A LOCALIZATION PROPERTY AT THE BOUNDARY FOR
MONGE-AMPERE EQUATION

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1. Introduction

In this paper we study the geometry of the sections for solutions to the Monge-Ampere equation
\[ \det D^2 u = f, \quad u : \overline{\Omega} \to \mathbb{R} \] convex,
which are centered at a boundary point \( x_0 \in \partial \Omega \). We show that under natural local assumptions on the boundary data and the domain, the sections
\[ S_h(x_0) = \{ x \in \overline{\Omega} \mid u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h \} \]
are "equivalent" to ellipsoids centered at \( x_0 \), that is, for each \( h > 0 \) there exists an ellipsoid \( E_h \) such that
\[ cE_h \cap \overline{\Omega} \subset S_h(x_0) - x_0 \subset CE_h \cap \overline{\Omega}, \]
with \( c, C \) constants independent of \( h \).

The situation in the interior is well understood. Caffarelli showed in [C1] that if
\[ 0 < \lambda \leq f \leq \Lambda \] in \( \Omega \),
and for some \( x \in \Omega \),
\[ S_h(x) \subset \subset \Omega, \]
then \( S_h(x) \) is equivalent to an ellipsoid centered at \( x \) i.e.
\[ kE \subset S_h(x) - x \subset k^{-1}E \]
for some ellipsoid \( E \) of volume \( h^{n/2} \) and for a constant \( k > 0 \) which depends only on \( \lambda, \Lambda, n \).

This property provides compactness of sections modulo affine transformations. This is particularly useful when dealing with interior \( C^{2,\alpha} \) and \( W^{2,p} \) estimates of strictly convex solutions of
\[ \det D^2 u = f \]
when \( f > 0 \) is continuous (see [C2]).

Sections at the boundary were also considered by Trudinger and Wang in [TW] for solutions of
\[ \det D^2 u = f \]
but under stronger assumptions on the boundary behavior of \( u \) and \( \partial \Omega \), and with \( f \in C^{\alpha}(\overline{\Omega}) \). They proved \( C^{2,\alpha} \) estimates up to the boundary by bounding the mixed derivatives and obtained that the sections are equivalent to balls.

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2. Statement of the main Theorem.

Let Ω be a bounded convex set in \( \mathbb{R}^n \). We assume throughout this note that
\[
(2.1) \quad B_\rho(\rho e_n) \subset \Omega \subset \{ x_n \geq 0 \} \cap B_1,
\]
for some small \( \rho > 0 \), that is \( \Omega \subset (\mathbb{R}^n)^+ \) and \( \Omega \) contains an interior ball tangent to \( \partial \Omega \) at 0.

Let \( u : \Omega \to \mathbb{R} \) be convex, continuous, satisfying
\[
(2.2) \quad \det D^2 u = f, \quad \lambda \leq f \leq \Lambda \text{ in } \Omega.
\]
We extend \( u \) to be \( \infty \) outside \( \Omega \).

By subtracting a linear function we may assume that
\[
(2.3) \quad x_{n+1} = 0 \text{ is the tangent plane to } u \text{ at } 0,
\]
in the sense that
\[
\begin{align*}
\n(x_n) & \geq 0, \quad u(0) = 0, \\
\text{and any hyperplane } x_{n+1} = \epsilon x_n, & \quad \epsilon > 0 \text{ is not a supporting hyperplane for } u.
\end{align*}
\]

In this paper we investigate the geometry of the sections of \( u \) at 0 that we denote for simplicity of notation
\[
S_h := \{ x \in \Omega : u(x) < h \}.
\]

We show that if the boundary data has quadratic growth near \( \{ x_n = 0 \} \) then, as \( h \to 0 \), \( S_h \) is equivalent to a half-ellipsoid centered at 0.

Precisely, our main theorem reads as follows.

**Theorem 2.1.** Assume that \( \Omega \), \( u \) satisfy (2.1)-(2.3) above and for some \( \mu > 0 \),
\[
(2.4) \quad \mu |x|^2 \leq u(x) \leq \mu^{-1} |x|^2 \text{ on } \partial \Omega \cap \{ x_n \leq \rho \}.
\]
Then, for each \( h < c(\rho) \) there exists an ellipsoid \( E_h \) of volume \( h^{n/2} \) such that
\[
kE_h \cap \Omega \subset S_h \subset k^{-1}E_h.
\]
Moreover, the ellipsoid \( E_h \) is obtained from the ball of radius \( h^{1/2} \) by a linear transformation \( A_h^{-1} \) (sliding along the \( x_n = 0 \) plane)
\[
A_h E_h = h^{1/2} B_1
\]
\[
A_h(x) = x - \nu x_n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}, 0),
\]
with
\[
|\nu| \leq k^{-1} |\log h|.
\]
The constant \( k \) above depends on \( \mu, \lambda, \Lambda, n \) and \( c(\rho) \) depends also on \( \rho \).

Theorem 2.1 is new even in the case when \( f = 1 \). The ellipsoid \( E_h \), or equivalently the linear map \( A_h \), provides information about the behavior of the second derivatives near the origin. Heuristically, the theorem states that in \( S_h \) the tangential second derivatives are bounded from above and below and the mixed second derivatives are bounded by \( |\log h| \). This is interesting given that \( f \) is only bounded and the boundary data and \( \partial \Omega \) are only \( C^{1,1} \) at the origin.

**Remark.** Given only the boundary data \( \varphi \) of \( u \) on \( \partial \Omega \), it is not always easy to check condition (2.4). Here we provide some examples when (2.4) is satisfied:

1) If \( \varphi \) is constant and the domain \( \Omega \) is included in a ball included in \( \{ x_n \geq 0 \} \).
2) If the domain $\partial \Omega$ is tangent of order 2 to \( \{x_n = 0\} \) and the boundary data $\varphi$ has quadratic behavior in a neighborhood of 0.

3) $\varphi, \partial \Omega \in C^3$ at the origin, and $\Omega$ is uniformly convex at the origin.

We obtain compactness of sections modulo affine transformations.

**Corollary 2.2.** Under the assumptions of Theorem 2.1, assume that
\[
\lim_{x \to 0} f(x) = f(0)
\]
and
\[
u(x) = P(x) + o(|x|^2) \quad \text{on} \; \partial \Omega
\]
with $P$ a quadratic polynomial. Then we can find a sequence of rescalings
\[
\tilde{u}_h(x) := \frac{1}{h} u(h^{1/2} A_h^{-1} x)
\]
which converges to a limiting continuous solution $\tilde{u}_0 : \overline{\Omega}_0 \to \mathbb{R}$ with
\[
k B_1^+ \subset \Omega_0 \subset k^{-1} B_1^+
\]
such that
\[
\det D^2 \tilde{u}_0 = f(0)
\]
and
\[
\tilde{u}_0 = P \quad \text{on} \; \overline{\Omega}_0 \cap \{x_n = 0\},
\]
\[
\tilde{u}_0 = 1 \quad \text{on} \; \partial \overline{\Omega}_0 \cap \{x_n > 0\}.
\]

In a future work we intend to use the results above and obtain $C^{2, \alpha}$ and $W^{2,p}$ boundary estimates under appropriate conditions on the domain and boundary data.

3. Preliminaries

Next proposition was proved by Trudinger and Wang in [TW]. Since our setting is slightly different we provide its proof.

**Proposition 3.1.** Under the assumptions of Theorem 2.1, for all $h \leq c(\rho)$, there exists a linear transformation (sliding along $x_n = 0$)
\[
A_h(x) = x - \nu x_n,
\]
with
\[
\nu_n = 0, \quad |\nu| \leq C(\rho) h^{- \frac{n}{n+1}}
\]
such that the rescaled function
\[
\tilde{u}(A_h x) = u(x),
\]
satisfies in
\[
\tilde{S}_h := A_h S_h = \{\tilde{u} < h\}
\]
the following:
(i) the center of mass of $\tilde{S}_h$ lies on the $x_n$-axis;
(ii) $k_0 h^{n/2} \leq |\tilde{S}_h| = |S_h| \leq k_0^{-1} h^{n/2};$
(iii) the part of $\partial \tilde{S}_h$ where $\{\tilde{u} < h\}$ is a graph, denoted by

$$\tilde{G}_h = \partial \tilde{S}_h \cap \{\tilde{u} < h\} = \{(x', g_h(x'))\}$$

that satisfies

$$g_h \leq C(\rho)|x'|^2$$

and

$$\frac{\mu}{2} |x'|^2 \leq \tilde{u} \leq 2\mu^{-1}|x'|^2 \quad \text{on } \tilde{G}_h.$$

The constant $k_0$ above depends on $\mu, \lambda, \Lambda$ and the constants $C(\rho), c(\rho)$ depend also on $\rho$.

In this section we denote by $c, C$ positive constants that depend on $n, \mu, \lambda, \Lambda$. For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion. Constants that depend also on $\rho$ are denote by $c(\rho), C(\rho)$.

Proof. The function

$$v := \mu|x'|^2 + \frac{\Lambda}{\mu^{n-1}} x_n^2 - C(\rho)x_n$$

is a lower barrier for $u$ in $\Omega \cap \{x_n \leq \rho\}$ if $C(\rho)$ is chosen large.

Indeed, then

$$v \leq u \quad \text{on } \partial \Omega \cap \{x_n \leq \rho\},$$

$$v \leq 0 \leq u \quad \text{on } \Omega \cap \{x_n = \rho\},$$

and

$$\det D^2v > \Lambda.$$

In conclusion,

$$v \leq u \quad \text{in } \Omega \cap \{x_n \leq \rho\},$$

hence

$$S_h \cap \{x_n \leq \rho\} \subset \{v < h\} \subset \{x_n > c(\rho)(\mu|x'|^2 - h)\}.$$  \hfill (3.1)

Let $x_h^*$ be the center of mass of $S_h$. We claim that

$$x_h^* \cdot e_n \geq c_0(\rho)h^\alpha, \quad \alpha = \frac{n}{n+1},$$

for some small $c_0(\rho) > 0$.

Otherwise, from (3.1) and John’s lemma we obtain

$$S_h \subset \{x_n \leq C(n)c_0h^\alpha \leq h^\alpha\} \cap \{|x'| \leq C_1h^{\alpha/2}\},$$

for some large $C_1 = C_1(\rho)$. Then the function

$$w = \epsilon x_n + \frac{h}{2} \left( \frac{|x'|}{C_1h^{\alpha/2}} \right)^2 + \Lambda C_1^{2(n-1)}h \left( \frac{x_n}{h^\alpha} \right)^2$$

is a lower barrier for $u$ in $S_h$ if $c_0$ is sufficiently small.

Indeed,

$$w \leq \frac{h}{4} + \frac{h}{2} + \Lambda C_1^{2(n-1)}(C(n)c_0)^2h < h \quad \text{in } S_h,$$

and for all small $h$,

$$w \leq \epsilon x_n + \frac{h^{1-\alpha}}{C_1^2} |x'|^2 + C(\rho)hc_0 \frac{x_n}{h^\alpha} \leq \mu|x'|^2 \leq u \quad \text{on } \partial \Omega,$$
and

$$\det D^2w = 2\Lambda.$$  

Hence

$$w \leq u \text{ in } S_h,$$

and we contradict that 0 is the tangent plane at 0. Thus claim (3.2) is proved.

Now, define

$$A_hx = x - \nu x_n, \quad \nu = x_h^* \cdot e_n,$$

and

$$\hat{u}(A_hx) = u(x).$$

The center of mass of $\tilde{S}_h = A_hS_h$ is

$$\hat{x}_h^* = A_hx_h^*$$

and lies on the $x_n$-axis from the definition of $A_h$. Moreover, since $x_h^* \in S_h$, we see from (3.1)-(3.2) that

$$|\nu| \leq C(\rho)\frac{(x_h^* \cdot e_n)^{1/2}}{(x_h^* \cdot e_n)} \leq C(\rho)h^{-\alpha/2},$$

and this proves (i).

If we restrict the map $A_h$ on the set on $\partial \Omega$ where $\{u < h\}$, i.e. on

$$\partial S_h \cap \partial \Omega \subset \{x_n \leq \frac{|x'|^2}{\rho}\} \cap \{|x'| < Ch^{1/2}\}$$

we have

$$|A_hx - x| = |\nu|x_n \leq C(\rho)h^{-\alpha/2}|x'|^2 \leq C(\rho)h^{1-\alpha} |x'|,$$

and part (iii) easily follows.

Next we prove (ii). From John’s lemma, we know that after relabeling the $x'$ coordinates if necessary,

$$D_hB_1 \subset \tilde{S}_h - \hat{x}_h^* \subset C(n)D_hB_1$$

where

$$D_h = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$  

Since

$$\tilde{u} \leq 2\mu \mu^{-1}|x'|^2 \quad \text{on } \tilde{G}_h = \{(x', g_h(x'))\},$$

we see that the domain of definition of $g_h$ contains a ball of radius $(\mu h/2)^{1/2}$. This implies that

$$d_i \geq c_1 h^{1/2}, \quad i = 1, \cdots, n - 1,$$

for some $c_1$ depending only on $n$ and $\mu$. Also from (3.2) we see that

$$\hat{x}_h^* \cdot e_n = x_h^* \cdot e_n \geq c_0(\rho)h^\alpha$$

which gives

$$d_n \geq c(n)\hat{x}_h^* \cdot e_n \geq c(\rho)h^\alpha.$$
We claim that for all small $h$, 
\[ \prod_{i=1}^{n} d_i \geq k_0 h^{n/2}, \]
with $k_0$ small depending only on $\mu, n, \Lambda$, which gives the left inequality in (ii).

To this aim we consider the barrier, 
\[ w = \epsilon x_n + \sum_{i=1}^{n} c h \left( \frac{x_i}{d_i} \right)^2. \]
We choose $c$ sufficiently small depending on $\mu, n, \Lambda$ so that for all $h < c(\rho)$, 
\[ w \leq h \quad \text{on} \quad \partial \tilde{S}_h, \]
and on the part of the boundary $\tilde{G}_h$, we have $w \leq \tilde{u}$ since 
\[ w \leq \epsilon x_n + \frac{c}{c_1^4} |x'|^2 + ch \left( \frac{x_n}{d_n} \right)^2 \]
\[ \leq \frac{H}{4} |x'|^2 + chC(n) \frac{x_n}{d_n} \]
\[ \leq \frac{H}{4} |x'|^2 + ch^{1-n} C(\rho) |x'|^2 \]
\[ \leq \frac{H}{2} |x'|^2. \]
Moreover, if our claim does not hold, then 
\[ \det D^2 w = (2ch)^n (\prod d_i)^{-2n} > \Lambda, \]
thus $w \leq \tilde{u}$ in $\tilde{S}_h$. By definition, $\tilde{u}$ is obtained from $u$ by a sliding along $x_n = 0$, hence $0$ is still the tangent plane of $\tilde{u}$ at $0$. We reach again a contradiction since $\tilde{u} \geq w \geq \epsilon x_n$ and the claim is proved.

Finally we show that 
\[ |\tilde{S}_h| \leq C h^{n/2} \]
for some $C$ depending only on $\lambda, n$. Indeed, if 
\[ v = h \quad \text{on} \quad \partial \tilde{S}_h, \]
and 
\[ \det D^2 v = \lambda \]
ten 
\[ v \geq u \geq 0 \quad \text{in} \quad \tilde{S}_h. \]
Since 
\[ h \geq h - \min_{\tilde{S}_h} v \geq c(n, \lambda) |\tilde{S}_h|^{2/n} \]
we obtain the desired conclusion.

In the proof above we showed that for all $h \leq c(\rho)$, the entries of the diagonal matrix $D_h$ from (3.3) satisfy 
\[ d_i \geq ch^{1/2}, \quad i = 1, \ldots, n - 1 \]
\[ d_n \geq c(\rho) h^\alpha, \quad \alpha = \frac{n}{n+1} \]

\[ ch^{n/2} \leq \prod d_i \leq Ch^{n/2}. \]

The main step in the proof of Theorem 2.1 is the following lemma that will be proved in the remaining sections.

**Lemma 3.2.** There exist constants \( c, c(\rho) \) such that

\[ d_n \geq ch^{1/2}, \]

for all \( h \leq c(\rho) \).

Using Lemma 3.2 we can easily finish the proof of our theorem.

**Proof of Theorem 2.1.** Since all \( d_i \) are bounded below by \( ch^{1/2} \) and their product is bounded above by \( Ch^{n/2} \) we see that

\[ Ch^{1/2} \geq d_i \geq ch^{1/2} \quad i = 1, \ldots, n \]

for all \( h \leq c(\rho) \). Using (3.3) we obtain

\[ \tilde{S}_h \subset Ch^{1/2}B_1. \]

Moreover, since

\[ \tilde{x}_h^* \cdot e_n \geq d_n \geq ch^{1/2}, \quad (\tilde{x}_h^*)' = 0, \]

and the part \( \tilde{G}_h \) of the boundary \( \partial \tilde{S}_h \) contains the graph of \( \tilde{g}_h \) above \( |x'| \leq ch^{1/2} \), we find that

\[ ch^{1/2}B_1 \cap \tilde{\Omega} \subset \tilde{S}_h, \]

with \( \tilde{\Omega} = A_h\Omega, \tilde{S}_h = A_hS_h \). In conclusion

\[ ch^{1/2}B_1 \cap \tilde{\Omega} \subset A_hS_h \subset Ch^{1/2}B_1. \]

We define the ellipsoid \( E_h \) as

\[ E_h := A_h^{-1}(h^{1/2}B_1), \]

hence

\[ cE_h \cap \tilde{\Omega} \subset S_h \subset CE_h. \]

Comparing the sections at levels \( h \) and \( h/2 \) we find

\[ cE_{h/2} \cap \tilde{\Omega} \subset CE_h \]

and we easily obtain the inclusion

\[ A_hA_{h/2}^{-1}B_1 \subset CB_1. \]

If we denote

\[ A_hx = x - \nu_h x_n \]

then the inclusion above implies

\[ |\nu_h - \nu_{h/2}| \leq C, \]

which gives the desired bound

\[ |\nu_h| \leq C|\log h| \]

for all small \( h \).
We introduce a new quantity $b(h)$ which is proportional to $d_n h^{-1/2}$ and which is appropriate when dealing with affine transformations.

**Notation.** Given a convex function $u$ we define

$$b_u(h) = h^{-1/2} \sup_{S_h} x_n.$$  

Whenever there is no possibility of confusion we drop the subindex $u$ and use the notation $b(h)$.

Below we list some basic properties of $b(h)$.

1) If $h_1 \leq h_2$ then

$$\left( \frac{h_1}{h_2} \right)^{1/2} \leq \frac{b(h_1)}{b(h_2)} \leq \left( \frac{h_2}{h_1} \right)^{1/2}.$$  

2) A rescaling

$$\tilde{u}(Ax) = u(X)$$

given by a linear transformation $A$ which leaves the $x_n$ coordinate invariant does not change the value of $b$, i.e

$$b_{\tilde{u}}(h) = b_u(h).$$  

3) If $A$ is a linear transformation which leaves the plane $\{x_n = 0\}$ invariant the values of $b$ get multiplied by a constant. However the quotients $b(h_1)/b(h_2)$ do not change values i.e

$$\frac{b_{\tilde{u}}(h_1)}{b_{\tilde{u}}(h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.$$  

4) If we multiply $u$ by a constant, i.e.

$$\tilde{u}(x) = \beta u(x)$$

then

$$b_{\tilde{u}}(\beta h) = \beta^{-1/2} b_u(h),$$

and

$$\frac{b_{\tilde{u}}(\beta h_1)}{b_{\tilde{u}}(\beta h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.$$  

From (3.3) and property 2 above,

$$c(n) d_n \leq b(h) h^{1/2} \leq C(n) d_n,$$

hence Lemma 3.2 will follow if we show that $b(h)$ is bounded below. We achieve this by proving the following lemma.

**Lemma 3.3.** There exist $c_0$, $c(\rho)$ such that if $h \leq c(\rho)$ and $b(h) \leq c_0$ then

$$\frac{b(th)}{b(h)} > 2,$$

for some $t \in [c_0, 1]$.  

This lemma states that if the value of \( b(h) \) on a certain section is less than a critical value \( c_0 \), then we can find a lower section at height still comparable to \( h \) where the value of \( b \) doubled. Clearly Lemma 3.3 and property 1 above imply that \( b(h) \) remains bounded for all \( h \) small enough.

The quotient in (3.5) is the same for \( \hat{u} \) which is defined in Proposition 3.1. We normalize the domain \( \hat{S}_h \) and \( \hat{u} \) by considering the rescaling

\[
v(x) = \frac{1}{h} \hat{u}(h^{1/2}Ax)
\]

where \( A \) is a multiple of \( D_h \) (see (3.3)), \( A = \gamma D_h \) such that

\[
\det A = 1.
\]

Then

\[
ch^{-1/2} \leq \gamma \leq Ch^{-1/2},
\]

and the diagonal entries of \( A \) satisfy

\[
a_i \geq c, \quad i = 1, 2, \ldots, n - 1,
\]

\[
cb_n(h) \leq a_n \leq Cb_n(h).
\]

The function \( v \) satisfies

\[
\lambda \leq \det D^2v \leq \Lambda,
\]

\[
v \geq 0, \quad v(0) = 0,
\]

is continuous and it is defined in \( \Omega_v \) with

\[
\Omega_v := \{ v < 1 \} = h^{-1/2}A^{-1}\hat{S}_h.
\]

Then

\[
x^* + cB_1 \subset \Omega_v \subset CB_1^+,
\]

for some \( x^* \), and

\[
ct^{n/2} \leq |S_t(v)| \leq C t^{n/2}, \quad \forall t \leq 1,
\]

where \( S_t(v) \) denotes the section of \( v \). Since

\[
\hat{u} = h \quad \text{in} \quad \partial\hat{S}_h \cap \{ x_n \geq C(\rho)h \},
\]

then

\[
v = 1 \quad \text{on} \quad \partial\Omega_v \cap \{ x_n \geq \sigma \}, \quad \sigma := C(\rho)h^{1-\alpha}.
\]

Also, from Proposition 3.1 on the part \( G \) of the boundary of \( \partial\Omega_v \) where \( \{ v < 1 \} \) we have

\[
\frac{1}{2^\mu} \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq v \leq 2\mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2.
\]

In order to prove Lemma 3.3 we need to show that if \( \sigma, a_n \) are sufficiently small depending on \( n, \mu, \lambda, \Lambda \) then the function \( v \) above satisfies

\[
b_v(t) \geq 2b_v(1)
\]

for some \( 1 > t \geq c_0 \).

Since \( \alpha < 1 \), the smallness condition on \( \sigma \) is satisfied by taking \( h < c(\rho) \) sufficiently small. Also \( a_n \) being small is equivalent to one of the \( a_i, 1 \leq i \leq n - 1 \) being large since their product is 1 and \( a_i \) are bounded below.
In the next sections we prove property (3.7) above by compactness, by letting \( \sigma \to 0 \), \( a_i \to \infty \) for some \( i \). First we consider the 2D case and in the last section the general case.

## 4. THE 2 DIMENSIONAL CASE.

In order to fix ideas, we consider first the 2 dimensional case.

We study the following class of solutions to the Monge-Ampere equation. Fix \( \mu > 0 \) small, \( \lambda, \Lambda \). We denote by \( D_\sigma \) the set of convex, continuous functions

\[
u : \overline{\Omega} \to \mathbb{R}
\]
such that

\[
\begin{align*}
\lambda \leq & \det D^2 \nu \leq \Lambda; \\
0 \in & \partial \Omega, \quad B_\mu(x_0) \subset \Omega \subset B_{1/\mu}^+ \quad \text{for some } x_0; \\
\mu h^{n/2} \leq & |S_h| \leq \mu^{-1} h^{n/2}; \\
u = & 1 \quad \text{on } \partial \Omega \setminus G, \quad 0 \leq u \leq 1 \quad \text{on } G, \quad u(0) = 0,
\end{align*}
\]

with \( G \) a closed subset of \( \partial \Omega \) included in \( B_\sigma \).

### Proposition 4.1.

Assume \( n = 2 \). For any \( M > 0 \) there exists \( c_0 \) small depending on \( M, \mu, \lambda, \Lambda \), such that if \( u \in D_\sigma \) and \( \sigma \leq c_0 \), then

\[
b(h) := (\sup_{S_h} x) h^{-1/2} > M
\]

for some \( h \geq c_0 \).

Property (3.7) easily follows from the proposition above. Indeed, by choosing

\[
M = 2 \mu^{-1} > 2b(1)
\]

we prove the existence of a section \( h \geq c_0 \) such that

\[
b(h) \geq 2b(1).
\]

Also, the function \( v \) of the previous section satisfies \( v \in D_{c_0} \) (after renaming the constant \( \mu \)) provided that \( \sigma \) is sufficiently small and \( a_1 \) sufficiently large.

We prove Proposition 4.1 by compactness. First we discuss briefly the compactness of bounded solutions to the Monge-Ampere equation. For this we need to introduce solutions with possibly discontinuous boundary data.

Let \( u : \Omega \to \mathbb{R} \) be a convex function with \( \Omega \subset \mathbb{R}^n \) bounded and convex. We denote by

\[
\Gamma_u := \{(x,x_{n+1}) \in \Omega \times \mathbb{R} | \quad x_{n+1} \geq u(x)\}
\]

the upper graph of \( u \).

### Definition 4.2.

We define the values of \( u \) on \( \partial \Omega \) to be equal to \( \varphi \) i.e

\[
u_{\partial \Omega} = \varphi,
\]

if the upper graph of \( \varphi : \partial \Omega \to \mathbb{R} \cup \{\infty\} \)

\[
\Phi := \{(x,x_{n+1}) \in \partial \Omega \times \mathbb{R}| \quad x_{n+1} \geq \varphi(x)\}
\]
is given by the closure of $\Gamma_u$ restricted to $\partial \Omega \times \mathbb{R}$,

$$\Phi := \Gamma_u \cap (\partial \Omega \times \mathbb{R}).$$

From the definition we see that $\varphi$ is always lower semicontinuous. The following comparison principle holds: if $w: \Omega \to \mathbb{R}$ is continuous and

$$\text{det } D^2 w \geq \Lambda \geq \text{det } D^2 u, \quad w|_{\partial \Omega} \leq u|_{\partial \Omega},$$

then

$$w \leq u \quad \text{in } \Omega.$$  

Indeed, from the continuity of $w$ we see that for any $\varepsilon > 0$, there exists a small neighborhood of $\partial \Omega$ where $w - \varepsilon < u$. This inequality holds in the interior from the standard comparison principle, hence $w \leq u$ in $\Omega$.

Since the convex functions are defined on different domains we use the following notion of convergence.

**Definition 4.3.** We say that the convex functions $u_m: \Omega_m \to \mathbb{R}$ converge to $u: \Omega \to \mathbb{R}$ if the upper graphs converge

$$\Gamma_{u_m} \to \Gamma_u \quad \text{in the Hausdorff distance.}$$

Similarly, we say that the lower semicontinuous functions $\varphi_m: \partial \Omega_m \to \mathbb{R}$ converge to $\varphi: \partial \Omega \to \mathbb{R}$ if the upper graphs converge

$$\Phi_m \to \Phi \quad \text{in the Hausdorff distance.}$$

Clearly if $u_m$ converges to $u$, then $u_m$ converges uniformly to $u$ in any compact set of $\Omega$, and $\Omega_m \to \Omega$ in the Hausdorff distance.

**Remark:** When we restrict the Hausdorff distance to the nonempty closed sets of a compact set we obtain a compact metric space. Thus, if $\Omega_m, u_m$ are uniformly bounded then we can always extract a subsequence $m_k$ such that $u_{m_k} \to u$ and $u_{m_k}|_{\partial \Omega_m} \to \varphi$.

Next lemma gives the relation between the boundary data of the limit $u$ and $\varphi$.

**Lemma 4.4.** Let $u_m: \Omega_m \to \mathbb{R}$ be convex functions, uniformly bounded, such that

$$\lambda \leq \text{det } D^2 u_m \leq \Lambda$$

and

$$u_m \to u, \quad u_m|_{\partial \Omega_m} \to \varphi.$$  

Then

$$\lambda \leq \text{det } D^2 u \leq \Lambda,$$

and the boundary data of $u$ is given by $\varphi^*$ the convex envelope of $\varphi$ on $\partial \Omega$.

**Proof.** Clearly $\Phi \subset \Gamma_u$, hence $\Phi^* \subset \Gamma_u$. It remains to show that the convex set $K$ generated by $\Phi$ contains $\Gamma_u \cap (\partial \Omega \times \mathbb{R})$.

Indeed consider a hyperplane

$$x_{n+1} = l(x)$$

which lies strictly below $K$. Then, for all large $m$

$$\{u_m - l \leq 0\} \subset \Omega_m,$$

and by Alexandrov estimate we have that

$$u_m - l \geq -C\delta_m^{1/n}$$
where \( d_m(x) \) represents the distance from \( x \) to \( \partial \Omega_m \). By taking \( m \to \infty \) we see that
\[
 u - l \geq -Cd_{1/n}
\]
thus no point on \( \partial \Omega \) below \( l \) belongs to \( \Gamma_u \).

In view of the lemma above we introduce the following notation.

**Definition 4.5.** Let \( \varphi : \partial \Omega \to \mathbb{R} \) be a lower semicontinuous function. When we write that a convex function \( u \) satisfies
\[
 u = \varphi \quad \text{on} \quad \partial \Omega
\]
we understand
\[
 u|_{\partial \Omega} = \varphi^*
\]
where \( \varphi^* \) is the convex envelope of \( \varphi \) on \( \partial \Omega \).

Whenever \( \varphi^* \) and \( \varphi \) do not coincide we can think of the graph of \( u \) as having a vertical part on \( \partial \Omega \) between \( \varphi^* \) and \( \varphi \).

It follows easily from the definition above that the boundary values of \( u \) when we restrict to the domain
\[
 \Omega_h := \{ u < h \}
\]
are given by
\[
 \varphi_h = \varphi \quad \text{on} \quad \partial \Omega \cap \{ \varphi \leq h \} \subset \partial \Omega_h
\]
and \( \varphi_h = h \) on the remaining part of \( \partial \Omega_h \).

The comparison principle still holds. Precisely, if \( w : \overline{\Omega} \to \mathbb{R} \) is continuous and
\[
 \det D^2 w \geq \Lambda \geq \det D^2 u, \quad w|_{\partial \Omega} \leq \varphi,
\]
then
\[
 w \leq u \quad \text{in} \quad \Omega.
\]

The advantage of introducing the notation of Definition 4.5 is that the boundary data is preserved under limits.

**Proposition 4.6 (Compactness).** Assume
\[
 \lambda \leq \det D^2 u_m \leq \Lambda, \quad u_m = \varphi_m \quad \text{on} \quad \partial \Omega_m,
\]
and \( \Omega_m, \varphi_m \) uniformly bounded.

Then there exists a subsequence \( m_k \) such that
\[
 u_{m_k} \to u, \quad \varphi_{m_k} \to \varphi
\]
with
\[
 \lambda \leq \det D^2 u \leq \Lambda, \quad u = \varphi \quad \text{on} \quad \partial \Omega.
\]

Indeed, we see that we can also choose \( m_k \) such that \( \varphi_{m_k}^* \to \psi \). Since \( \varphi_{m_k} \to \varphi \) we obtain
\[
 \varphi \geq \psi \geq \varphi^*,
\]
and the conclusion follows from Lemma 4.4.

Now we are ready to prove Proposition 4.1.

**Proof of Proposition 4.1.** If \( \epsilon_0 \) does not exist we can find a sequence of functions \( u_m \in D_{1/m} \) such that
\[
 b_{u_m}(h) \leq M, \quad \forall h \geq \frac{1}{m}.
\]
By Proposition 4.6 there is a subsequence which converges to a limiting function $u$ satisfying (4.1)-(4.2)-(4.3) and (see Definition 4.5) $u = \varphi$ on $\partial \Omega$ with

\begin{equation}
\varphi = 1 \quad \text{on } \partial \Omega \setminus \{0\}, \quad \varphi(0) = 0,
\end{equation}

and moreover $u$ has an obstacle by below in $\Omega$

\begin{equation}
u \geq \frac{1}{M^2} x_2^2.
\end{equation}

We consider the barrier

\[ w := \delta(|x_1| + \frac{1}{2} x_1^2) + \frac{\Lambda}{\delta} x_2^2 - N x_2 \]

with $\delta$ small depending on $\mu$, and $N$ large so that

\[ \frac{\Lambda}{\delta} x_2^2 - N x_2 \leq 0 \quad \text{in } B_1^+. \]

Then

\[ w \leq \varphi \quad \text{on } \partial \Omega, \]

and

\[ \det D^2 w > \Lambda. \]

Hence

\[ w \leq u \quad \text{in } \Omega \]

which gives

\[ u \geq \delta |x_1| - N x_2. \]

Next we construct another explicit subsolution $v$ such that whenever $v$ is above the two obstacles

\[ \delta |x_1| - N x_2, \quad \frac{1}{M^2} x_2^2, \]

we have

\[ \det D^2 v > \Lambda \quad \text{and} \quad v \leq 1. \]

Then we can conclude that

\[ u \geq v, \]

and we show that this contradicts the lower bound on $|S_h|$. We look for a function of the form

\[ v := r \phi(\theta) + \frac{1}{2M^2} x_2^2, \]

where $r, \theta$ represent the polar coordinates in the $x_1, x_2$ plane.

The domain of definition of $v$ is the angle

\[ K := \{ \theta_0 \leq \theta \leq \pi - \theta_0 \} \]

with $\theta_0$ small so that

\[ \frac{1}{2M^2} x_2^2 \leq \frac{1}{2} (\delta |x_1| - N x_2) \quad \text{on } \partial K \cap B_\mu. \]

In the set

\[ \{ v \geq \frac{1}{M^2} x_2^2 \} \]

i.e. where

\[ \frac{1}{r} \geq \frac{\sin^2 \theta}{2M^2 f} \]
we have
\begin{equation}
\det D^2 v = \frac{1}{f''(f'' + f)} \frac{\sin^2 \theta}{M^2} \geq \frac{1}{f(f'' + f)} \frac{\sin^4 \theta}{2M^4}.
\end{equation}

We let
\[ f(\theta) = \sigma e^{C_0 |\frac{\pi}{2} - \theta|}, \]
where $C_0$ is large depending on $\theta_0, M, \Lambda$ so that (see (4.7))
\[ \det D^2 v > \Lambda \]
in the set where
\[ \{ v \geq \frac{1}{M^2 x_2^2} \}. \]

On the other hand we can choose $\sigma$ small so that
\[ v \leq \delta |x_1| - N x_2 \quad \text{on } \partial K \cap B_{\mu} \]
and
\[ v \leq 1 \quad \text{on the set } \{ v \geq \frac{1}{M^2 x_2^2} \}. \]

In conclusion
\[ u \geq v \geq \epsilon x_2, \]
hence
\[ u \geq \max \{ \epsilon x_2, \delta |x_1| - N x_2 \}. \]
This implies
\[ |S_h| \leq C h^2 \]
for all small $h$ and we contradict that
\[ |S_h| \geq \mu h, \quad \forall h \in [0, 1]. \]

\[ \square \]

5. The higher dimensional case

In higher dimensions it is more difficult to construct an explicit barrier as in Proposition 4.1 in the case when in (3.6) only one $a_i$ is large and the others are bounded. We prove our result by induction depending on the number of large eigenvalues $a_i$.

Fix $\mu$ small and $\lambda, \Lambda$. For each increasing sequence
\[ \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1} \]
with
\[ \alpha_1 \geq \mu, \]
we consider the family of solutions
\[ D^\mu_\alpha(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \]
of convex, continuous functions $u : \overline{\Omega} \to \mathbb{R}$ that satisfy
\begin{align*}
\lambda &\leq \det D^2 u \leq \Lambda \quad \text{in } \Omega, \quad u \geq 0 \text{ in } \overline{\Omega}; \\
0 &\in \partial \Omega, \quad B_{\mu}(x_0) \subset \Omega \subset B_{1/\mu}^+ \quad \text{for some } x_0; \\
\mu h^{n/2} &\leq |S_h| \leq \mu^{-1} h^{n/2}; \\
u &= 1 \quad \text{on } \partial \Omega \setminus G.
\end{align*}
and
\[
\mu \sum_{i=1}^{n-1} \alpha_i^2 x_i^2 \leq u \leq \mu^{-1} \sum_{i=1}^{n-1} \alpha_i^2 x_i^2 \quad \text{on } G,
\]
where \(G\) is a closed subset of \(\partial \Omega\) which is a graph in the \(e_n\) direction and is included in boundary in \(\{x_n \leq \sigma\}\).

For convenience we would like to add the limiting solutions when \(\alpha_{k+1} \to \infty\) and \(\sigma \to 0\). We denote by
\[
D_{\mu}^\sigma(\alpha_1, \ldots, \alpha_k, \infty, \infty, \ldots, \infty)
\]
the class of functions \(u: \Omega \to \mathbb{R}\) that satisfy properties (5.1)-(5.2)-(5.3) and (see Definition 4.5) \(u = \varphi\) on \(\partial \Omega\) with
\[
\varphi = 1 \quad \text{on } \partial \Omega \setminus G;
\]

\[
\mu \sum_{i=1}^{k} \alpha_i^2 x_i^2 \leq \varphi \leq \min\{1, , \mu^{-1} \sum_{i=1}^{k} \alpha_i^2 x_i^2\} \quad \text{on } G,
\]
where \(G\) is a closed set
\(G \subset \partial \Omega \cap \{x_i = 0, \ i > k\}\),

and if we restrict to the space generated by the first \(k\) coordinates then
\[
\{ \mu^{-1} \sum_{i=1}^{k} \alpha_i^2 x_i^2 \leq 1 \} \subset G \subset \{ \mu \sum_{i=1}^{k} \alpha_i^2 x_i^2 \leq 1 \}.
\]

We extend the definition of \(D_{\mu}^\sigma(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})\) to include also the pairs with
\[
\mu \leq \alpha_1 \leq \ldots \leq \alpha_k < \infty, \quad \alpha_{k+1} = \cdots = \alpha_{n-1} = \infty
\]
for which \(\sigma = 0\) i.e. \(D_{\mu}^\sigma(\alpha_1, \alpha_2, \ldots, \alpha_k, \infty, \ldots, \infty)\).

Proposition 4.6 implies that if \(u_m \in D_{\sigma_m}^\mu(\alpha_1, \ldots, \alpha_{n-1})\) is a sequence with
\[
\sigma_m \to 0 \quad \text{and} \quad a_{k+1}^m \to \infty
\]
for some fixed \(0 \leq k \leq n - 2\), then we can extract a convergent subsequence to a function \(u\) with
\[
u \in D_{\sigma}^\mu(\alpha_1, \ldots, \alpha_{n-1})
\]
for some \(l \leq k\) and \(a_l \leq \ldots \leq a_{k+1}\).

**Proposition 5.1.** For any \(M > 0\) and \(1 \leq k \leq n - 1\) there exists \(C_k\) depending on \(M, \mu, \lambda, \Lambda, n, k\) such that if \(u \in D_{\sigma}^\mu(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})\) with
\[
\alpha_k \geq C_k, \quad \sigma \leq C_k^{-1}
\]
then
\[
b(h) = (\sup_{S_h} x_n) h^{-1/2} \geq M
\]
for some \(h\) with \(C_k^{-1} \leq h \leq 1\).

As we remarked in the previous section, property (3.7) and therefore Lemma 3.3 follow from Proposition 5.1 by taking \(k = n - 1\) and \(M = 2\mu^{-1}\).

We prove the proposition by induction on \(k\).
Lemma 5.2. Proposition 5.1 holds for $k = 1$.

Proof. By compactness we need to show that there does not exist $u \in \mathcal{D}_0^\mu(\infty, \ldots, \infty)$ with $b(h) \leq M$ for all $h$.

The proof is almost identical to the 2 dimensional case. One can see as before that

$$u \geq \max\{\delta|x'| - Nx_n, \frac{1}{M^2}x_n^2\}$$

and then construct a barrier of the form

$$v = rf(\theta) + \frac{1}{2M^2}x_n^2, \quad \theta_0 \leq \theta \leq \frac{\pi}{2}$$

where $r = |x|$ and $\theta$ represents the angle in $[0, \pi/2]$ between the ray passing through $x$ and the $\{x_n = 0\}$ plane.

Now,

$$\det D^2v = \frac{f'' + f}{r} \left( \frac{f \cos \theta - f' \sin \theta}{r \cos \theta} \right)^{n-2} \sin^2 \theta \frac{1}{M^2}.$$ 

We have

$$\frac{f}{r} > \frac{\sin^2 \theta}{2M^2}$$

on the set $\{v > \frac{1}{M^2}x_n^2\}$

and we choose a function of the form

$$f(\theta) := \nu e^{C_0(\frac{\pi}{2} - \theta)}$$

which is decreasing in $\theta$.

Then

$$\det D^2v > \frac{f'' + f}{f} \left( \frac{\sin^2 \theta_0}{2M^2} \right)^{n-1} > \Lambda$$

if $C_0$ is chosen large.

We obtain as before that

$$u \geq \max\{\delta|x'| - Nx_n, \epsilon x_n\}$$

which gives

$$|S_h| \leq Ch^n$$

and we reach a contradiction. □

Now we prove Proposition 5.1 by induction on $k$.

Proof of Proposition 5.1. In this proof we denote by $c$, $C$ positive constants that depend on $M, \mu, \lambda, \Lambda, n$ and $k$.

We assume that the statement holds for $k$ and we prove it for $k + 1$.

It suffices to show the existence of $C_{k+1}$ only in the case when $\alpha_k < C_k$, otherwise we use the induction hypothesis.

If no $C_{k+1}$ exists then we can find a limiting solution

$$u \in \mathcal{D}_0^\mu(1, 1, \ldots, 1, \infty, \ldots, \infty)$$

with

$$(5.8) \quad b(h) < Mh^{1/2}, \quad \forall h > 0$$

where $\mu$ depends on $\mu$ and $C_k$.

We show that such a function $u$ does not exist.
Denote 
\[ x = (y, z, x_n), \quad y = (x_1, \ldots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \ldots, x_{n-1}) \in \mathbb{R}^{n-1-k}. \]

On the \( \partial \Omega \) plane we have 
\[ \varphi \geq w := \delta |x'|^2 + \delta |z| + \frac{\Lambda}{\delta^{n-1}} x_n^2 - N x_n \]
for some small \( \delta \) depending on \( \tilde{\mu} \), and \( N \) large so that 
\[ \frac{\Lambda}{\delta^{n-1}} x_n^2 - N x_n \leq 0 \text{ on } B^+_{1/\tilde{\mu}}. \]

Since 
\[ \det D^2 w > \Lambda, \]
we obtain \( u \geq w \) on \( \Omega \) hence 
(5.9) 
\[ u(x) \geq \delta |z| - N x_n. \]

We look at the section \( S_h \) of \( u \). From (5.8)-(5.9) we see that 
(5.10) 
\[ S_h \subset \{ x_n > \frac{1}{N}(\delta |z| - h) \} \cap \{ x_n \leq Mh^{1/2} \}. \]

We notice that an affine transformation \( x \to Tx \), 
\[ Tx := x + \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_{n-k-1} z_{n-k-1} + \nu_{n-k} x_n \]
with 
\[ \nu_1, \nu_2, \ldots, \nu_{n-k} \in \text{span}\{e_1, \ldots, e_k\} \]

i.e a sliding along the \( y \) direction, leaves the \( z, x_n \) coordinate invariant together with the subspace \( (y, 0) \).

The section \( \tilde{S}_h := TS_h \) of the rescaling 
\[ \tilde{u}(Tx) = u(x) \]

satisfies (5.10) and \( \tilde{u} = \tilde{\varphi} \) on \( \partial \tilde{S}_h \) with 
\[ \tilde{\varphi} = \varphi \quad \text{on } \tilde{G} := \{ \varphi \leq h \} \subset G, \]
\[ \tilde{\varphi} = h \quad \text{on } \partial \tilde{S}_h \setminus \tilde{G}. \]

From John’s lemma we know that \( S_h \) is equivalent to an ellipsoid \( E_h \). We choose \( T \) an appropriate sliding along the \( y \) direction, so that \( TE_h \) becomes symmetric with respect to the \( y \) and \( (z, x_n) \) subspaces, thus 
\[ \tilde{x}_h^* + c(n)|\tilde{S}_h|^{1/n} AB_1 \subset \tilde{S}_h \subset C(n)|\tilde{S}_h|^{1/n} AB_1, \quad \det A = 1 \]
and the matrix \( A \) leaves the \( y \) and the \( (z, x_n) \) subspaces invariant.

By choosing an appropriate system of coordinates in the \( y \) and \( z \) variables we may assume 
\[ A(y, z, x_n) = (A_1 y, A_2(z, x_n)) \]
with 
\[ A_1 = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{pmatrix} \]

with \( 0 < \beta_1 \leq \cdots \leq \beta_k \), and
\[ A_2 = \begin{pmatrix} \gamma_{k+1} & 0 & \cdots & 0 & \theta_{k+1} \\ 0 & \gamma_{k+2} & \cdots & 0 & \theta_{k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{n-1} & \theta_{n-1} \\ 0 & 0 & \cdots & 0 & \theta_{n} \end{pmatrix} \]

with \( \gamma_j, \theta_n > 0 \).

Next we use the induction hypothesis and show that \( \tilde{S}_h \) is equivalent to a ball.

**Lemma 5.3.** There exists \( C_0 \) such that
\[ \tilde{S}_h \subset C_0 h^{n/2} B_1^+ . \]

**Proof.** Using that
\[ |\tilde{S}_h| \sim h^{n/2} \]
we obtain
\[ \tilde{x}_h + ch^{1/2} AB_1 \subset \tilde{S}_h \subset Ch^{1/2} AB_1 . \]

We need to show that
\[ \|A\| \leq C . \]

Since \( \tilde{S}_h \) satisfies (5.10) we see that
\[ \tilde{S}_h \subset \{ |(z,x_n)| \leq Ch^{1/2} \} , \]
which together with the inclusion above gives \( \|A_2\| \leq C \) hence
\[ \gamma_j, \theta_n \leq C, \; |\theta_j| \leq C . \]

Also \( \tilde{S}_h \) contains the set
\[ \{(y,0,0) \mid |y| \leq \tilde{\mu}^{1/2} h^{1/2} \} \subset \tilde{G} , \]
which implies
\[ \beta_i \geq c > 0, \; i = 1, \ldots, k . \]

We define the rescaling
\[ w(x) = \frac{1}{h} \tilde{\mu}(h^{1/2} Ax) \]
which is defined in a domain \( \Omega_w := h^{-1/2} A^{-1} \tilde{S}_h \) such that
\[ B_1(x_0) \subset \Omega_w \subset B_1^+, \; 0 \in \partial \Omega_w , \]
and \( w = \varphi_w \) on \( \partial \Omega_w \) with
\[ \varphi_w = 1 \; \text{ on } \partial \Omega_w \setminus G_w , \]
\[ \tilde{\mu} \sum \beta_i^2 x_i^2 \leq \varphi_w \leq \min \{ 1, \tilde{\mu}^{-1} \sum \beta_i^2 x_i^2 \} \; \text{ on } G_w , \]
where \( G_w := h^{-1/2} A^{-1} \tilde{G} . \)

This implies that
\[ w \in D_{\tilde{\mu}}^0 (\beta_1, \beta_2, \ldots, \beta_k, \infty, \ldots, \infty) \]
for some value \( \tilde{\mu} \) depending on \( \mu, M, \lambda, \Lambda, n, k . \)

We claim that
\[ b_u(h) \geq c_* . \]

First we notice that
\[ b_u(h) = b_0(h) \sim \theta_n . \]
Since
\[ \theta_n \prod \beta_i \prod \gamma_j = \det A = 1 \]
and
\[ \gamma_j \leq C, \]
we see that if \( b_u(h) \) (and therefore \( \theta_n \)) becomes smaller than a critical value \( c \), then
\[ \beta_k \geq C_k(\bar{\mu}, \bar{M}, \lambda, \Lambda, n), \]
with \( \bar{M} := 2\bar{\mu}^{-1} \), and by the induction hypothesis
\[ b_w(\hat{h}) \geq M \geq 2b_w(1) \]
for some \( \hat{h} > C_k^{-1} \). This gives
\[ \frac{b_u(h\hat{h})}{b_u(h)} = \frac{b_w(\hat{h})}{b_w(1)} \geq 2, \]
which implies \( b_u(h\hat{h}) \geq 2b_u(h) \) and our claim follows.

Next we claim that \( \gamma_j \) are bounded below by the same argument. Indeed, from
the claim above \( \theta_n \) is bounded below and if some \( \gamma_j \) is smaller than a small value \( \tilde{c}_* \) then
\[ \beta_k \geq C_k(\bar{\mu}, \bar{M}_1, \lambda, \Lambda, n) \]
with
\[ \bar{M}_1 := \frac{2M}{\mu c_*}. \]
By the induction hypothesis
\[ b_w(\hat{h}) \geq \bar{M}_1 \geq \frac{2M}{c_*}b_w(1), \]
thus
\[ \frac{b_u(h\hat{h})}{b_u(h)} \geq \frac{2M}{c_*} \]
which gives \( b_u(h\hat{h}) \geq 2M \), contradiction. In conclusion \( \theta_n, \gamma_j \) are bounded below which implies that \( \beta_i \) are bounded above. This shows that \( \|A\| \) is bounded and the lemma is proved.

\[ \square \]

Next we use the lemma above and show that the function \( u \) has the following property.

**Lemma 5.4.** If for some \( p, q > 0 \),
\[ u \geq p(|z| - qx_n), \quad q \leq q_0 \]
then
\[ u \geq p'(|z| - (q - \eta)x_n) \]
for some \( p' \ll p, and with \( \eta > 0 \) depending on \( q_0 \) and \( \mu, M, \lambda, \Lambda, n, k \).

**Proof.** From Lemma 5.3 we see that after performing a linear transformation \( T \) (siding along the \( y \) direction) we may assume that
\[ S_h \subset C_0h^{1/2}B_1. \]
Let
\[ w(x) := \frac{1}{h}u(h^{1/2}x) \]
for some small $h \ll p$.

Then

$$S_1(w) := \Omega_w = h^{-1/2}S_h \subset B^+_C$$

and our hypothesis becomes

(5.11) \quad \quad w \geq \frac{p}{h^{1/2}}(|z| - qx_n),

Moreover the boundary values $\varphi_w$ of $w$ on $\partial\Omega_w$ satisfy

$$\varphi_w = 1 \quad \text{on} \quad \partial\Omega_w \setminus G_w$$

$$\bar{\mu}|y|^2 \leq \varphi_w \leq \min\{1, \bar{\mu}^{-1}|y|^2\} \quad \text{on} \quad G_w,$$

where $G_w := h^{-1/2}\{\varphi \leq h\}$.

Next we show that $\varphi_w \geq v$ on $\partial\Omega_w$ where $v$ is defined as

$$v := \delta|z|^2 + \frac{\Lambda}{\delta^{n-1}}(z_1 - qx_n)^2 + N(z_1 - qx_n) + \delta x_n,$$

and $\delta$ is small depending on $\bar{\mu}$ and $C_0$, and $N$ is chosen large such that

$$\frac{\Lambda}{\delta^{n-1}}t^2 + Nt$$

is increasing in the interval $|t| \leq (1 + q_0)C_0$.

From the definition of $v$ we see that

$$\det D^2v > \Lambda.$$

On the part of the boundary $\partial\Omega_w$ where $z_1 \leq qx_n$ we use that $\Omega_w \subset B_{C_0}$ and obtain

$$v \leq \delta(|z|^2 + x_n) \leq \varphi_w.$$

On the part of the boundary $\partial\Omega_w$ where $z_1 > qx_n$ we use (5.11) and obtain

$$1 = \varphi_w \geq C(|z| - qx_n) \geq C(z_1 - qx_n)$$

with $C$ arbitrarily large provided that $h$ is small enough. We choose $C$ such that the inequality above implies

$$\frac{\Lambda}{\delta^{n-1}}(z_1 - qx_n)^2 + N(z_1 - qx_n) < \frac{1}{2}.$$

Then

$$\varphi_w = 1 > \frac{1}{2} + \delta(|z|^2 + x_n) \geq v.$$

In conclusion $\varphi_w \geq v$ on $\partial\Omega_w$ hence the function $v$ is a lower barrier for $w$ in $\Omega_w$. Then

$$w \geq N(z_1 - qx_n) + \delta x_n$$

and, since this inequality holds for all directions in the $z$-plane, we obtain

$$w \geq N(|z| - (q - \eta)x_n), \quad \eta := \frac{\delta}{N}.$$

Scaling back we get

$$u \geq p'(|z| - (q - \eta)x_n) \quad \text{in} \quad S_h.$$

Since $u$ is convex and $u(0) = 0$, this inequality holds globally, and the lemma is proved.

□
We remark that Lemma 5.4 can be used directly to prove Proposition 4.1 and Lemma 5.2.

End of the proof of Proposition 5.1. From (5.9) we obtain an initial pair \((p, q_0)\) which satisfies the hypothesis of Lemma 5.4. We apply this lemma a finite number of times and obtain that

\[ u \geq \epsilon(|z| + x_n), \]

and we contradict that \(\tilde{S}_h\) is equivalent to a ball of radius \(h^{1/2}\).

□

References


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