C_2h}(x_0),$$

with $C_1, C_2$ depending only on $\lambda, \Lambda$.

\[ \text{d) If } M > 1 \text{ then } \]

$$S_{Mh}(x) - x \subset M(S_h(x) - x) \subset S_{C(M)h} - x,$$

for some constant $C(M)$ depending on $\lambda, \Lambda$ and $M$.

Caffarelli and Gutierrez proved in \[3\] the Harnack inequality for solutions of the linearized operator

$$L_\varphi u = \sum a^{ij}u_{ij}.$$ 

Precisely, if $u \geq 0$ in $S_h(x_0)$ then

$$\inf_{S_{h/2}(x_0)} u \geq c \sup_{S_{h/2}(x_0)} u,$$

with $c > 0$ a small constant depending only on $\lambda, \Lambda$. We need the following weak Harnack inequality for supersolutions which was proved also in \[3\] (Theorem 2).

**Theorem.** If $L_\varphi (u) \leq 0$ and $u \geq 0$ in $S_h(x_0)$ then,

$$\inf_{S_{h/2}(x_0)} u \geq c(\tau) \inf_{S_{\tau h}(x_0)} u$$

with $c(\tau) > 0$ a small constant depending on $\lambda, \Lambda$.

Applying the theorem repeatedly we see that for any $\tau \leq 1/4$,

$$\inf_{S_{h/2}(x_0)} u \geq c(\tau) \inf_{S_{\tau h}(x_0)} u$$

with $c(\tau) > 0$ depending also on $\tau$.

Before we state the next lemmas we introduce the following notation. We define $A_\delta$ as the set

$$A_\delta := \{(x,t) | \quad \text{diam } S_t(x) \geq \delta, \quad S_t(x) \subset B_{1/\delta}\}. \quad \text{(7)}$$

**Lemma 2.1.** Let $S_h(0)$ be the maximal section at 0 which is included in $B_1$. Assume that (see (3))

$$|E_h(0)| \leq M,$$

for some constant $M$. Then

$$|E_t(x)| \leq C(M, \delta) \quad \forall (x,t) \in A_\delta,$$
with \( C(M, \delta) \) a large constant depending on \( M, \delta, \lambda, \Lambda \).

**Proof.** Since \( S_h(0) \) is the maximal section included in \( B_1 \) and satisfies \( |E_h(0)| \leq M \) we see from property b) that for a small constant \( c(M) \) depending on \( M \) (and on \( \lambda, \Lambda \))

\[
B_c \subset S_h(0) \subset B_1.
\]

Then by property d) we can find a large constants \( C_1(M, \delta) \) such that

\[
B_{2/\delta} \subset S_{C_1h}(0) \subset B_{C_1}.
\]

Now by c), there exist \( C_2(M, \delta), C_3(M, \delta) \) such that for any \( x \in B_{1/\delta} \),

\[
B_1 \subset S_{C_2h}(x) - x \subset B_{C_3}.
\]

Now, we use d) and find \( c_1(M, \delta), c_2(M, \delta) \) small such that

\[
B_{c_1} \subset S_{c_2h}(x) - x \subset B_{\delta/2}.
\]

This shows, by property a), that any section \( S_t(x) \) with \( x \in B_{1/\delta} \) and \( \text{diam} S_t(x) \leq \delta \) contains a ball of radius \( c_1 \) in the interior, and the conclusion of the lemma follows easily. \( \square \)

**Lemma 2.2.** Let \( S_h(0) \) be the maximal section at 0 included in \( B_1 \), and assume \( |E_h(0)| \leq M \). Let \( u \) be defined on \( B_r(x) \) for some \( x \in B_1 \) and \( \delta \leq r \leq 1 \).

If \( u \geq 0 \) in \( B_r(x) \) and

\[
L_\varphi u \leq 0,
\]

then

\[
\inf_{B_{\delta/2}(x)} u \geq c(M, \delta) \inf_{B_{\delta/4}(x)} u
\]

with \( c(M, \delta) \) a small positive constant depending on \( M, \delta, \lambda, \Lambda \).

**Proof.** It suffices to show that if \( u \geq 0 \) in \( B_\delta(x) \) then

\[
\inf_{B_\epsilon(x)} u \geq c(M, \delta) \inf_{B_{\epsilon/2}(x)} u,
\]

for some \( \eta(M, \delta) \) small.

By Lemma 2.1 there exists \( \eta(M, \delta) \) small and a section \( S_t(x) \) with \( (x, t) \in A_\delta \) such that

\[
B_\eta \subset S_{t/2}(x) - x \subset S_t(x) - x \subset B_\delta.
\]

By property d), we can find \( \tau(M, \delta) > 0 \) such that

\[
S_{\epsilon t}(x) - x \subset B_{\eta/2}.
\]

Now the weak Harnack inequality (6) applied to \( u \) in \( S_t \) gives

\[
\inf_{B_{\epsilon}(x)} u \geq \inf_{S_{t/2}(x)} u \geq c(\tau) \inf_{S_{\epsilon t}(x)} u \geq c(\tau) \inf_{B_{\eta/2}(x)} u.
\]

\( \square \)

**Lemma 2.3.** Let \( S_h(0) \) be the maximal section included in \( B_1 \) and assume

\[
|E_h(0)| \geq M.
\]

There exists \( \sigma(M, \delta) \) such that for all \( (x, t) \in A_\delta \) (see (5), (7))

\[
|E_t(x)| \geq \sigma^{-1}, \quad \angle(\xi_1(x), \xi_h(0)) \leq \sigma
\]

and \( \sigma(M, \delta) \to 0 \) as \( M \to \infty \).

Here \( \angle(\xi_1, \xi_2) \) denotes the angle \( (\in [0, \pi/2]) \) between the lines of directions \( \xi_1 \) and \( \xi_2 \).
Proof. We need to show that, for \( \delta \) fixed,
\[
\inf_{(x,t) \in \mathcal{A}_\delta} |E_t(x)| \to \infty, \quad \sup_{(x,t) \in \mathcal{A}_\delta} \angle(\xi_t(x), \xi_h(0)) \to 0 \quad \text{as} \quad M \to \infty.
\]
From Lemma 2.1 it follows that if \( |E_t(x)| \leq N \) for some \((x, t) \in \mathcal{A}_\delta\), then
\[
|E_h(0)| \leq C(\delta, N).
\]
This shows that
\[
\inf_{(x,t) \in \mathcal{A}_\delta} |E_t(x)| \to \infty \quad \text{as} \quad M \to \infty.
\]
Now assume that for some \((x, t) \in \mathcal{A}_\delta\)
\[
\angle(\xi_t(x), \xi_h(0)) \geq \sigma_0 > 0.
\]
Let \( x^* \) be the point of intersection of the line passing through \( x \) and of direction \( \xi_t(x) \) with the line passing through \( 0 \) and direction \( \xi_h(0) \). Clearly \( |x^*| \leq C(\delta, \sigma_0) \). Moreover by the properties c) and d), there exists a section \( S_t'(x^*) \) with
\[
S_h(0) \subset S_t'(x^*), \quad S_t(x) \subset S_t'(x^*), \quad \text{diam} \ S_t'(x^*) \leq C(\delta, \sigma_0).
\]
Since \( S_t'(x^*) \) contains 2 segments of length \( \delta \) at an angle \( \sigma_0 \), it contains also a small ball of radius \( c(\delta, \sigma_0) \), hence
\[
|E_t'(x^*)| \leq C(\sigma_0, \delta).
\]
By Lemma 2.1, this implies that \( |E(0, h)| \leq C(\sigma_0, \delta) \). In conclusion
\[
\angle(\xi_t(x), \xi_h(0)) \to 0 \quad \text{as} \quad M \to \infty.
\]

3. Proof of Theorem 1.1. Without loss of generality assume \( u(0) = 0 \) and \( \|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq 1 \). Let
\[
K := \nabla u(\mathbb{R}^2).
\]
We need to show that \( K \) consists of a single point. As in the proof of the theorem of Bernstein, the key will be to use the following 2D theorem.

**Theorem.** Assume \( u \in C^2(\mathbb{R}^2) \) satisfies \( L_\varphi u = 0 \). If \( x \) is a nondegenerate point i.e \( D^2 u(x) \neq 0 \), then the set
\[
\{ y \in \mathbb{R}^2 \mid u(y) > u(x) + \nabla u(x) \cdot (y - x) \}
\]
contains at least two disconnected unbounded components that have \( x \) as a boundary point.

From now on we assume by contradiction that \( u \) is not linear.

First we remark that the set of nondegenerate points is dense in \( \mathbb{R}^2 \). Indeed, otherwise \( D^2 u = 0 \) in a neighborhood, and by unique continuation (since \( \varphi \in C^\infty \)) \( D^2 u = 0 \) in whole \( \mathbb{R}^2 \) and we reach a contradiction.

Clearly, the images of the gradients of these nondegenerate points form a dense open subset of \( K \).

Let \( R_n \) be a sequence converging to \( \infty \) and let
\[
u_n(x) := \frac{u(R_n x)}{R_n}
\]

represent the corresponding rescalings of $u$. The functions $u_n$ satisfy

$$L_{\varphi_n}u_n = 0, \quad \varphi_n(x) := \frac{\varphi(R_n x)}{R_n^2}.$$ 

The function $\varphi_n$ also satisfies (3), and its sections are obtained by $1/R_n$-dilations of the original sections of $u$. Denote

$$e_n := |E_{h_n}(0)|$$

where $S_{h_n}(0)$ is the maximal section of $u$ included in $B_{R_n}$. We distinguish 2 cases:

1) There exists a sequence of $R_n \to \infty$ such that $e_n$ remains bounded;

2) $e_n \to \infty$ as $R_n \to \infty$.

We show that we reach a contradiction in both cases.

**Case 1.** By assumption, there exists $M$ such that $|e_n| \leq M$ for all $n$. Without loss of generality we can assume that

$$u_n \to u^* \quad \text{uniformly on compact sets.}$$

From Lemma 2.2, each $u_n$ satisfies the weak Harnack inequality (8), thus the same inequality holds for $u^*$ if $u^* \geq 0$ in $B_r(x)$.

**Lemma 3.1.** Let $\nu \in \mathbb{R}^2$, $|\nu| = 1$ be a unit direction, and assume

$$\min_{p \in K} \nu \cdot p$$

is achieved for $p = p_\nu \in \bar{K}$. Then

$$u^*(t \nu) = t\nu \cdot p_\nu$$

either for all $t \geq 0$ or for all $t \leq 0$.

**Proof.** The equation is invariant under addition with linear functionals, thus we may assume for simplicity that $\nu = e_2$ and $p_\nu = 0$, that is

$$K \subset \{x \cdot e_2 \geq 0\}, \quad 0 \in \bar{K}.$$ 

This implies that the functions $u$, $u_n$, and $u^*$ are all increasing in the $e_2$ direction and that there exists a sequence of nondegenerate points for $u$ whose gradients approach 0. By passing if necessary to a subsequence we may assume that there exists $x_n \to 0$ with $\nabla u_n(x_n) = \nabla u(R_n x_n) \to 0$ and $x_n$ is a nondegenerate point for $u_n$. Define

$$l_n(x) := u_n(x_n) + \nabla u_n(x_n) \cdot (x - x_n),$$

then clearly $l_n \to 0$ uniformly on compact sets. By the theorem above, the set $\{u_n > l_n\}$ contains at least 2 unbounded connected components that have $x_n$ as a boundary point.

Since $u^*$ is increasing in the $e_2$ direction, it suffices to show that either $u^*(e_2) = 0$ or $u^*(-e_2) = 0$. Assume by contradiction that

$$u^*(-e_2) < 0, \quad u^*(e_2) > 0.$$ 

Then we can find $\delta$ (depending on $u^*$) and a rectangle

$$\mathcal{R} := [-2\delta, 2\delta] \times [-1, 1]$$
such that \( u^* \) is positive on the top of \( R \) and negative on the bottom. This implies that for all \( n \) large, \( u_n \) is positive on the top of \( R \) and negative on the bottom. We conclude that the set \( \{u_n > l_n\} \) has an unbounded connected component \( U \) that does not intersect the top or the bottom of, say the rectangle
\[
R_1 := [\delta, 2\delta] \times [-1, 1],
\]
but intersects both lateral sides of \( R_1 \). Let \( P \) be a nonintersecting polygonal line included in \( U \) which connects the lateral sides. This polygonal line splits \( R_1 \) into two disjoint domains \( R_1^+ \) (containing the top) and \( R_1^- \).

From each \( u_n \) we create a supersolution \( \tilde{u}_n : R_1 \rightarrow \mathbb{R} \) to \( L_{\varphi_n} \tilde{u}_n \leq 0 \) as follows.

First we replace \( u_n \) by \( l_n \) in the set \( U \). Clearly the new function is a supersolution. Then we modify this function to be equal to \( l_n \) in \( R_1^- \).

Notice that \( \tilde{u}_n \) converges uniformly in \( R_1 \) to
\[
(u^*)^+ := \max\{u^*, 0\}.
\]
Since \( \tilde{u}_n \) satisfies the weak Harnack inequality of Lemma 2.2, we see that the same conclusion holds for \( (u^*)^+ \) as well. This implies that \( (u^*)^+ > 0 \) in the interior of \( R_1 \). On the other hand \( (u^*)^+ = 0 \) in a neighborhood of the bottom of \( R_1 \) since \( u^* \) is negative there. We reached a contradiction and the lemma is proved.

By the lemma above, \( u^*(x) = p_{\nu} \cdot x \) (and \( u(x) = p_{-\nu} \cdot x \)) on at least half of the line \( tv \). Since the set \( K \) has nonempty interior, by the definition of \( p_{\nu} \) we have
\[
\nu \cdot (p_{\nu} - p_{-\nu}) < 0,
\]
which implies that \( u^*(tv) \) is linear both for \( t \geq 0 \) and \( t \leq 0 \) but with different slopes. We conclude that \( u^* \) is homogenous of degree one.

Since \( u^* \) is continuous, homogenous of degree 1 but not linear, we can easily find a ball \( B_r(x) \) and a linear function \( l \) such that \( u^* - l \geq 0 \) in \( B_r(x) \), \((u^* - l)(x) = 0 \) but \( u^* - l \) is not identically 0 in \( B_r(x) \). This contradicts weak Harnack inequality for \( u^* - l \), and concludes Case 1.

**Case 2.** By passing to a subsequence, we can assume that the directions
\[
\xi_n := \xi_{h_n}(0) \rightarrow e_2 \quad \text{as} \quad n \rightarrow \infty,
\]
and as before \( u_n \rightarrow u^* \) uniformly on compact sets. First we show that \( u^* \) satisfies weak Harnack inequality in the \( e_2 \) direction.

**Lemma 3.2.** Assume
\[
u^*(x + te_2) \geq 0, \quad \text{for all} \quad |t| \leq r,
\]
for some \( x \in B_1 \) and \( 0 < r \leq 1 \). Then
\[
\inf_{|t| \leq \frac{r}{4}} u^*(x + te_2) \geq \inf_{|t| \leq \frac{r}{4}} u(x + te_2), \tag{9}
\]
where \( c > 0 \) depends only on \( \lambda, \Lambda \).

**Proof.** It suffices to prove (9) with \( r/2 \) replaced by \( \eta r \) and \( r/4 \) by \( \eta r/2 \) with \( \eta \) a small constant depending on \( \lambda, \Lambda \).

Since \( |\nabla u| \leq 1 \), \( u_n \) and \( u^* \) are Lipschitz functions with Lipschitz constant 1. By hypothesis, \( u^* \geq 0 \) on the segment \([x - re_2, x + re_2]\), hence
\[
u^* + 2\varepsilon > 0 \quad \text{on} \quad R = [-\varepsilon, \varepsilon] \times [x - re_2, x + re_2],
\]
and the same inequality holds for $u_n$ for all $n$ large.

Let $S_{t_n}(x)$ be the maximal section of $u_n$ at $x$ which is included in $\mathcal{R}$. From the hypotheses $e_n \to \infty$, $\xi_n \to e_2$ and Lemma 2.3 we see that

$$2r + 2\varepsilon \geq \text{diam } S_{t_n}(x) \geq 2r$$

for all large $n$. From the properties of sections we see that there exist constants $\eta$, $\tau$ such that $S_{t_n}(x)$ contains a segment of length $2\eta$ centered at $x$ and

$$S_{\tau t_n}(x) \subset [-\varepsilon, \varepsilon] \times [x - \frac{\eta}{2} e_2, x + \frac{\eta}{2} e_2].$$

We apply weak Harnack inequality (6) for $u_n + 2\varepsilon$ in $S_{t_n}(x)$ and use that $u_n$ is Lipschitz to obtain

$$\inf_{|t| \leq \frac{2r}{\tau}} u_n(x + te_2) \geq c \inf_{|t| \leq \frac{2r}{\tau}} u_n(x + te_2) - C\varepsilon.$$

The lemma is proved by letting $n \to \infty$ and then $\varepsilon \to 0$. 

**Lemma 3.3.** If $\nu = \pm e_2$ we have

$$u^*(t\nu) = tv \cdot p_\nu$$

either for all $t \geq 0$ or $t \leq 0$.

**Proof.** The proof is essentially identical to the proof of Lemma 3.1. We need to remark that the supersolutions $\tilde{u}_n$ obtained from $u_n$ are uniformly Lipschitz. Hence, as in the proof of Lemma 3.3 above, the weak Harnack inequality for $\tilde{u}_n$ implies the weak Harnack inequality for their limit $(u^*)^+$ in the $e_2$ direction. This gives that $(u^*)^+ > 0$ in $\mathcal{R}_1$ and we reach a contradiction as before.

Now we are ready to reach a contradiction in Case 2.

The previous lemma implies (as in Case 1) that on the line $te_2$ the function $u^*$ is linear on both half lines $t \geq 0$ and $t \leq 0$, but with different slopes. This contradicts that $u^* - l$ satisfies weak Harnack inequality in the $e_2$ direction for an appropriate linear function $l$.

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