Definition. A representation \( V \) of \( Y_k g \) is said to be highest weight representation \((\text{of h.w. } \{d_i \in C \}_{i \in I}, r \in \mathbb{N})\) if \( \exists v \in V \) s.t.

1. \( x_i^+ v = 0 \quad \forall i \in I, r \in \mathbb{N} \)
2. \( \xi_i v = d_i v \)
3. \( V = (Y_k g)_v \)

Given \( d = \{ d_i \in C \}_{i \in I, r \in \mathbb{N}} \) define \( M(d) \) (Verma module) as

quotient of \( Y_k g \) by left ideal generated by \( \{ \xi_i - d_i x_i, x_i^+ \}_{i \in I, r \in \mathbb{N}} \)

\( M(d) \) has a unique maximal proper submodule. Define \( L(d) = M(d) / M'(d) \).

Theorem. (1) Every irreducible f.d. repn of \( Y_k g \) is a highest weight repn
and hence isomorphic to \( L(d) \) for some collection \( d \).

(2) \( L(d) \) is finite-dim if and only if there exist \( P_i(u) \in C[\mathbb{C}] \)
monic

\[ \sum_{r \geq 0} d_i r u^{r+1} = \frac{P_i(u + d_i + 1)}{P_i(u)} \]

Proof of (1). Let \( V \) be a f.d. irr. repn of \( Y_k g \). \( V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu \) as \( g \)-module.
Let \( \lambda \in \mathfrak{h}^* \) be max' weight of \( V \) (i.e. \( V_\mu = 0 \forall \mu > \lambda \))
Then \( x_i^+ V_\lambda = 0 \quad \forall i \in I, r \in \mathbb{N} \).
\{\xi_i\} are commuting operators on f.d. \( V_\lambda \) and hence \( \exists v \in V_\lambda \) i.e.

\[ \xi_i v = d_i v \quad V' = (Y_k g)_v, \quad v \subset V \text{ is proper submodule} \]

(\( i \in I \))

By im of \( V \), \( V' = V \). \( \square \)
(a) is proved by reduction to sl₂-case. Let us assume the statement for sl₂ and prove it for arbitrary g.

(15.1) Easy part. Assume $L(d)$ is f.d. Then so is the irreducible quotient (as a $\gamma_{d,\ell}(\mathfrak{sl}_2)$-module) of $\gamma_{d,\ell}(\mathfrak{sl}_2) \cdot 1$ (here 1 is the h.w. vector).

Notation $\gamma_i = \text{subalgebra generated by } \{ \xi_{i,r}, \gamma_{i,r}^1, r \geq 0 \} \cong \gamma_{d,\ell}(\mathfrak{sl}_2)$.

Hence $1 + h \sum_{r \geq 0} d_{i,r} \gamma_{i,r}^1 = \frac{P_i(u + d + h)}{P_i(u)}$ for some polynomial $P_i(u) \in \mathbb{C}[u]$ (monic).

(15.2) Now assume the existence of $\{ P_i(u) \}_{i \in I}$. That is,

$$1 + h \sum_{r \geq 0} d_{i,r} \gamma_{i,r}^1 = \frac{P_i(u + d + h)}{P_i(u)} \quad (\forall i \in I)$$

We want to show that $L = L(d)$ is finite-dimensional.

We will prove the following two assertions:

$(15.2.1)$ $\forall \mu \leq \lambda$ s.t. $L_{\mu} \neq 0$ and $i \in I$, $\exists N > 0$ s.t.

$$L_{\mu - n\cdot i} = 0 \quad \forall n \geq N.$$

$(15.2.2)$ $\dim L_{\mu} < \infty \quad \forall \mu \leq \lambda$

Note $\gamma_{i,j}$ admits triangular decomposition. Thus $L_{\mu}$ is a span of $x_{i_1, r_1} \cdots x_{i_k, r_k} \cdot 1$ where $\sum_{i=1}^{k} r_i = \lambda - \mu$.  

This implies f.d. of o.g. mod. $L$, using $W$-action and the fact that dominant chamber is fundamental domain for $W \subset G/\mathfrak{g}^\ast$. 

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Lemma. \( \forall i \in I, \ Y_i \cdot 1 \) is an irreducible repn of \( Y_{d,h} \) (5.1).

Proof. Let \( V = Y_i \cdot 1 \). Assume on the contrary that there is a proper submodule \( V' \subseteq V \) (as \( Y_i \cdot \) modules).

\[ V = \bigoplus_{n \geq 0} (L_{\lambda \cdot n\alpha_i} \otimes V) \text{.} \]

Similarly for \( V' \). Then there exists \( 0 \neq v \in V'_{\lambda \cdot m\alpha_i} \) (\( m \geq 1 \)) s.t. \( x_{i,j}^+ v = 0 \) \( \xi_{i,r} v = \alpha_i v \) (\( \alpha_i, r \in \mathbb{C} \)).

Now \( x_{j,i}^+ v \in L_{\lambda \cdot m\alpha_i + \alpha_j} = 0 \) (\( \forall j \neq i \)).

Fix \( \{\xi_{i,r}\}_{r \geq 0} \) and let \( W' = \{ v' \in V' \mid x_{j,i}^+ v' = 0 \ \forall j \in I, \text{ read} \} \).

\( W' \) is non-zero space preserved by \( \{\xi_{j,i}\}_{j \in I} \) and hence we can find a joint eigenvector which \( \text{ will be h.v. vector for } L, \text{ contradicting its irreducibility.} \)

Lemma. For any \( r > 0 \), \( L_{\mu \cdot r\alpha_i} \) is spanned by vectors of the form

\[ X_1 \alpha_{i,k_1} X_2 \alpha_{i,k_2} \ldots X_h \alpha_{i,k_h} X_{h+1} 1 \]

\( (\lambda - \mu = \alpha_{i,1} + \ldots + \alpha_{i,h}) \), \( k_1, \ldots, k_h \in \mathbb{N} \) arbitrary and

\[ X_p = \alpha_{j,l_{i,p}} \ldots \alpha_{c,l_{r_{i,p}}} l_{i,p} \ldots l_{r_{i,p}} \in \mathbb{N} \]

\( \gamma_1, \ldots, \gamma_{h+1} \in \mathbb{N} \) and \( \gamma_1 + \ldots + \gamma_{h+1} = r \)

\( r_i, \ldots, r_h \leq r^* = \max \{ -\alpha_{j,i} \}_{j \neq i} \)

Proof: Use (Y6) and induction.
(15.5) Proof of (15.2.1). By previous lemma $L_{\mu-\nu_c}$ is spanned by vectors of the form (4) with $r_{k+1} \geq n - h \cdot r^v$ ($h = \text{height}(\lambda-\mu)$). Thus if $n - h \cdot r^v > \lambda(h) = \deg P$, we get $X_{k+1}^+ 1 = 0$ by Lemma (15.3). Hence $L_{\mu-\nu_c} = 0$ for $n > \lambda(h) + h \cdot r^v$.

(15.6) Proof of (15.2.2) is by induction on $h = \text{height}(\lambda-\mu)$.

$h = 0$ : \[ L_{\mu} = L_{\lambda} \text{ is 1-dim}. \]

$h = 1$ : $L_{\mu} = L_{\lambda} \cdot \alpha_c \subset Y_i \cdot 1$ is f.d. by Lemma (15.3).

(for some $i$)

Assume $h \geq 2$. \[ \lambda-\mu = \alpha_i_1 + \ldots + \alpha_i_h \] fix an ordering $i_1 \ldots i_h$ and define

$V_{i_1 \ldots i_h} = \text{span of } \{ X_{i_1,k_1} \ldots X_{i_h,k_h} 1 \}$ for some $k_1 \ldots k_h$.

It is enough to show that $V_{i_1 \ldots i_h}$ is finite-dim.

$V_{i_1 \ldots i_h} = \sum_{k \geq 0} x_{i_1}^{k_1} V_{i_2 \ldots i_h}$

Now using (44) $x_{i_1}^{k_1} X_{i_2}^{k_2}$ can be written as a linear combination of $X_{i_1,k_1}^{i_2,k_2}$.

Also $V_{i_1 \ldots i_h}$ is spanned by $x_{i_2}^{k_2} \ldots x_{i_h}^{k_h} 1$ with $k_2 \ldots k_h < M$ (for some fixed $M \in \mathbb{N}$)

$\Rightarrow x_{i_1}^{k_1} V_{i_2 \ldots i_h} \subset \sum_{j = i_2 \ldots i_h} (x_{i_1}^{j_1} L_{\mu+\alpha_j} + \bar{x}_{i_1}^{j_1} L_{\mu+\alpha_i})$

and we are done by induction.