The aim of this lecture is to prove the classification theorem for \( sl_2 \).

\[ Y = Y_h \otimes \mathfrak{sl}_2. \]

Thm. Given \( d \in \mathbb{C} \setminus \mathbb{Z}_0 \), let \( L(d) \) be the unique \( \mathfrak{g}_h \)-module with highest weight \( \{ d_r \}_{r \geq 0} \). Then \( L(d) \) is finite-dimensional if, and only if, there exists a nonzero \( \mathbb{C}(u) \in \mathcal{L}(u) \) such that

\[ 1 + \frac{h}{2} \sum_{r \geq 0} d_r u^{-r-1} = \frac{p(u+h)}{p(u)}. \]

(16.1) Evaluation homomorphism. For each \( a \in \mathbb{C} \) we have an algebra homomorphism \( \text{ev}_a : Y \to U \otimes \mathfrak{sl}_2 \) defined as

\[ \text{ev}_a : \begin{cases} \text{ev}_a(e) = h \\ \text{ev}_a(f) = k \\ \text{ev}_a(x) = e/f \\ \text{ev}_a(t) = a h - \frac{k}{2} (e f + f e) \end{cases} \]

For \( \lambda \in \mathbb{C} \) and \( a \in \mathbb{C} \), let \( M_\lambda(a) \) (resp. \( L_\lambda(a) \)) be the pull-back of the Verma module \( M_\lambda \) (resp. \( \mathfrak{g}_h \)-module \( L_\lambda \)) of \( \mathfrak{sl}_2 \) under \( \text{ev}_a \). Recall: \( M_\lambda \) has basis \( \{ m_\lambda(r) \}_{r \geq 0} \) with \( \mathfrak{sl}_2 \)-action given by

\[ h m_\lambda(r) = (\lambda - 2r) m_\lambda(r), \quad e m_\lambda(r) = (\lambda + r + 1) m_\lambda(r-1), \quad f m_\lambda(r) = (r + 1) m_\lambda(r+1). \]

\[ L_\lambda = M_\lambda \text{ if } \lambda \notin \mathbb{N}. \text{ If } \lambda \in \mathbb{N}, \text{ then } L_\lambda = \text{span of } m_\lambda(r) \text{ for } 0 \leq r \leq \lambda \text{ is } (\lambda+1) \text{-dim}. \]

Lemma. \( \gamma \)-action of \( M_\lambda(a) \) (or \( L_\lambda(a) \)) is given by the following
\[ \xi(u) \, m_\lambda(r) = \frac{(u-a_0)(u-a_{r+1})}{(u-a_r)(u-a_{r+1})} \, m_\lambda(r) \]

\[ x^+(u) \, m_\lambda(r) = \frac{\frac{t}{u}(\lambda-r+1)}{u-a_r} \, m_\lambda(r-1) ; \quad \bar{x}^-(u) \, m_\lambda(r) = \frac{\frac{t}{u}(r+1)}{u-a_{r+1}} \, m_\lambda(r+1) \]

where \( a_r = a + \frac{t}{2}(\lambda-2r+1) \).

Proof: Using the formula for \( ev_\lambda \), we have:

\[ t_1 \, m_\lambda(r) = (a(\lambda-2r) - \frac{t}{2} \lambda - t \, r(\lambda-r)) \, m_\lambda(r) \]

\[ \Rightarrow \text{ad} \, t_1 \text{ on } \text{Hom}_\mathbb{C}(Cm_\lambda(r), Cm_\lambda(r+1)) \text{ is given by the scalar } \]

\[ 2a + \frac{t}{u}(\lambda-2r+1) = 2a_r. \]

Hence we get \( x^+(u) \, m_\lambda(r) = (u-a_r) \, t_0 \, m_\lambda(r) \)

\[ = \frac{\frac{t}{u}(\lambda-r+1)}{u-a_r} \, m_\lambda(r). \]

Similarly for \( \bar{x}^- \). Finally, \( \xi(u) = 1 + [x^+(u) \, x^-] \).

(Recall the proof of rationality prop from previous location:

\[ x^+(u) \, x^- = (u - \frac{\text{ad} \, t_1}{2})^{-1} \, t_0 \, x^\pm. \]

□)

Change of notation. If we set \( b = a_1 = a + \frac{t}{2}(\lambda-1) \) then the

formulae of this lemma become

\[ \xi(u) \, m_\lambda(0) = \frac{u-b+\lambda h}{u-b} \, m_\lambda(0) \]

\[ \xi(u) \, m_\lambda(r) = \frac{(u-b)(u-b+2h)}{(u-b+(r+1)h)(u-b+rh)} \, m_\lambda(r) \]

\[ x^+(u) \, m_\lambda(r) = \frac{\frac{t}{u}(\lambda-r+1)}{u-b+(r+1)h} \, m_\lambda(r-1) \]

\[ \bar{x}^-(u) \, m_\lambda(r) = \frac{\frac{t}{u}(r+1)}{u-b+rh} \, m_\lambda(r+1) \]

Let us denote this repn. by \( V(\lambda,b) = ev_\lambda^* \, L_\lambda \).

\[ \sigma(V(\lambda,b)) = \left\{ \begin{array}{ll}
\{ b, b-h, \ldots, b-(\lambda-1)h \} & \text{if } \lambda \in \mathbb{N} \\
\{ b-rh : r \geq 0 \} & \text{if } \lambda \notin \mathbb{N}
\end{array} \right. \]

\[ = \bigoplus_{k=0}^{\lambda-1} \{ b-kh \} \]
(16.2) Coproduct  \( \Delta(y_0) = y_0 \otimes 1 + 1 \otimes y_0 \)
\[ \Delta(t_0) = t_0 \otimes 1 + 1 \otimes t_0 - 2t \otimes x_0 \otimes x_0^+ \]
extends to a unique algebra hom. \( Y \rightarrow Y \otimes Y \).

Proof: Using relations of the Yangian we get
\[ \Delta(x^+_i) = x^+_i \otimes 1 + 1 \otimes x^+_i + h \xi_0 \otimes x_0^+ \]
\[ \Delta(x^-_i) = x^-_i \otimes 1 + 1 \otimes x^-_i + h x_0 \otimes \xi_0 \]
Let \( t_2 = \xi_2 - h \xi_0 \xi_1 + \frac{h^2}{3} \xi_2 \).
Then \( \Delta(t_2) = t_2 \otimes 1 + 1 \otimes t_2 - 2t (x_0 \otimes x_1^+ + x^- \otimes x_1^-) \).

Now we can easily check Levednikov's relations.

Lemma: \( \Delta(x^+(u)) = x^+(u) \otimes 1 + \xi(u) \otimes x^+(u) + \ldots \)
where \( \ldots \) consists of terms of weight \((-2k) \otimes 2k+2\) \((u21)\).
Every wt of \( x^+_r \) is \( \pm 2 \) and wt. of \( \xi_r = 0 \).

Proof. We need to prove that \( \Delta(x^+_r) = x^+_r \otimes 1 + 1 \otimes x^+_r + h \sum_{k=0}^{r-1} \xi_k \otimes x^+_{r-k-1} + \ldots \)
This is proved by induction on \( r \) as follows
\[ 2 \Delta(x^+_{r+1}) = [t_1, x^+_r] \otimes 1 + 1 \otimes [t_1, x^+_r] + h \sum_{k=0}^{r-1} \xi_k \otimes [t_1, x^+_{r-k-1}] \]
\[ - 2h [x^-_0, x^+_1] \otimes x_0^+ + \ldots \]
Using \([t_1, x^+_r] = 2 x^+_r\) and \([x^-_0, x^+_1] = -\xi_1\) we are done. \( \Box \)

(16.3) Antipode and counit. \( E(y_r) = 0 \) \( y = \xi, x^+ \); \( r \geq 0 \)
defines an algebra hom. \( E : Y \rightarrow \mathbb{C} \) called the counit.
\( S(y_0) = -y_0 \) \( (y = \xi, x^+) \)
\( S(t_0) = -t_0 - 2t x_0 x_0^+ \)
defines algebra (and coalgebra) anti-homomorphism.
(16.4) Proof of Theorem. Part I.

Assume \( L(d) \) is the irreducible repn. with
\[
1 + \hbar \sum_{r \geq 0} d_r \tilde{u}^{r-1} = \frac{P(u+\hbar)}{P(u)}.
\]
Write \( P(u) = \prod_{i=1}^{n} u - a_i \); and consider \( V = C^2_{a_1} \otimes \cdots \otimes C^2_{a_N} \) (where
\( C^2_{a_i} = ev^*_\mu (C^2) \)). Then \( \chi^+_r \mid \uparrow \cdots \uparrow \rangle = 0 \ \forall \ r \geq 0 \) and
\[
\xi(u) \mid \uparrow \cdots \uparrow \rangle = \prod_{i=1}^{n} \frac{u - a_i + \hbar}{u - a_i} \mid \uparrow \cdots \uparrow \rangle.
\]

Let \( V' \) be the submodule generated by \( \mid \uparrow \cdots \uparrow \rangle \). By universal property of \( M(d) \) we have a surjective map \( M(d) \xrightarrow{\varphi} V' \) and hence \( 1 \xrightarrow{\varphi} \mid \uparrow \cdots \uparrow \rangle \).

\( L(d) \cong V' / \varphi(M'(d)) \) quotient of a free space, hence finite-dimensional.

(16.5) Proof of Theorem. Part II.

The other implication follows from a very general and useful result due to A. Malcev.

Let \( L \) be an irreducible highest weight repn. of \( T \), with highest weight vector \( 1 \) s.t.
\[
\xi(u) \mid 1 \rangle = \frac{P_1(u)}{P_2(u)} \mid 1 \rangle \text{ where } \deg P_1 = \deg P_2. \text{ Enumerate } \text{zeros of } P_1 (\ell_1, \ell_2) \text{ and } P_2 (\ell_3, \ell_4) \text{ s.t. the following condition holds }: \forall k, \text{ s.t. } 1 \leq i \leq N
\]
\[
\{ b_k - a_j \} \text{ contains a positive integer multiple of } h, \text{ then } (\ast).
\]

If \( b_k - a_k \) is minimal among those.

Thm. \( L \cong V(\lambda_1, b_1) \otimes \cdots \otimes V(\lambda_N, b_N) \)

where
\[
\lambda_i = \frac{b_i - a_i}{h} \quad (1 \leq i \leq N).
\]
This theorem completes the classification as follows.

Let \( L(d) \) be a f.d. irr repn. of \( \mu = \{ \delta r \}_{r \geq 0} \). By rationality prop.

\[
\xi(u) \cdot 1 = \left( 1 + h \sum \delta r \bar{u}^{-r} \right) 1 = \frac{P_1(u)}{P_2(u)} 1. \quad \text{Hence}
\]

\[
L(d) \cong V(\lambda_1, b_1) \oplus \ldots \oplus V(\lambda_N, b_N). \quad \text{Since} \ L \ \text{is f.d.} \ \lambda_i \in \mathbb{N} \ (\forall i)
\]

\[
\Rightarrow \alpha_i = b_i - \lambda_i h \quad (\lambda_i \in \mathbb{N}) \quad \text{and hence} \quad \frac{P_i(u)}{P_2(u)} = \frac{P(u + b_i)}{P(u)}
\]

where

\[
P(u) = \prod_{i=1}^{N} (u - b_i) \ldots (u - b_i + \delta_i - 1) b_i.
\]

(16.6) Proof of Theorem (16.5). Let \( L = V_0(\lambda_0, b_0) \oplus \ldots \oplus V(\lambda_N, b_N) \) (not needed)

Claim 1. If \( \eta \in L \) is s.t. \( \chi^*_r \eta = 0 \) (\( \forall r \geq 0 \)) \( \quad \text{and} \quad \xi_r \eta = x_r \eta \)

(\( \text{Hence} \ L \ \text{has only one submodule} \ K, \ \text{namely the one generated} \)

\( \text{by} \ m_\lambda(0) \otimes \ldots \otimes m_\lambda(\mathbb{N}) \))

Proof. Write \( \eta = \sum_{p=0}^{M} \eta_p \otimes m_\lambda(p) \) \( (\text{if} \ \lambda \in \mathbb{N} \ \text{then} \ \sum_{p=0}^{M} \chi^*_p \eta \otimes m_\lambda(p) + (\lambda - p + 1) \eta_p \otimes m_\lambda(p) = 0 \)

\( \chi^*_0 \eta = 0 \Rightarrow \sum_{p=0}^{M} \chi^*_0 \eta_p \otimes m_\lambda(p) = -(\lambda_n - M + 1) \eta_M \)

\( \chi^*_0 \eta_M = 0 \quad - (1) \)

\( \text{Coeff. of} \ m_\lambda(M) \ \text{gives} \ x_0 \eta_M = 0 \quad - (1) \)

\( \text{Coeff. of} \ m_\lambda(M-1) \ \text{gives} \ x_0 \eta_{M-1} = -(\lambda_n - M + 1) \eta_M \quad - (2) \)

\( (\eta \text{ is an e.v. for} \ t_1 \ \Rightarrow \ \eta_M \text{ is an eigenvector for} \ t_1 \) \quad \text{not needed} \)
Using $\Delta(x^+_r)$ and the same argument as (1) we get $x^+_r \eta_M = 0$

By induction on number of tensor factors: $\eta_M = m_{\lambda_1}(0) \otimes \ldots \otimes m_{\lambda_{N-1}}(0)$

(or a scalar multiple).

Now we show that $M = 0$. If $M \geq 1$ then apply $\Delta(x^+_r) \eta_M$ from Lemma 16.2 to get

$$x^+_r(\eta_{M-1}) + \xi(\eta_M = 0 \quad (***)$$

$$u - b_N + (M-1)k$$

$$\eta_{M-1} \in \text{Span of} \left\{ m_{\lambda_1}(0) \otimes \ldots \otimes m_{\lambda_{i-1}}(0) \otimes m_{\lambda_i}(1) \otimes m_{\lambda_{i+1}}(0) \otimes \ldots \otimes m_{\lambda_{N-1}}(0) \right\}$$

$$\xi(\eta_M \in \text{by (2)}$$

$$x^+_r(\xi) = \left( \prod_{j=1}^{i-1} \frac{u - b_j + \lambda_j k}{u - b_j} \right) \frac{t}{u - b_i} \eta_M$$

Cleaning denominator in (***), we get

$$\prod_{i=1}^{N-1} b_N - (M-1)k - a_i = 0$$

$$\Rightarrow \exists j \text{ s.t. } b_N - a_j = (M-1)k \in \mathbb{N}$$

by assumption (**) we get $t \lambda_N = b_N - a_N \leq (M-1)k$ 

$\lambda_N \in \mathbb{N}$ contradicts

Claim 2. $K = L$. If $K$ is proper submodule $M \leq \lambda_N$.

of $L$ then $\text{Ann}(K)$ is non-zero proper submodule of $L^*$, which

doesn't contain the h.w. vector (see below). This contradicts Claim 1 since

$$L^* \cong V(\lambda_1, b_1 - k) \otimes \ldots \otimes V(\lambda_N, b_N - k)$$

as assumption (**) holds.
Using the expression for the antipode $S$, one can easily check that

$$L^*_\lambda(a)^* \cong L^*_\lambda(a^{-h}) - (1)$$

As $sl_2$-modules we can identify $L^*_\lambda$ with $L_\lambda$ using Shapovalov form. Hence we get the following identifications as $\gamma$-modules

$$(M_{\lambda_1}(a_1) \otimes \ldots \otimes M_{\lambda_n}(a_n))^* \cong M_{\lambda_n}(a_n)^* \otimes \ldots \otimes M_{\lambda_1}(a_1)^*$$

$$\cong M_{\lambda_n}^*(a_{n-1}) \otimes \ldots \otimes M_{\lambda_1}^*(a_1) \cong M_{\lambda_1}(a_{1-1}) \otimes \ldots \otimes M_{\lambda_n}(a_{n-1}).$$

Verification of (1):

$M^*_\lambda$ : basis $\overline{m}_\lambda(r)$ with $sl_2$-action

$$\begin{cases}
    e \overline{m}_\lambda(r) = -(\lambda-r) \overline{m}_\lambda(r+1) \\
    f \overline{m}_\lambda(r) = -r \overline{m}_\lambda(r-1) \\
    h \overline{m}_\lambda(r) = -(\lambda-2r) \overline{m}_\lambda(r)
\end{cases}$$

$\Rightarrow \ t_1$ acting on $M^*_\lambda(b)$ can be written as

$$t_1 \overline{m}_\lambda(r) = -b(\lambda-2r) - \frac{\hbar \lambda}{2} - \hbar r(\lambda-r)$$

On the other hand $t_1$ acting on $M_\lambda(a)^*$:

$$t_1 \overline{m}_\lambda(r) = -a(\lambda-2r) + \frac{\hbar \lambda}{2} + \hbar r(\lambda-r) - 2(\lambda-r+1) +$$

$$= -a(\lambda-2r) - \frac{\hbar \lambda}{2} - \hbar r(\lambda-r).$$