(18.0) Recall: $Y_k q = \text{Yangian of a simple f.d. Lie algebra } q$. Here $t \in \mathbb{C}^*$ is a deformation parameter.

$\text{Rep}_Y = \text{category of f.d. reps. of } Y = Y_k q$.

Last time we introduced Drinfeld coproduct which endows $\text{Rep}_Y$ with a monoidal tensor structure. For $\sigma(V_1) \cap \sigma(V_2) = \emptyset$, the Yangian acts on $V_1 \otimes V_2$ via

$$\xi_i(u) \mapsto \xi_i(u) \otimes \xi_i(u)$$

$$x_i^+(u) \mapsto x_i^+(u) \otimes 1 + \int \frac{1}{u - v} \xi_i(v) \otimes x_i^+(v) \, dv$$

$$x_i^-(u) \mapsto 1 \otimes x_i^-(u) + \int \frac{1}{u - v} x_i^-(v) \otimes \xi_i(v) \, dv$$

$u$ is outside of $C_j$ \quad \left( j = 1, 2 \right)$

$C_j$ encloses $\sigma(V_j)$ and avoids $\sigma(V_j)$.

Let $V_1 \otimes V_2$ denote the resulting representation. For arbitrary $V_1$ and $V_2$ we obtain a rational family of reps. $\left\{ V(s) \otimes V_2 \right\}$ with poles at $s \in \sigma(V_2) - \sigma(V_1) = \left\{ a_j - a_i \mid a_j \in \sigma(V_j), j = 1, 2 \right\}$.

(18.1) Thm. Then exist meromorphic functions $R^{0, \pm}_{V_1, V_2}(s) : \mathbb{C} \to \text{End}(V_1 \otimes V_2)$ s.t.

(1) $R^{0, \pm}_{V_1, V_2}(s)$ is holomorphic and invertible for $\pm \text{Re}(s/k) \gg 0$, and

$$\left[ R^0(s), R^0(s') \right] = 0$$

$$R^0(s) \sim 1 \pm \kappa \Delta^0 s^\pm + \cdots \quad \text{as } s \to \infty$$

$$\pm \text{Re}(s/k) \gg 0$$

(2) $\sigma \circ R^{0, \pm}_{V_1, V_2}(s) : V_1(s) \otimes V_2 \to V_2 \otimes V_1(s)$ is a morphism in $\text{Rep}_Y$.

(3) $R^{0, +}_{V_1, V_2}(s) \circ R^{0, +}_{V_1, V_2}(-s) \circ \sigma$
(4) For \(V_1, V_2, V_3 \in \text{Rep} Y\)
\[
R^o_{V_1 V_2, V_3}(s) = R^o_{V_1 V_3}(s_1 + s_2) R^o_{V_2 V_3}(s_2)
\]
\[
R^o_{V_1, V_2(s_1) \otimes V_3}(s_1 + s_2) = R^o_{V_1 V_3}(s_1 + s_2) R^o_{V_1 V_2}(s_1)
\]
\[
(5) \quad R^o_{V_1(a), V_2(b)}(s) = R^o_{V_1 V_2}(s + a - b)
\]

Proof: We construct \(R^o_{V_1 V_2}(s)\) in the same way as for \(U_q(\mathfrak{g})\) case (see Lecture 12).

Step 1. Define \(B(T) = \left( [d_{i} a_{j}]_{T} \right)_{i,j \in I}\). Then (Khoroshkin-Teleskey)
\[
B(T)^{-1} = \frac{1}{[2]_T} C(T)
\]
where \(l = m h^\vee \), \(h^\vee = \text{dual Coxeter number}\)
\(\Theta = 1 + p(\theta)\)
and \(m = 1, 2, 3\) if \(g_j\) is of type ADE, BCF or G resp.
\(p \in \mathfrak{g}^*\) is such that \(p(h_i) = 1, 1 \leq i \leq r\).
\(\Theta \in \mathfrak{g}^*\) is the longest root.
\[
C(T) \text{ has entries from } \mathbb{Z} [T^\pm] , \quad C_{ij}(T) \quad i,j \in I
\]
\[
c_{ij}(T) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} T^r
\]
(finite sum)

Step 2. Define
\[
A(s) = \exp \left[ - \sum_{i,j \in I} \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} \int_{C_i} \left( t_i'(v) \otimes t_j(v + s + \frac{r(1 + r)}{2}) \right) dv \right]
\]
\[
t_i'(u) = \log \xi_i(u)
\]
We explained in Lecture 12 how to make sense of this logarithm.

Check (see §12.11) \(A(s)\) is a rat'l fn. of \(s\), regular at \(0\)
\[
A(s) = 1 - \frac{1}{h^2} \sum_{i} \frac{s^2}{\xi_i(s)} + O(s^3)
\]
Step 3. \( R^0(s + 2lh) = A(s) R^0(s) \)

We will study additive difference equations in detail. The one above can be solved as

\[
R^0_{s+}(s) = A(s) A(s + 2lh) A(s + 4lh) \ldots
\]

\[
R^0_{s-}(s) = A(s - 2lh) A(s - 4lh) A(s - 6lh) \ldots
\]

The theorem (except for (ii)) is proved exactly as its \( \mathbb{U}_q(kG) \) counterpart.

(18.2) Conjecture. There exist meromorphic twist (tensor structure) on the identity functor \( (\text{Rep} Y, \otimes) \rightarrow (\text{Rep} \mathbb{U}_q(kG), \otimes) \), which is in fact rational.

In more detail, we have iso. of reps., rati'd in \( s \), natural in \( V_1, V_2 \):

\[
R^0_{V_1, V_2}(s) : V_1(s) \otimes V_2 \rightarrow V_1(s) \otimes V_2
\]

s.t. \( \forall V_1, V_2, V_3 \in \text{Rep} Y \) the following diagram is commutative:

\[
\begin{array}{ccc}
V_1(s + s_2) \otimes V_2(s_2) \otimes V_3 & \xrightarrow{\text{Id} \otimes R^0_{V_1, V_2}(s_2)} & V_1(s + s_2) \otimes V_2(s_2) \otimes V_3 \\
\downarrow & & \downarrow \\
(V_1(s) \otimes V_2)(s_2) \otimes V_3 & \xrightarrow{R^0_{V_1, V_2}(s_2)} & V_1(s + s_2) \otimes V_2(s_2) \otimes V_3
\end{array}
\]

Conjecture is true for \( q = s1_2 \).
Fix a subset $\Pi \subseteq C$ s.t. $\Pi + \text{diag} \frac{k}{2} \subseteq \Pi \quad (\forall j \in I)$.

**Definition.** $\text{Rep}^\Pi Y$ is the full subcategory of $\text{Rep} Y$ consisting of representations, the Drinfeld polynomials of irreducible factors of whose composition series have roots in $\Pi$. That is, let $V \in \text{Rep} Y$ and let $0 = V_0 \subset V_1 \subset \ldots \subset V_r = V$ be its composition series.

Let $\{ P_{i,j}(u) \}_{i \in I}$ be Drinfeld poly. of $V_j / V_{j-1}$ $(0 \leq j \leq r)$. Then $V \in \text{Rep}^\Pi Y \iff \text{zeros of } P_{i,j} \subset \Pi \quad (\forall 0 \leq j \leq r, i \in I)$.

**Theorem.** Let $V \in \text{Rep} Y$. Then the following are equivalent:

1. $V \in \text{Rep}^\Pi Y$.
2. $\sigma(V) \subseteq \Pi$.
3. Poles of $\xi_i(u)$ are contained in $\Pi$ $(\forall i \in I)$.
4. Poles of eigenvalues of $\xi_i(u)$ are contained in $\Pi$ $(\forall i \in I)$.

$(2) \implies (3) \implies (4) \implies (1)$ are clear.

We prove $(1) \implies (2)$ by induction on the length of composition series of $V$.

**Remark.** By Knight's result, zeros of $\xi_i(u)$ are obtained by shifting the poles of its diagonal entries by $\pm \text{d} \Delta$ and $\Pi$ is stable under these shifts. So $(3)$ (or $(4)$) $\implies \xi_i(u)$ are regular and invertible on $C \setminus \Pi$. 
(18.4) Base Case: \( V \) is irreducible. Let \( \{P_i\}_{i \in I} \) be Drinfeld polynomials of \( V \). Since \( V \in \text{Rep}^\mathbb{C} \), zeros of \( P_i \) lie in \( \Pi \) \((\forall i \in I)\). 

To prove: \( \{\xi_i(u), \chi^+_i(u)\} \) have poles in \( \Pi \).

Let us write \( V = \bigoplus V[\mu] \) as a \( \mathfrak{g} \)-module. Let \( \lambda \in \mathfrak{g}^* \) be the highest weight. We will use the following properties of \( V \):

(P1) For a weight \( \mu < \lambda \), the weight space \( V[\mu] \) is spanned by \( \{x_{\alpha_i}^{-}, V[\mu+\alpha_i]\}_{i \in I, r \in \mathbb{R}} \).

(P2) If \( u \in V[\mu] \) is annihilated by \( x_{\alpha_i}^{-} \) \((\forall i \in I, \text{ real})\) and \( \mu < \lambda \), then \( u = 0 \).

Recall: \( \mu < \lambda \) means \( \lambda - \mu = \sum \{n_i \alpha_i\} \) \((n_i \in \mathbb{N})\). Define

\[
\text{ht} (\lambda - \mu) = \sum n_i.
\]

S(k): \( \forall i \in I \), \( \xi_i(u), \chi^+_i(u) \) have poles in \( \Pi \), \( \forall \mu \) s.t. \( \text{ht}(\lambda - \mu) < k \), and \( x_{\alpha_i}^{-}(u) \) has poles in \( \Pi \) \( \forall \mu \) s.t. \( \text{ht}(\lambda - \mu) < k \).

We will prove \( \text{S(k)} \) by induction on \( k \).

\[
\text{S(k)} \quad \begin{cases}
\text{S(k)} \quad \text{is the statement for this subspace} \\
\bigoplus V[\mu] : \text{ht}(\lambda - \mu) = k
\end{cases}
\]

\( \text{S(0)} \) is clear since \( \chi^+_i(u) V[\lambda] = 0 \)

\[
\xi_i(u) = \frac{P_i(u + d \lambda)}{P_i(u)} \quad \text{and zeros of} \quad P_i(u) < \Pi.
\]

\[
\xi_i(u)_{\lambda} = \frac{P_i(u + d \lambda)}{P_i(u)} \quad \text{and zeros of} \quad P_i(u) < \Pi.
\]
Assume \( S(k') \) for every \( k' \leq k \), when \( k \geq 0 \). Let us prove \( S(k+1) \). Take \( v \) to be a wt. of \( V \) st. \( \text{ht}(\lambda - v) = k+1 \).

- Use (Y5)

\[
\chi_j^+(u) \chi_j^-(v)_{u+a_j} = \chi_j^-(v)_{v+u+a_j} \chi_j^+(u)_{u+a_j} + \xi_j k \left( \xi_i(v)_{v+a_i} - \xi_i(u)_{u+a_i} \right)
\]

RHS has poles \( f \) in \( \Pi \times \Pi \). (Induction hypothesis).

Assume \( \chi_j^+(v)_{v+a_j} \) has a pole at \( z \in \Pi \) of order \( n \). Multiply by \((v-z)^n\) and let \( v = z \) to get

\[
\chi_j^+(u)_{u+a_j} \left[ (v-z)^n \chi_j^-(v)_{v+a_j} \bigg|_{v=z} \right] = 0
\]

i.e. Image of \((v-z)^n \chi_j^-(v)_{v+a_j} \bigg|_{v=z}\) is annihilated by all \( \chi_i^r \). Hence

it must be 0, by property (P2). Contradicts the fact that \( n \) was order of the pole.

Similarly if \( \chi_i^+(u)_{u+a_j} \) has a pole of order \( n \) at \( z \in \Pi \) we get

\[
\left( (v-z)^n \chi_i^+(u)_{u+a_j} \bigg|_{v=z} \right) \chi_j^-(v)_{v+a_j} = 0
\]

\[
\text{X}
\]

i.e. \( X \) vanishes on the image of all \( \chi_j^r \neq V \chi_j^r \). But this image is \( V\chi_j \)

by (P1). So \( X = 0 \).

- Use (Y23)

\[
\xi_i(u)_{u+a_j} \chi_j^-(v)_{v+a_j} = \frac{u-v-a}{u-v+a} \chi_j^-(v)_{v+a_j} \xi_i(u)_{v+a_j}
\]

\[
+ \frac{2a}{u-v+a} \chi_j^-(u+a)_{u+a_j} \xi_i(u)_{v+a_j}
\]

\( \Rightarrow \) the image of \( \chi_j^-(v)_{v+a_j} \), poles of \( \xi_i(u) \) are contained in those of \( \xi_i(u)_{v+a_j} \) or \( \chi_j^-(u+a)_{u+a_j} \), hence contained in \( \Pi \).
Let $0 \to V_1 \to V \to V_2 \to 0$ be a short exact seq. If

\[ \sigma(V_1), \sigma(V_2) \subset \Pi \] then $\sigma(V) \subset \Pi$.

Enough to prove it for $V_2$. (by fixing index $i \in I$).

Write $V = V_1 \oplus V_2$ as vector space and every $y \in V$ has the form

\[ y = \begin{bmatrix} y^1 & y^2 \\ 0 & y^{22} \end{bmatrix}. \]

We know that $\xi^{k}(u)$, $x^{\pm}(u)$ have poles in $\Pi$. \hspace{1cm} (k = 1, 2).

(T.S.) $\xi^{12}(u)$, $x^{12}(u)$ have poles in $\Pi$.

Assume the contrary. Choose $z_0$ outside of $\Pi$ where $\xi^{12}(u)$, $x^{12}(u)$ have poles.

Max $c(t)$ such that these functions are regular at $z + c(t) r$. \hspace{1cm} (r > 0).

Let $N = \max$ of the orders of the pole of $\xi^{12}(u)$, $x^{12}(u)$ at $u = z$.

\[ H := (u - z)^N \xi^{12}(u) \bigg|_{u = z} \quad \text{and} \quad X^{\pm} := (u - z)^N x^{\pm}(u) \bigg|_{u = z}. \]

**Use (Y5) and (1,2) entry**

\[ (u - v) \left( x^{+}(u) x^{-}(v) + x^{+}(u) x^{-}(v) - x^{-}(v) x^{+}(u) - x^{-}(v) x^{+}(u) \right) \]

\[ = \frac{k}{h} \left( \xi^{12}(u) - \xi^{12}(v) \right) \]

\[ \Rightarrow (z - v) \left( X^{+} - x^{-}(v) x^{+}(v) \right) = -\frac{k}{h} H \]

Multiply by $(u - z)^N$ and let $u \to z$.

**Set** $v = z$ to get $H = 0$.

**Use (Y23) for $+$:**

\[ (u - v) \xi^{+}(u) x^{+}(v) = \quad (u - v + h) \xi^{+}(v) \xi^{+}(u) = -2k x^{+}(u) \xi(v) - 2k x^{+}(u) \xi(v) \]

Take 1,2 entry. Multiply by $(u - h - z)^N$ and let $u \to z + h$. Using assumptions on $z$ and $H = 0$ we get

\[ X^{+} \xi^{(z + h)^2} = 0. \]

By $\xi^{(z + h)^2}$ is inv. outside of $\Pi$.

\[ \Rightarrow X^{+} = 0. \]

**Use (Y23) for $-$:**

\[ (u - v + h) \xi^{-}(u) x^{-}(v) = \quad (u - v - h) \xi^{-}(u) \xi^{-}(v) = 2h \xi^{-}(u) x^{-}(u) \]

\[ \Rightarrow \quad X^{-} = 0. \]

Contradict the fact that $N$ was order of the pole of one of $\xi^{12}(u)$, $x^{12}(u)$.