(5.0) Recall: last time we defined $U_{q}\mathfrak{g} = \tilde{U}/\text{radical}$. $R \in U_{q}\mathfrak{g}^{\otimes 2}$
the canonical element of the non-degenerate pairing $\langle , , \rangle : U^+ \times U^- \to \mathbb{C}$.
Thm. $(U_{q}\mathfrak{g}, R)$ is a quasi-triangular Hopf algebra.
Let $u = \mu \circ (S \otimes 1)(R_{21})$ be the Drinfeld element. Then $S^2(x) = ux u^{-1}$
\forall x \in U_{q}\mathfrak{g}$.

$$S^2(h) = h$$
$$S^2(e_i) = S(-e_i \tilde{e}_i) = k_i e_i \tilde{e}_i = q_i e_i$$
$$S^2(\tilde{e}_i) = S(-\tilde{e}_i e_i) = k_i \tilde{e}_i e_i = q_i \tilde{e}_i$$

Choose $p \in \mathfrak{g}^*$ st. $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) = \alpha_i \rho_i$. Then above calculation shows that $S^2(x) = k^p x \tilde{k}^p = q^p x q^{-p}$.
Define $C_q := q^p \tilde{u}$ (quantum Casimir element).
Cor. $C_q = q^p \tilde{u}$ is central.

(5.1) $R \in U_{q}\mathfrak{g}^{\otimes 2}$ has the form

$$R = q \sum \alpha \otimes \beta \quad \text{when} \quad \{x_i\} \text{ is a basis of } \mathfrak{g}$$

$$\{\tilde{x}^i\} \text{ dual basis of } \mathfrak{g} \text{ w.r.t. } (\cdot, \cdot) \quad \{a_\alpha\} \text{ is a homogeneous basis of } U^+ = \bigoplus U_{-\alpha}$$

of $U = \bigoplus U_\mu$ and $\{b_\alpha\}$ (dual) basis of $\tilde{U} = \bigoplus \tilde{U}_\mu$.

Proof: Let $n = |\mathfrak{g}|$ and $\{x_i\}_{i=1}^n$ be an. basis of $\mathfrak{g}$ w.r.t $(\cdot, \cdot)$.

Claim 1: $\langle x_1^{m_1} \cdots x_n^{m_n}, x_1^{m_1'} \cdots x_n^{m_n'} \rangle = \delta_{m_1, m_1'} \cdots \delta_{m_n, m_n'} \prod_{t \neq m_1, \cdots, m_n} t$.

Proof: easy induction on $\min (\sum m_i, \sum m_i')$.
Claim 2. For $p, p' \in U^0$, $a \in U^*_p$ be $U^{-}_p$ we have

$$\langle p a, p' b \rangle = \langle p, p' \rangle \langle a, b \rangle$$

Proof of claim 2: let us assume $p$ and $p'$ are monomials in $\{x_i\}$.

We first prove that $\langle p a, b \rangle = 0$ if $p \neq 1$. This is clear since

$$\langle p a, b \rangle = \langle p \otimes a, \Delta^t(b) \rangle = \langle p \otimes a, 1 \otimes b \rangle = \varepsilon(p) \langle a, b \rangle.$$  

Finally $\langle p a, p' b \rangle = \langle \Delta(p), \Delta(a), p \otimes b \rangle = \langle p \otimes a, p' \otimes b \rangle$

(since by weight reasons only relevant term of $\Delta(a)$ is $1 \otimes a$ and

$\Delta(p) = p \otimes 1 + \ldots$ where ... has non-trivial monomials $p^n$ on the second factor and $\langle p^n a, b \rangle = 0$).

Hence if $\{a_{\ell}\}$ $\{b_{\ell}\}$ are bases (hgs) of $U^*_p$ and $\overline{U^*_p}$ dual to each other

then

$$R = \sum_{m_1, \ldots, m_n \geq 0} \frac{t^{m_1 + \ldots + m_n}}{m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n} a_{\ell} \otimes x_1^{m_1} \ldots x_n^{m_n} b_{\ell}$$

$$= \exp \left( \sum_{i} x_i \otimes x_i \right) \sum_{\ell} a_{\ell} \otimes b_{\ell} = q^t \sum_{\ell} a_{\ell} \otimes b_{\ell} \quad \square$$

(5.2) For each $i, j \in I$, let $t_i = 1 - u_{ij}$ and define

$$\Theta^{+}_{ij} = \sum_{s=0}^{r} (-1)^s \left[ \begin{array}{c} r \\ s \end{array} \right] q_i^{r-s} e_j e_i^s$$

$$\Theta^{-}_{ij} = \sum_{s=0}^{r} (-1)^s \left[ \begin{array}{c} r \\ s \end{array} \right] q_i^{s-r} f_j f_i^s$$

Recall

$$\binom{n}{q}_{q} = q^{n-q} \frac{[n]_{q}!}{[q]_{q}!}$$

$$[n]_{q}! = [n]_{q} \ldots [1]_{q}$$

$$\binom{n}{m}_{q} = \frac{[n]_{q}!}{[n-m]_{q}! [m]_{q}!}$$

and $q_i = q d_i \quad (\forall i \in I)$. 
Prop. \( \forall i \neq j \in I, \; \Theta^*_i \in \text{Rad}^* \)

Proof. Let us prove it for + case. - case is proved similarly. (or one could use automorphism \( \omega \) of \( \tilde{U} \) : \( \omega(e_i) = e_i \), \( \omega(p_i) = e_j \), \( \omega(h) = -h \)).

Recall that we introduced derivations \( \Gamma_i, \Gamma'_i \) of \( \tilde{U}^* \) last time:

\[
\begin{align*}
\Gamma_i(e_j) &= \delta_{ij} \\
\Gamma_i(x^{(x')}) &= q^{(p_i x')} \Gamma_i(x) x' + x \Gamma_i(x') \\
\Gamma'_i(e_j) &= \delta_{ij} \\
\Gamma'_i(x^{(x')}) &= \Gamma'_i(x) x' + q^{(p_i x')} x \Gamma'_i(x')
\end{align*}
\]

It suffices to prove that \( \Gamma_k(\Theta^*_j) = 0 \) \( \forall k \in I \).

For \( k \neq i, j \), \( \Gamma_k(\Theta^*_j) = 0 \) is clear.

For \( k = j \):

\[
\Gamma_j(\Theta^*_j) = \left[ \sum_{s=0}^{r} \left( \begin{array}{c} r \\ s \end{array} \right) q_i^{s} \right] e_i = \left[ \sum_{s=0}^{r} \left( \begin{array}{c} r \\ s \end{array} \right) q_i^{s} \right] e_i
\]

Use the identity \( \left[ n+1 \atop m \right] = \left[ n \atop m \right] + \left[ n \atop m-1 \right] \) and prove by induction on \( r \) that

\[
\sum_{s=0}^{r} (-1)^s \left[ r \atop s \right] q_i^{s} = 0.
\]

For \( k = i \):

\[
\Gamma_i(e_i^n e_j e_i^m) = q_i^{2m+n-1+a_{ij}} \left[ n \atop c_i \right] e_i^n e_j e_i^m
\]

Thus coeff of \( e_i^{r-a} e_j e^a \) in \( \Gamma_i(\Theta^*_j) \) is given by \((0 \leq a \leq r - 1)\)

\[
(-1)^a \left[ r \atop a \right] q_i^{2a+r-a-1+a_{ij}} \left[ r-a \atop c_i \right] + (-1)^{a+1} \left[ r \atop a+1 \right] q_i^a \left[ a+1 \atop c_i \right]
\]

\[
= (-1)^a \frac{[r]!}{[r-a-1]! [a]!} \left( q_i^a - q_i^a \right) = 0
\]
Remark: in fact \( \text{rad}^s = \langle \Theta_j^* : i \neq j \rangle \), but we won’t need it
so I will not prove it.

(5.3) Representations of \( U_q g \).

A representation \( V \) of \( U_q g \) is called \( g \)-diagonalizable if \( V = \bigoplus_{\mu \in g^*} V[\mu] \)

\[ V[\mu] = \{ v \in V \mid h \cdot v = \mu(h) v \ \forall \ h \in g \} \]

Category \( \Theta \): An \( g \)-diagonalizable repn. of \( U_q g \) is in category \( \Theta \) if

\[ \exists \lambda_1, \ldots, \lambda_r \in g^* \quad \text{such that} \quad V[\mu] \neq 0 \Rightarrow \mu = \lambda_i \text{ for some } i = 1, \ldots, r. \]

\( P(V) := \{ \mu \in g^* \mid V[\mu] \neq 0 \} \) weights of \( V \). Then the condition

above can be written as: \( P(V) \subset \bigcup_{i=1}^{r} \lambda_i - \mathbb{Q}^+ \)

Verma Modules: \( \forall \lambda \in g^* \) define \( M_\lambda \in \Theta \) as repn. gen by \( 1_\lambda \) s.t.

\[ e_i \cdot 1_\lambda = 0 \quad (\forall i \in I) \quad h \cdot 1_\lambda = \lambda(h) 1_\lambda. \]

In other words, \( M_\lambda \) is quotient of \( U_q g \) by left ideal generated

by \( e_i (i \in I) \) and \( h = \lambda(h) (h \in g) \). It is the universal highest

weight representation of highest weight \( \lambda \). That is, if \( V \in \Theta \)

and \( v \in V[\mu] \) is a primitive vector (i.e. \( e_i v = 0 \quad \forall i \in I \)) then

\( \exists! \) \( U_q g \)-module homomorphism \( M_\mu \rightarrow V \)

\[ 1_\mu \rightarrow v \]

Any proper submodule of \( M_\lambda \) avoids the highest weight vector. Hence

\( \exists! \) max’l proper submodule \( I_\lambda \) of \( M_\lambda \). Define \( L_\lambda = M_\lambda / I_\lambda \)

Clearly \( L_\lambda \) is irreducible. Conversely if \( V \) is an \( \text{corr. repn.} \in \Theta \),

choose \( \mu \in P(V) \) to be a max’l element. Then \( e_i v = 0 \quad \forall i \in I \quad \forall v \in V[\mu] \).

\[ \Rightarrow M_\mu \rightarrow V \text{ and hence } L_{1\mu} \simeq V \text{ (by irreducibility).} \]
Remark: Category $\mathcal{O}$ is a tensor category. Moreover for $V, W \in \mathcal{O}$

$$R_{V,W} = \pi_V \otimes \pi_W (R)$$ is well defined ($\pi_V, \pi_W : U_q g \to \text{End}(V), \text{End}(W)$)
element of $\text{End}(V \otimes W)$. Thus $\mathcal{O}$ is braided by commutativity

constraint $C_{V,W} = \circ \circ R_{V,W}$.

Lemma $C_q$ acts as $q^{(\lambda + 2p, \lambda)} \cdot \text{Id}$ on $M_{\lambda}$.

Proof $C_q = q^{2p} \tilde{u}$ and $\tilde{u} = \mu \circ S' \otimes S (R_{x_1})$. Then $\tilde{u}$ is

of the form $(\sum S' (b_1) S (a_1)) q^{\sum x_i \otimes x'_i}$

On $1_{\lambda}$, $\sum x_i x'_i$ acts by $\sum \lambda(x_i) \lambda(x'_i) = (\lambda, \lambda)$

$S(a_1) \in U^+ \Rightarrow S(a_1) 1_{\lambda} = 0$ unless $a_1 = 1$.

$S(a_1) \in U^+ \Rightarrow S(a_1) 1_{\lambda} = 0$ unless $a_1 = 1$.

$\Rightarrow C_q 1_{\lambda} = q^{(\lambda, 2p)} q^{(\lambda, \lambda)} 1_{\lambda} = q^{(\lambda + 2p, \lambda)} 1_{\lambda}$. Finally

$M_{\lambda}$ is generated by $1_{\lambda}$ and $C_q$ is central.

Example of $sl_2$. Let $\lambda \in \mathbb{C}$. Then $M_{\lambda}$ has the form

$$M_{\lambda} = \text{span of } \{ 1_{\lambda}, f 1_{\lambda}, f^2 1_{\lambda}, \ldots \}$$

Notation $f^{(r)} = f^{r \choose [r]}$. $U_q sl_2$ action on $M_{\lambda}$ is given by

$$f \cdot \left( f^{(r)} 1_{\lambda} \right) = [r + 1] f^{(r + 1)} 1_{\lambda}$$

$$h \cdot f^{(r)} 1_{\lambda} = (\lambda - 2r) f^{(r)} 1_{\lambda}$$

Now we have the following commutation relation

$$e f^{(r)} = f^{(r)} e + \frac{q^{r-1} - q^{r+1}}{q - q^{-1}} f^{(r+1)}$$
\[ \Rightarrow e \cdot \left( f^{(r)} \mathbf{1}_\lambda \right) = \left[ \lambda - r + 1 \right] f^{(r-1)} \mathbf{1}_\lambda \]

If \( \lambda \in \mathbb{N} \), \( M_\lambda \) has a submodule generated by \( f^{(\lambda+1)} \mathbf{1}_\lambda \). Otherwise, \( M_\lambda \) is irreducible. Thus for \( \lambda \in \mathbb{N} \) we get simple \( L_\lambda \) of dimension \( \lambda + 1 \).

(5.4) Integrable representations. \( V \in \mathfrak{g} \) is said to be integrable if each \( f_i \) acts locally nilpotently on \( V \).

\[ \mathfrak{g}_{\text{int}} = \text{subcategory of integrable representations (extension closed subquotient/intertwining)} \]

(again a braided tensor category).

Theorem. Simple (or irreducible) objects of \( \mathfrak{g}_{\text{int}} \) are \( L_\lambda \)'s where \( \lambda \in \mathcal{P}_+ \) (i.e. \( (\lambda, \alpha_i) \in \mathbb{N} \forall i \)). \( \mathfrak{g}_{\text{int}} \) is a semisimple category.

Proof. We prove this theorem in the following steps:

(a) \( L_\lambda \) is integrable iff \( \lambda \in \mathcal{P}_+ \). In this case

\[ L_\lambda = M_\lambda / \langle f_i^{\lambda(h_i) + 1} \mathbf{1}_\lambda \rangle \]

(b) \( V \in \mathfrak{g}_{\text{int}} \) is irn. \( \Rightarrow V \cong L_\lambda \) for some \( \lambda \in \mathcal{P}_+ \)

(c) \( V \in \mathfrak{g}_{\text{int}} \Rightarrow V \) is sum of simple objects (and hence a direct sum).

(a) Let \( \lambda \in \mathcal{P}_+ \). Then \( J_\lambda = \text{submodule gen. by } f_i^{\lambda(h_i) + 1} \mathbf{1}_\lambda \) (i.e.) is a proper submodule

\[ e_i f_i^{(r)} \mathbf{1}_\lambda = \left[ \lambda(h_i) - r + 1 \right] q_i f_i^{(r-1)} \mathbf{1}_\lambda \]

\[ \overline{L}_\lambda = M_\lambda / J_\lambda \quad \Rightarrow \quad L_\lambda. \]
Claim. \( \widetilde{L}_\lambda \) is integrable. This is clear since by some relations each product \( f_i^N f_j \) can be written in terms of \( f_i^a f_j f_i \) with \( 0 < a < r \) \((r = 1-a_{ij})\).

Hence \( L_\lambda \) is integrable being a quotient of \( \widetilde{L}_\lambda \). Moreover if \( L_\lambda \neq L_{\lambda'} \)
then will exist \( \nu \in \widetilde{L}_\lambda [X] \) s.t. \( e_i \nu = 0 \ \forall \ i \in I \), \( \lambda' \in \mathbb{P}^+ \).

But then using \( q \)-Casimir element \( (\lambda' + 2 \rho, \lambda') = (\lambda + 2 \rho, \lambda) \Rightarrow \lambda = \lambda' \)

Now assume \( \lambda \in \mathbb{P}^+ \) is s.t. \( L_\lambda \) is integrable. \( \forall \ i \in I \ \exists \ N_i \) s.t.
\[
f_i^{N_i+1} \lambda = 0 \ \land \ f_i^{N_i} \lambda \neq 0.
\]
Again we get
\[
0 = e_i f_i^{(N_i+1)} \lambda = \left[ \lambda(h_i) - N_i \right] f_i^{N_i} \lambda \Rightarrow \lambda(h_i) = N_i \ \forall i \in I.
\]

(6) has already been proved.

(c) Let \( V \in \Theta^0 \). \( V^0 = \{ v \in V \mid e_i v = 0 \ \forall i \} = \bigoplus_{\lambda \in \mathbb{P}^+} V^0[\lambda] \)
each \( \lambda \in \mathbb{P}^+ \) and set \( V_1 = \) submodule gen by \( v \in V^0 \ (\forall v \in V^0) \).

Then \( V_1 \) is sum of simple modules and hence a direct sum. We claim \( V_1 = V \). If not \( V_2 := V/V_1 \) is again integrable and hence contains a primitive vector \( \nu \in V_2 [X] \quad \lambda \in \mathbb{P}^+ \).

\( e_i v \in V_1 \Rightarrow \lambda' \leq \lambda \) for some \( \lambda \in \mathbb{P}^+ \) s.t. \( V^0[\lambda] \neq 0 \).

q. Casimir \( \Rightarrow (\lambda' + 2 \rho, \lambda') = (\lambda + 2 \rho, \lambda) \Rightarrow \lambda = \lambda' \) contradiction.

Proof of (\( \ast \)):
\[
\lambda, \lambda' \in \mathbb{P}^+ \quad \lambda \geq \lambda' \quad (\lambda + 2 \rho) = (\lambda' + 2 \rho, \lambda) \Rightarrow \lambda = \lambda'.
\]
\[
\lambda - \lambda' = \sum_{i \in I} n_i \alpha_i, \ n_i \geq 0 \quad \text{Now} \quad (\lambda, 2) - (\lambda', 2) \geq (\lambda, \lambda - \lambda') = \sum_{i \in I} n_i (\lambda + 2 \rho, \alpha_i)
\]
\[
\Rightarrow \quad 0 = (\lambda + 2 \rho, \lambda) - (\lambda' + 2 \rho, \lambda') \geq 0 \Rightarrow n_i = 0 \ \forall i.
\]