DIFFEOMORPHISMS AND NONLINEAR HEAT FLOWS*  

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Abstract. We show that the gradient flow $u$ on $L^2$ generated by the energy functional $I[u] := \int_U \Phi(\det Du) \, dx$ for vector-valued mappings is in some sense “integrable,” meaning that (i) the inverse Jacobian $\beta := (\det Du)^{-1}$ satisfies a scalar nonlinear diffusion equation, and (ii) we can recover $u$ by solving an ODE determined by $\beta$.

Key words. quasi convexity, gradient flow, Jacobian

AMS subject classifications. 35K45, 49J40

DOI. 10.1137/04061386X

1. Introduction.

1.1. Gradient flows for quasi-convex energies. This paper is a contribution to the mostly unsolved problem of understanding the gradient flow dynamics on $L^2$ generated by integral functionals having the form

\begin{equation}
I[v] := \int_U F(Dv) \, dx,
\end{equation}

defined for functions $v : U \to \mathbb{R}^m$, where $U$ is an open subset of $\mathbb{R}^n$. The gradient $Dv$ belongs to $\mathbb{M}^{m \times n}$, the space of $m \times n$ matrices, and we are given the nonlinearity $F : \mathbb{M}^{m \times n} \to (-\infty, +\infty]$.

Quasi convexity. As is well known, the critical assumption for the existence of minimizers of $I[\cdot]$, subject to appropriate boundary conditions, is that $F$ be quasi-convex in the sense of C. B. Morrey, Jr. This is the condition that

\begin{equation}
\int_U F(A) \, dx \leq \int_U F(A + Dv) \, dx
\end{equation}

for all matrices $A \in \mathbb{M}^{m \times n}$ and all $C^1$ functions $v : U \to \mathbb{R}^m$ vanishing on $\partial U$.

Dynamics. As the existence and (partial) regularity theories for minimizers are fairly well understood, it has long seemed natural to turn attention to related dynamical problems. The corresponding flow on $L^2$ generated by $I[\cdot]$ is the initial-value problem for the system of PDEs

\begin{equation}
\begin{cases}
    u_t = \text{div}(DF(Du)) & (t > 0), \\
    u = u^0 & (t = 0),
\end{cases}
\end{equation}

*Received by the editors August 25, 2004; accepted for publication February 1, 2005; published electronically November 9, 2005.
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with appropriate boundary conditions.

Given the quasi-convexity hypothesis (1.2), the system (1.3) is parabolic, at least in some weak sense. However, it is extremely nonlinear, so much so that it remains to date a challenging open problem to prove existence of even weak solutions, to understand uniqueness issues, and/or to show partial regularity.

**Time-step approximations.** One obvious approach is to approximate by an implicit time-step approximation. For this, we fix a step size \( h > 0 \) and recursively find \( u_{k+1} \) to minimize

\[
I_k[v] := \frac{1}{2} \int_U |v - u_k|^2 \, dx + h \int_U F(Dv) \, dx,
\]

with appropriate boundary conditions, given \( u_k \). The Euler–Lagrange equations read

\[
\frac{u_{k+1} - u_k}{h} = \text{div}(DF(Du_{k+1})) \quad (k = 0, 1, \ldots),
\]

This procedure generates a strong candidate for an approximation to the full dynamics (1.3). The fundamental point is that under our quasi-convexity assumption we can in fact iteratively find minimizers of (1.4).

The really hard task is passing to limits as \( h \to 0 \). Since our approximations \( u_k \) are minimizers, and not just critical points, of \( I_k[\cdot] \), the expectation and hope is that we obtain in the limit some sort of reasonable weak solution of (1.3). It has, however, proved in practice impossible to carry out this program in general, owing to the usual problem in nonlinear PDE that we do not have very good uniform estimates on the approximate solutions \( u_k \). (The paper [E] demonstrates a completely different minimization principle, but we have not been able to exploit this usefully.)

### 1.2. Nonlinearities depending only on the determinant

This paper documents some progress in this matter for the case \( m = n \) and nonlinearities \( F \) with the special structure

\[
F(P) = \Phi(\text{det} P) \quad (P \in \mathbb{M}^{n \times n}),
\]

where \( \Phi \) is a convex function and “det” means determinant. Such a nonlinearity is quasi-convex, and it has long been known that for the static calculus of variations the particular hypothesis (1.6) has strong implications; see, for instance, Dacorogna [D].

We begin by reviewing the issue of minimizing the functional

\[
I[v] := \int_U F(Dv) \, dx = \int_U \Phi(\text{det} Dv) \, dx
\]

among mappings \( v = (v^1, \ldots, v^n) \) from a connected, open set \( U \subset \mathbb{R}^n \) into \( \mathbb{R}^n \). We write the gradient matrix of \( v \) as

\[
Dv = \begin{pmatrix}
v^1_{x_1} & \cdots & v^1_{x_n} \\
\vdots & \ddots & \vdots \\
v^n_{x_1} & \cdots & v^n_{x_n}
\end{pmatrix}.
\]

If \( u = (u^1, \ldots, u^n) \) is a smooth minimizer of \( I[\cdot] \), subject to boundary conditions which for the moment we do not specify, then \( u \) solves the Euler–Lagrange system of PDEs

\[
\text{div}(DF(Du)) = \text{div}(\Phi'(\text{det} Du)(\text{cof} Du)^T) = 0,
\]
where $\text{cof } Du$ is the cofactor matrix formed from $Du$. To derive (1.8) we employed the formula
\begin{equation}
\frac{\partial \det P}{\partial p^k_i} = (\text{cof } P)_i^k \quad (1 \leq i, k \leq n)
\end{equation}
for the $n \times n$ matrix $P$, whose $(i, k)$ entry is denoted $p^k_i$. Likewise, $(\text{cof } P)^T_i$ means the $(i, k)$ entry of cof $P$. Formula (1.9) is a consequence of the matrix identity
\begin{equation}
(\text{cof } P)^T P = I \det P,
\end{equation}
but for any $C^2$ function $w = (w^1, \ldots, w^n)$ we have
\begin{equation}
\text{div}(\text{cof } Dw)^T \equiv 0;
\end{equation}
that is,
\begin{equation}
(\text{cof } Dw)^k_{i, x_i} = 0 \quad (k = 1, \ldots, n).
\end{equation}
Therefore (1.8) implies
\begin{equation}
0 = \Phi''(\det Du)D(\det Du)(\text{cof } Du)^T.
\end{equation}
In view of (1.10), our multiplying (1.10) by $Du$ gives
\[
0 = \Phi''(\det Du)D(\det Du)(\det Du) = \frac{1}{2} \Phi''(\det Du)D(\det Du)^2.
\]
Assuming next the strict convexity condition that $\Phi'' > 0$, we deduce that $(\det Du)^2$ is constant within $U$. Thus, if $u$ is smooth, we conclude that
\begin{equation}
\det Du \equiv C \quad \text{within } U
\end{equation}
for some constant $C$.

1.3. A gradient flow. We study in this paper the corresponding “heat flow” governed by the function $I[\cdot]$, that is, the system of PDEs
\begin{equation}
u_t = \text{div}(DF(Du)) = \text{div}(\Phi'(\det Du)(\text{cof } Du)^T),
\end{equation}
plus appropriate initial and boundary conditions, detailed later.
We are especially interested in the case that $\Phi(d) < \infty$ for $d > 0$, $\Phi(d) = \infty$ for $d < 0$, and $\lim_{d \to 0^-} \Phi(d) = +\infty$. Then (1.14) enforces the constraint
\[
\det Du > 0.
\]
We can hope therefore that for each time $t$ the mapping $x \mapsto y = u(x, t)$ is a diffeomorphism, with inverse $y \mapsto x = v(y, t)$. And since the static problem, recalled in section 1.1, is so simple, we hope as well that the analysis of the system (1.14) may not be so complicated.
This is in fact so, for as we will see in section 2, the quantity
\begin{equation}
\beta := (\det Du)^{-1} > 0,
\end{equation}
regarded as a function of \( y \) and \( t \), solves the nonlinear parabolic PDE

\[
\beta_t = \text{div} \left( \frac{1}{\beta} \frac{D\beta}{\beta^2} \right) = \text{div} \left( \beta \Psi'(\beta) D\beta \right) \tag{1.16}
\]

with Neumann boundary conditions, where

\[
\Psi(d) := d \Phi \left( \frac{1}{d} \right) \quad \text{for } d > 0.
\]

Now (1.16) is singular in regimes where \( \beta \to 0 \) or \( \infty \), but the maximum principle implies that if the initial data \( \beta^0 \) is bounded away from 0 and \( \infty \), then so is the solution.

We will show furthermore that given \( \beta \), the solution of (1.16) with appropriate initial conditions, we can then recover the mappings \( u \) by solving a system of ODEs governed by \( \beta \) and proving then that the PDE (1.14) holds. In this sense, we can regard the parabolic system of PDEs (1.14) as being somehow “integrable.”

1.4. Outline. Our paper introduces in section 2 the formal computations showing how (1.16) results from (1.14). Section 3 then reverses this process to provide careful proofs: we start with the solution \( \beta \) of the nonlinear diffusion equation and build from it the mappings \( u(\cdot,t) \) for \( t > 0 \).

Section 4 introduces some interesting variants of our construction, the first for more general integrands than in (1.7). We discuss also a situation when the range of the initial mapping \( u^0 \) is a proper subset \( W_0 \) of the target \( V \). In this case we can design \( \Phi \) so that the flow “fills up” \( V \) in finite time. Interesting complications occur if \( U \) and \( V \) are not in fact diffeomorphic.

The concluding section 5 introduces and analyzes a related “time-stepping” dynamic variational principle. This discussion will make much clearer the connections between our PDE (1.16) and (1.14).

2. Calculations for smooth solutions. Suppose now \( U \) is a smooth, open, bounded, connected subset of \( \mathbb{R}^n \), and

\[ u : \bar{U} \times [0, \infty) \to \mathbb{R}^n \]

is smooth, \( u = (u^1, \ldots, u^n) \). In this section we suppose as well that \( u \) solves the system (1.14). Let \( u^0 = u(\cdot, 0) \) denote the initial mapping.

2.1. Changing variables. Suppose that for each time \( t \geq 0 \), the mapping

\[ u(\cdot, t) : \bar{U} \to \bar{V} \]

is a diffeomorphism, where \( V \subset \mathbb{R}^n \) is a fixed open subset of \( \mathbb{R}^n \). We can then invert the relationship

\[ y = u(x, t) \quad (x \in \bar{U}, y \in \bar{V}) \tag{2.1} \]

to give

\[ x = v(y, t) \quad \text{for } v := u^{-1}. \tag{2.2} \]

Set

\[ \beta(y, t) := \det Dv(y, t) = (\det Du(x, t))^{-1}. \tag{2.3} \]
2.2. A PDE for $\beta$. Our main observation is that $\beta$ solves a scalar, nonlinear diffusion equation.

**Theorem 2.1.** We have

\begin{align*}
\beta_t &= \text{div} \left( \Phi' \left( \frac{1}{\beta} \right) \frac{1}{\beta^2} D\beta \right) \quad \text{in } V \times (0, \infty), \\
\frac{\partial \beta}{\partial \nu} &= 0 \quad \text{on } \partial V \times (0, \infty),
\end{align*}

$\nu$ denoting the unit outward-pointing normal vectorfield to $\partial V$.

**Proof.**

1. Fix any time $T > 0$ and select a smooth function $\zeta : \bar{V} \times [0, T] \to \mathbb{R}$ such that

\begin{align*}
(2.5) \quad \zeta(\cdot, 0) &\equiv \zeta(\cdot, T) \equiv 0.
\end{align*}

Then, employing (2.1), we compute

\begin{align*}
\int_0^T \int_V \beta \zeta_t + D_y \left( \Phi' \left( \frac{1}{\beta} \right) \right) \cdot D_y \zeta \, dy \, dt \\
(2.6) \quad &= \int_0^T \int_U \left[ \beta(u, t) \zeta_t(u, t) + D_x \left( \Phi' \left( \frac{1}{\beta} \right) \right) (D_u)^{-1} \cdot D_y \zeta \right] \frac{dx}{\beta(u, t)} \, dt \\
&\quad - \int_0^T \int_U D_y \zeta \cdot \left[ u_t - D_x \left( \Phi' \left( \frac{1}{\beta} \right) \right) (D_u)^{-1} \beta \right] (D_u)^{-1} \, dx \, dt.
\end{align*}

Now our PDE (1.14) reads

$$u_t = \text{div}_x \left( \Phi' (\det D_u) \det D_u (D_u)^{-1} \right) = D_x \left( \Phi' \left( \frac{1}{\beta} \right) \right) \frac{(D_u)^{-1}}{\beta},$$

since $\text{div}(\det D_u (D_u)^{-1}) = \text{div} (\text{cof} D_u^T) \equiv 0$. Consequently the expression within the square brackets in the last term of (2.6) vanishes. So

$$\int_0^T \int_V \beta \zeta_t + D_y \left( \Phi' \left( \frac{1}{\beta} \right) \right) \cdot D_y \zeta \, dy \, dt = 0$$

for all test functions $\zeta$ as above.

2. If also $\zeta \equiv 0$ on $\partial V \times [0, T]$, we may integrate by parts to deduce

\begin{align*}
(2.7) \quad \beta_t + \text{div}_y \left( D_y \Phi' \left( \frac{1}{\beta} \right) \right) &\equiv 0,
\end{align*}

and this is the PDE in (2.4). Now drop the assumption that $\zeta = 0$ on the boundary and again integrate by parts:

$$\int_0^T \int_{\partial V} \frac{\partial}{\partial \nu} \left( \Phi' \left( \frac{1}{\beta} \right) \right) \zeta \, d\mathcal{H}^{n-1} \, dt = 0.$$

It follows that

$$\frac{\partial}{\partial \nu} \left( \Phi' \left( \frac{1}{\beta} \right) \right) \equiv -\Phi'' \left( \frac{1}{\beta} \right) \frac{\partial \beta}{\partial \nu} \equiv 0 \quad \text{on } \partial V \times (0, T).$$

Since $\Phi'' > 0$, the proof is done. \(\Box\)
2.3. Recovering the mapping $u$ from $\beta$. We next address the question of how to recover the mapping $u$ from knowledge of $\beta$. One possibility is for each time $t$ to try to find $x \mapsto u(x, t)$ solving

$$
\begin{cases}
\beta(u(x, t), t) \det Du(x, t) \equiv 1 & \text{in } \bar{U}, \\
\mathbf{u}(\cdot, t) \in \text{Diff}(\bar{U}, \bar{V}),
\end{cases}
$$

(2.8)

where $\text{Diff}(\bar{U}, \bar{V})$ denotes the set of all diffeomorphisms of $\bar{U}$ onto $\bar{V}$. As we will discuss later in section 5, this approach works, provided $U$ and $V$ are convex sets.

However, there is a simpler construction available. First, define the new nonlinearity

$$
\Psi(d) := d \Phi \left( \frac{1}{d} \right) \quad (d > 0).
$$

(2.9)

Then

$$
\Psi'(d) = \Phi \left( \frac{1}{d} \right) - \frac{1}{d} \Phi' \left( \frac{1}{d} \right), \quad \Psi''(d) = \frac{1}{d^3} \Phi'' \left( \frac{1}{d} \right);
$$

(2.10)

and so $\Psi : (0, \infty) \to \mathbb{R}$ is convex.

Next, perform these calculations:

$$
\begin{align*}
\mathbf{u}_t &= \text{div}_x (\Phi'(\det D\mathbf{u}) \det D\mathbf{u}(Du)^{-1}) \\
&= D_x (\Phi'(\det D\mathbf{u}) \cdot (\det D\mathbf{u}(Du)^{-1}) \\
&= \Phi''(\det D\mathbf{u}) D_x (\det D\mathbf{u}) \cdot (\det D\mathbf{u}(Du)^{-1}) \\
&= \Phi'' \left( \frac{1}{\beta} \right) \frac{1}{\beta^3} D_x (\det D\mathbf{u})^{-1} \\
&= -\Phi''(\beta) D_y \beta = -D_y \Psi'(\beta).
\end{align*}
$$

(2.11)

This computation suggests that we fix a point $x \in \bar{U}$ and then solve the ODE

$$
\begin{cases}
\dot{y}(t) = -\Psi''(\beta(y(t), t)) D\beta(y(t), t) & \text{for } t > 0, \\
y(0) = y = u^0(x),
\end{cases}
$$

(2.12)

where $\cdot = \frac{d}{dt}$. Then by uniqueness of solutions we have $u(x, t) = y(t)$ for $t \geq 0$.

3. Building diffeomorphisms. The formal calculations from the previous section done with, we turn now to building rigorously a smooth solution

$$
\mathbf{u} : \bar{U} \times [0, \infty) \to \bar{V}
$$

of our system

$$
\begin{cases}
\mathbf{u}_t = \text{div}(\Phi'(\det D\mathbf{u})(\det D\mathbf{u})^T) & \text{in } \bar{U} \times (0, \infty), \\
\mathbf{u} = u^0 & \text{on } \bar{U} \times \{t = 0\}, \\
\mathbf{u}(\cdot, t) \in \text{Diff}(\bar{U}, \bar{V})
\end{cases}
$$

(3.1)

under some additional assumptions.
3.1. Hypotheses. We require that the initial mapping $u^0 : \bar{U} \to \bar{V}$ be a diffeomorphism, mapping $\partial U$ onto $\partial V$. We write

$$\beta^0 := \det Dv^0$$

for $v^0 := (u^0)^{-1}$ and assume that there exist positive constants $0 < C_1 \leq C_2$ such that

$$(H1) \quad C_1 \leq \beta^0 \leq C_2 \quad \text{on } \bar{V}.$$ 

We ask also that the following compatibility condition hold:

$$(H2) \quad \frac{\partial \beta^0}{\partial \nu} = 0 \quad \text{on } \partial V.$$ 

Finally we require that $\Phi$ be smooth and convex on $(0, \infty)$, with the lower bound

$$(H3) \quad \Phi'' \left( \frac{1}{\beta} \right) > 0 \quad \text{for } C_1 \leq \beta \leq C_2.$$ 

3.2. Solving PDE and ODE. In view of $(H1)$, $(H2)$, the initial/boundary-value problem

$$\begin{cases}
\beta_t = \text{div} \left( \Phi'' \left( \frac{1}{\beta} \right) \frac{\partial \beta}{\partial \nu} \right) \quad \text{in } V \times (0, \infty), \\
\frac{\partial \beta}{\partial \nu} = 0 \quad \text{on } \partial V \times [0, \infty), \\
\beta = \beta^0 \quad \text{on } \bar{V} \times \{ t = 0 \}
\end{cases}$$

has a unique, smooth solution $\beta$, with

$$(4.4) \quad 0 < C_1 \leq \beta \leq C_2 \quad \text{in } \bar{V} \times [0, \infty).$$

Next, for each $y \in \bar{V}$, solve the ODE (2.12):

$$\begin{cases}
\dot{y}(t) = -\Phi''(\beta(y(t), t))D\beta(y(t), t) & \text{for } t > 0, \\
y(0) = y.
\end{cases}$$

We write $y(t) = y(t, y)$ to display dependence on the initial point $y$.

Theorem 3.1. (i) For each given $x \in \bar{U}$, the ODE (3.5) has a unique solution $y : [0, \infty) \to \bar{V}$, existing for all times $t \geq 0$.

(ii) If $y \in \partial V$, then $y(t) \in \partial V$ for all times $t \geq 0$.

(iii) For each $t \geq 0$, the mapping

$$(3.6) \quad u(x, t) := y(t, u^0(x)) \quad (x \in \bar{U}, \ t \geq 0)$$

is a smooth diffeomorphism from $\bar{U}$ to $\bar{V}$, mapping $\partial U$ onto $\partial V$.

Proof. Since $\frac{\partial \beta}{\partial \nu} = 0$ on $\partial V$, $D\beta$ is tangent to $\partial V$ and consequently the flow does not leave $\bar{V}$. In particular, if $u^0(x) \in \partial V$, then $x(t) \in \partial V$ for times $t \geq 0$.

Assertion (iii) is standard. □

Define $u : \bar{U} \times [0, \infty) \to \bar{V}$ by (3.6) and set $v(\cdot, t) := u^{-1}(\cdot, t)$ for each time $t \geq 0$.

Theorem 3.2. (i) We have

$$(3.7) \quad \beta \equiv \det Dv.$$
Furthermore, \( u \) solves the system of PDEs (2.1), and the mapping
\[
t \mapsto \int_U \Phi(\det Du)(x, t) \, dx
\]
is nonincreasing.

Proof. 1. As before, set \( \alpha = \det Du, \alpha = \alpha(x, t) \). Then
\[
(3.8) \quad \alpha_t = \alpha D_x u_t(Du)^{-1}.
\]
Now
\[
u_t = -D_y \Psi'(\beta)
\]
and so
\[
D_x u_t = -D_y^2 \Psi'(\beta)(D_x u).
\]
Hence
\[
(3.9) \quad \alpha_t = -\alpha \Delta_y \Psi'(\beta).
\]
Next, regarding \( \beta = \beta(u, t) \) as a function of \( (x, t) \), we have
\[
(\alpha \beta)_t = \alpha_t \beta + \alpha \beta_t + \alpha D_y \beta \cdot u_t
\]
\[
= -\alpha \beta \Delta_y \Psi'(\beta) + \alpha \div(\Psi''(\beta) D_y \beta) - \alpha D_y \beta \cdot (\Psi''(\beta) D_y \beta)
\]
\[= 0.
\]
Since \( \alpha \beta \equiv 1 \) at \( t = 0 \), we deduce that
\[
\beta = \alpha^{-1} = \det Dv.
\]
2. We have shown that \( \beta \equiv \det Dv \), where \( v = u^{-1} \) and \( u \) is defined by (3.6).
We then return to the computation (2.11) to deduce that
\[
(3.10) \quad u_t = \dot{x} = -\Psi''(\beta) D \beta = \div(\Phi'(\det Du)(\cof Du)^T).
\]
Finally let us calculate
\[
\frac{d}{dt} \int_U \Phi(\det Du) \, dx = \frac{d}{dt} \int_V \Phi \left( \frac{1}{\beta} \right) \beta \, dy
\]
\[
= \int_V \left( \Phi \left( \frac{1}{\beta} \right) - \frac{1}{\beta} \Phi' \left( \frac{1}{\beta} \right) \right) \beta_t \, dy
\]
\[
= \int_V \Psi'(\beta) \div(\beta \Psi''(\beta) D \beta) \, dy
\]
\[
= -\int_V \Psi''(\beta)^2 \beta |D \beta|^2 \, dy \leq 0. \quad \square
\]

4. Some variants.

4.1. More general nonlinearities. Our methods extend with little difficulty to the functional
\[
(4.1) \quad I[v] := \int_U \Phi(f(v) \det Dv) \, dx
\]
for \( \Phi \) as before and \( f : \bar{V} \to (0, \infty) \).
**Euler–Lagrange equation.** The corresponding Euler–Lagrange equation is

\[ - \text{div}(\Phi'(f \det Du)f(\text{cof} Du)^T) + \Phi'(f \det Du)(\det Du)Df = 0, \]

which simplifies to read

\[ \Phi''(f \det Du)D(f \det Du)f(\text{cof} Du)^T = 0. \]

As in section 1.1 this implies that

\[ f(u) \det Du \equiv C \text{ within } U \]

for some constant \( C \).

**A gradient flow.** The evolution associated with (4.1) is

\[ u_t - \text{div}(\Phi'(f \det Du)f(\text{cof} Du)^T) + \Phi'(f \det Du)(\det Du)Df = 0, \]

plus initial and boundary conditions.

As before, assume \( v := u^{-1} \) exists and write

\[ \beta := \det Du. \]

**Theorem 4.1.** We have

\[
\begin{cases}
\beta_t = -\text{div} \left( \Phi'' \left( \frac{f}{\beta} \right) f D \left( \frac{f}{\beta} \right) \right) & \text{in } V \times (0, \infty), \\
\frac{\partial}{\partial \nu} \left( \frac{f}{\beta} \right) = 0 & \text{on } \partial V \times (0, \infty).
\end{cases}
\]

**Proof.** 1. Fix any time \( T > 0 \) and select a smooth function \( \zeta : \bar{V} \times [0, T] \to \mathbb{R} \) satisfying (2.5).

Then

\[
\int_0^T \int_U \beta \zeta_t + D_y \left( \Phi' \left( \frac{f}{\beta} \right) f D \left( \frac{f}{\beta} \right) \right) \cdot D_y \zeta f \ dy dt
\]

\[ = \int_0^T \int_U \left[ \beta(u, t) \zeta_t(u, t) + D_x \left( \Phi' \left( \frac{f}{\beta} \right) \right) (Du)^{-1} \cdot D_y \zeta f \right] \frac{dx}{\beta(u, t)} \ dt
\]

\[ = \int_0^T \int_U \frac{\partial}{\partial t} (\zeta(u, t)) - D_y \zeta \cdot u_t + D_x \left( \Phi' \left( \frac{f}{\beta} \right) \right) \frac{(Du)^{-1}}{\beta} \cdot D_y \zeta f \ dx dt
\]

\[ = -\int_0^T \int_U D_y \zeta \cdot \left[ u_t - D_x \left( \Phi' \left( \frac{f}{\beta} \right) \right) \frac{(Du)^{-1}}{\beta} f \right] \ dx dt.
\]

But according to (4.3), we have

\[ u_t = D(\Phi'(f \det Du))f(\det Du)(Du)^{-1}
\]

\[ = D \left( \Phi' \left( \frac{f}{\beta} \right) \right) \frac{f}{\beta} (Du)^{-1}.
\]

Consequently the expression within the square brackets in the last term of (4.5) vanishes. \( \Box \)
4.2. “Filling up” the target domain. An interesting variant of our construction is as follows. Select \( u^0 : \bar{U} \to \bar{W}_0 \) to be a diffeomorphism, where \( W_0 \subset \subset V \) is given. We will choose \( \Phi \) and \( u \) so that

\[
\begin{aligned}
W(t) := u(U, t) & \quad (t \geq 0), \\
W(0) = W_0 & 
\end{aligned}
\]

(4.7)

expands to “fill up” the target \( V \) in finite time.

For this, let us take \( m > 0 \) and

\[
\Phi(d) := \begin{cases} 
\frac{1}{m} d^{-m} & (d > 0), \\
+\infty & (d \leq 0).
\end{cases}
\]

Therefore

\[
\Psi(d) = d\Phi \left( \frac{1}{d} \right) = \frac{1}{m} d^{m+1}
\]

for \( d > 0 \). Then \( \beta \) solves the porous medium equation

\[
\beta_t = \text{div}(\Psi''(\beta)\beta D\beta) = \Delta(\beta^{m+1}).
\]

5. Connections with optimal mass transfer problems. As noted in the introduction, the time-step minimization method (1.4) and (1.5) provides an extremely natural approximation method, but one which we have not been able to prove converges. This section recalls more about this procedure, to highlight the connections with Monge–Kantorovich mass transfer theory.

We are primarily motivated by Otto [O] and Jordan, Kinderlehrer, and Otto [J-K-O]. The novelty of Otto’s paper [O] was to interpret (5.8) as a gradient flux of the “entropy” \( S(\beta) := \int_V \Psi(\beta) dy \) with respect to the Wasserstein distance.

5.1. Time-step approximations. Assume for this section that \( U \) and \( V \) are two bounded, open, convex sets with smooth boundaries.

We discuss a time-discrete algorithm for the flow

\[
\begin{aligned}
\mathbf{u}_t &= \text{div}(DF(D\mathbf{u})), \\
\mathbf{u}(\cdot, 0) &= \mathbf{u}_0, \\
\mathbf{u}(\cdot, t) &\in \text{Diff}(\bar{U}, \bar{V}),
\end{aligned}
\]

(5.1)

where, as before,

\[
F(P) = \begin{cases} 
\Phi(\det P), & \det P > 0, \\
+\infty, & \det P \leq 0.
\end{cases}
\]

The system (5.1) is a gradient flux of the functional \( I[\cdot] \) with respect to the \( L^2 \)-metric. In section 2 we have shown that (5.1) is related to (1.16), which, as we will recall below, is the gradient flow governed by \( \int_V \Psi(\beta) dy \) with respect to the Wasserstein distance. The algorithm which we discuss is another way to view that relation.

A discrete-time approximation. First, let us fix a time-step size \( h > 0 \). We introduce the implicit scheme of recursively finding \( \mathbf{u}_{k+1} \) to solve

\[
\begin{aligned}
\frac{\mathbf{u}_{k+1} - \mathbf{u}_k}{h} &= \text{div}(DF(D\mathbf{u}_{k+1})), \\
\mathbf{u}_{k+1} &\in \text{Diff}(\bar{U}, \bar{V}),
\end{aligned}
\]

(5.2)
given $u_k$. More precisely, set

$$I_k[v] := \frac{1}{2} \int_U |v - u_k|^2 \, dx + h \int_U F(Dv) \, dx.$$  

We intend to find $u_{k+1}$ to be the unique minimizer of

$$\min_v \{I_k[v] \mid v \in \text{Diff} \bar{U}, \bar{V}\}.$$  

**Changing variables.** Since our nonlinearity $F$ is neither coercive nor convex, standard calculus of variations methods do not apply. However, recent papers by Gangbo and Van der Putten [G-VP] and Maroofi [Ma] demonstrate how to exploit the special structure of $F(P) = \Phi(\det P)$ to find minimizers.

Indeed, if we apply a change of variables $y = u(x)$ and set $\beta := \det(Du - 1)$, $\beta_k := \det(Du_k^{-1})$, we discover that

$$I_k[v] = \frac{1}{2} \int_V |y - u_k(v^{-1}(y))|^2 \, dy + h \int_V \Psi(\beta) \, dy.$$  

Consequently

$$\min_{v \in \text{Diff}(\bar{U}, \bar{V})} I_k[v] = \inf_{\beta} \left\{ h \int_V \Psi(\beta) \, dy + \inf_v \left\{ \frac{1}{2} \int_V |y - u_k(v^{-1}(y))|^2 \, dy \mid \beta = \det(Dv^{-1}) \right\} \right\}$$

$$= \inf_{\beta} \left\{ h \int_V \Psi(\beta) \, dy + \inf_w \left\{ \frac{1}{2} \int_V |y - w(y)|^2 \, dy \mid \beta_k = \beta(w) \right\} \right\}$$

$$= \inf_{\beta} \left\{ h \int_V \Psi(\beta) \, dy + W_2^2(\beta_k, \beta) \right\},$$

where $W_2$, the Wasserstein distance between two Borel measures $\mu$ and $\nu$, is defined as

$$W_2^2(\mu, \nu) := \frac{1}{2} \inf_{\gamma \in \Gamma(\mu, \nu)} \iint |x - y|^2 \, d\gamma(x, y).$$

Here $\Gamma(\mu, \nu)$ is the set of Borel measures $\gamma$ on $\mathbb{R}^{2n}$ that have $\mu$ and $\nu$ as their marginals. The notation $W_2^2(\beta_k, \beta)$ means that we have identified $\beta$ with the measure whose density is $\beta$.

We assume for $k = 0$ that

$$\int_V \beta_0 \, dy = 1,$$

where $\beta_0 = \det(Du_0^{-1})$. This reduces the last three problems in (5.5) to minimization problems over $\mathcal{P}_a(V)$, the set of probability densities supported in $V$.

Define the new functional

$$J_k(\beta) := W_2^2(\beta, \beta_k) + h \int_V \Psi(\beta) \, dy.$$  

Now $W_2^2(\beta_k, \cdot)$ is convex and is weakly-∗ lower semicontinuous. Since $\Psi$ is strictly convex, we see also that $\beta \rightarrow \int_V \Psi(\beta) \, dy$ is a strictly convex functional of $\beta$ and is
weakly-* lower semicontinuous on subsets of \( L^1 \) that are weakly-* compact. Consequently, the minimization problem

\[
\inf_{\beta \in \mathcal{P}_a(V)} J_k(\beta)
\]

has a unique solution \( \beta_{k+1} \).

5.2. Time-step approximations for \( \beta \). This subsection quickly reviews a time-discrete algorithm based on the Wasserstein distance for solving

\[
\beta_t = \text{div} \left( \beta D[\Psi'(\beta)] \right),
\]

\( \beta(\cdot, 0) = \beta_0 \).

Let us now deal with the following nonlinear problem appearing in (5.5), where we replace \( \beta \) by \( \beta_{k+1} \). We study the minimization problem

\[
\inf_{\nu} \left\{ \int_V |y - \nu(y)|^2 dy \mid \beta_k = \beta_{k+1}(\nu) \det D\nu \right\},
\]

which, thanks to the Monge–Kantorovich theory, is known to admit a unique minimizing \( v_{k+1} \) (see Brenier [B]). Furthermore, \( v_{k+1} \) is the gradient of a convex function \( \psi_{k+1} : \bar{V} \to \mathbb{R} \), satisfying the Monge–Ampere problem

\[
\beta_k = \beta_{k+1}(D\psi_{k+1}) \det D^2 \psi_{k+1}, \quad D\psi_{k+1}(\bar{V}) = \bar{V}
\]

in the sense that

\[
D\psi_{k+1} : \bar{V} \to \bar{V} \text{ a.e. and } \int_V f(D\psi_{k+1})\beta_k dx = \int_V f\beta_{k+1} dy
\]

for all \( f \in C(\mathbb{R}^n) \). Equivalently, if \( \phi_{k+1} \) is the Legendre transform of \( \psi_{k+1} \), then

\[
D\phi_{k+1} : \bar{V} \to \bar{V} \text{ a.e. and } \int_V g(D\phi_{k+1})\beta_{k+1} dy = \int_V g\beta_k dx
\]

for all \( g \in C(\mathbb{R}^n) \). We write that

\[
(D\psi_{k+1})_\# \beta_k = \beta_{k+1}, \quad (D\phi_{k+1})_\# \beta_{k+1} = \beta_k,
\]

the symbol \( \# \) denoting push-forward. Agueh [A] has shown that

\[
C_1 \leq \beta_{k+1} \leq C_2,
\]

provided

\[
C_1 \leq \beta_k \leq C_2
\]

for constants \( 0 < C_1 \leq C_2 \).

The Euler–Lagrange equations of (5.7) read

\[
D\phi_{k+1}(y) = y + hD[\Psi'(\beta_{k+1}(y))],
\]

and we conclude from (5.15) that

\[
\beta_{k+1}(y) = (\Psi^*)' \left( \left( \phi_{k+1}(y) - \frac{|y|^2}{2} \right) / h \right),
\]
where $\Psi^*$ is the Legendre transform of $\Psi$.

Assume that $\beta_k \in C^{l,\alpha}(V)$ for some $\alpha > 0$ and some integer $l \geq 0$. By (5.16), $\beta_{k+1} \in W^{1,\infty}(V) \subset C^{0,\alpha}(V)$. Regularity theory for the Monge–Ampère equations (see [C1], [C2], [C3], [C4]) and (5.10) imply that $\psi_{k+1}, \phi_{k+1} \in C^{2,\alpha}(V)$. This and (5.16) demonstrate that $\beta_k \in C^{2,\alpha}(V)$. Thus

$$
\gamma_{k+1} := D\phi_{k+1} \circ D\phi_k \circ \cdots \circ D\phi_1 \in C^{l+1,\alpha}(\bar{V}, \bar{V}).
$$

The map

$$
u_{k+1} = \gamma_{k+1} \circ u_0
$$

is then the unique solution to (5.2), and $u_{k+1} \in C^{l+1,\alpha}(\bar{U}, \bar{V})$ if $u_0 \in C^{l+1,\alpha}(\bar{U}, \bar{V})$.

We record next that the time-step approximations converge as $h \to 0$.

**Theorem 5.1.** For $h > 0$, inductively define $\beta_{k+1}$ to be the unique minimizer of $J_k[\cdot]$ over $\mathcal{P}_\alpha(V)$. Set

$$
\beta_h(y, t) = \begin{cases} 
\beta_0(y) & \text{if } t = 0, \\
\beta_k(y) & \text{if } t \in ((k - 1)h, kh].
\end{cases}
$$

Fix $T > 0$ and assume that $T = Mh$ for an integer $M > 0$.

Then the following hold:

(i) For each test function $\eta \in C^2_0$, we have

$$
\left| \int_{V_T} \partial^h_t \eta(\beta_h - \beta_0) \, dx \, dt + \int_{V_T} \text{div} \left( \beta_h D[\Psi'(\beta_h)] \right) \, dx \, dt \right| \leq C_\eta h,
$$

where $\partial^h_t \eta(x, t) = (\eta(x, t + h) - \eta(x, t))/h$ and $V_T = V \times (0, T)$.

(ii) There exists a subsequence $\{h_m\}_{m=1}^\infty$ converging to 0 and $\beta \in L^1(V_T)$ such that $\{\beta_{h_m}\}_{m=1}^\infty$ converges to $\beta$. Furthermore, $\beta$ satisfies the parabolic equation (5.8).

**5.3. Time-step approximations for $u$.** Finally, we return to the approximation scheme (5.2) and consider the convergence problem as $h \to 0$.

We first record some uniform estimates.

**Theorem 5.2.** Fix $h > 0$ and inductively define $u_{k+1}$ to be the unique minimizer of $I_k[\cdot]$ over Diff$(\bar{U}, \bar{V})$. Define

$$
u^h(\cdot, t) = \begin{cases} 
u_0(\cdot) & \text{if } t = 0, \\
u_k(\cdot) & \text{if } t \in ((k - 1)h, kh].
\end{cases}
$$

Fix $T > 0$ and assume that $T = Mh$ for an integer $M > 0$. Set $U_T = U \times (0, T)$.

Then the following hold:

(i) For each $t \in [0, T]$ we have that $\nu^h(\cdot, t) \in \text{Diff}(\bar{U}, \bar{V}) \cap C^{l+1,\alpha}(\bar{U}, \mathbb{R}^n)$ and there are constants $C_1, C_2 > 0$ depending only on $u_0$ such that

$$
C_1 \leq \det D(\nu^h)^{-1} \leq C_2.
$$

(ii) There exists a constant $C > 0$, depending only on $u_0$, such that

$$
\sum_{k=0}^{M-1} \int_U \left| u_{k+1} - u_k \right|^2 \, dx \leq Ch.
$$
(iii) For each test function $v \in C^2$, we have

\begin{equation}
\left| \int_U u^h \cdot v_t - DF(Du^h) : Dv \, dx \, dt + \int_U u_0 \cdot v(\cdot, 0) \, dx \right| \leq \frac{h}{2} C \sqrt{T} \|v_t\|_{L^\infty(U_T)}.
\end{equation}

**Proof.** 1. Set $\beta_0 = \det Du_0^{-1}$. Since $u_0 \in \text{Diff}(\bar{U}, \bar{V})$ we have that

\[ 0 < C_1 := \min_{\bar{V}} \beta_0, C_2 := \max_{\bar{V}} \beta_0 < +\infty. \]

According to the discussion above, we can choose inductively $u_{k+1}$ to be the unique minimizer of $I_k$ over $\text{Diff}(\bar{U}, \bar{V})$.

2. The inequality $I_k(u_{k+1}) \leq I_k(u_k)$ implies that

\[ \sum_{k=0}^{M-1} I_k(u_{k+1}) \leq \sum_{k=0}^{M-1} I_k(u_k). \]

Therefore

\begin{equation}
\frac{1}{2} \sum_{k=0}^{M-1} \int_V |u_{k+1} - u_k|^2 \, dx \leq h \int_U \Phi(\det Du_0) - \Phi(\det Du_M) \, dx \leq 2h|U| \max_{\{\tau^2, \tau^1\}} |\Phi|.
\end{equation}

This proves (ii).

3. Suppose now that $v \in C^2$, and set $t_k = kh$, $v_k = v(\cdot, kh)$, and $U_k = U \times (t_k, t_{k+1})$. Then

\[ \int_U u^h \cdot v_t - DF(Du^h) : Dv \, dx \, dt = \sum_{k=0}^{M-1} \int_{U_k} u_{k+1} \cdot v_t \, dx \, dt - DF(Du_{k+1}) : Dv \, dx \, dt. \]

We recall that $(u_{k+1} - u_k)/h = \text{div}(DF(Du_{k+1}))$ and continue to calculate that

\[ \int_U u^h \cdot v_t - DF(Du^h) : Dv \, dx \, dt = \sum_{k=0}^{M-1} \int_{U_k} u_{k+1} \cdot v_t + \left( \frac{u_{k+1} - u_k}{h} \right) \cdot v \, dx \, dt \]

\[ = \sum_{k=0}^{M-1} \int_U u_{k+1} \cdot (v_{k+1} - v_k) \, dx + \int_U (u_{k+1} - u_k) \cdot v_k \, dx + \frac{1}{2} \sum_{k=0}^{M-1} \int_U \left( \frac{u_{k+1} - u_k}{h} \right) \cdot \left( \int_{t_k}^{t_{k+1}} v - v_k \, dt \right) \, dx \]

\[ = \int_U u_M \cdot v_M - u_0 \cdot v_0 \, dx + \frac{1}{2} \sum_{k=0}^{M-1} \int_U \left( \frac{u_{k+1} - u_k}{h} \right) \cdot \left( \int_{t_k}^{t_{k+1}} v - v_k \, dt \right) \, dx. \]
Taking into account \( v_M = v(T) = 0 \) and 
\[
\left| \int_{t_k}^{t_{k+1}} v - v_k \, dt \right| \leq \frac{h^2}{2} \max_{U_T} |v_t|,
\]
we conclude that
\[
\left| \int_{U_T} u^h \cdot v_t - DF(Du^h) : Dv \, dx dt + \int_U u_0 \cdot v(\cdot, 0) \, dx \right|
\leq \frac{h}{2} ||v_t||_{L^\infty(U_T)} \sum_{k=0}^{M-1} \int_U |u_{k+1} - u_k| \, dx
\leq \frac{h}{2} ||v_t||_{L^\infty(U_T)} \left( \sum_{k=0}^{M-1} \int_U |u_{k+1} - u_k|^2 \, dx \right)^{\frac{1}{2}} M^{\frac{1}{2}}.
\]
(5.19)

We combine (5.18) and (5.19) to finish up the proof of (iii).

This theorem provides some uniform estimates, but it remains an unsolved problem to show that as \( h \to 0 \), the approximation \( u^h \) converges somehow to a solution \( u \) of (1.3). One particular issue is that we do not know if the gradients \( Du^h \) converge strongly in \( L^2 \).

Our belief is that although the scheme (5.2), (5.3), and (5.4) is obviously extremely natural, we do not currently know how fully to exploit the minimization structure. We have here a problem in the “time-dependent calculus of variations,” but we do not have enough experience to understand, for instance, the proper choices of comparison functions to employ in our variational principles. The direct PDE and ODE methods in sections 2 and 3 provide a way around this difficulty for the special case of the nonlinearity (1.6).

REFERENCES


