Motivation. Let us define another coproduct on $U = U_q(L^g)$ as follows:

$$\Delta_s(\psi_{i,\pm m}) = \sum_{s=0}^{m} \sum_{i \in I} \psi_{i,\pm (m-s)} \otimes \psi_{i,s}$$

$$\Delta_s(E_{i,k}) = \sum_{l \geq 0} S^k E_{i,k} \otimes 1 + \sum_{l \geq 0} \text{ coassoc.}$$

For $s = 1$, this coproduct was introduced by Drinfeld, and is called Drinfeld coproduct in his honor. The expression of $\Delta_s$ given above is due to Hernandez who coined the term deformed Drinfeld coproduct for $\Delta_s$ and proved that

- $\Delta_s : U \rightarrow U \otimes U$ is an algebra homomorphism
- Coassociativity $\Delta_s \otimes 1 \circ \Delta_s = 1 \otimes \Delta_s \circ \Delta_s$

Expression of $\Delta_s$ as a contour integral was obtained only recently by G. and Toledano Laredo. Namely let $U$ and $W$ be f.d. reps of $U_q(L^g)$

$$\sigma(U) = \text{ poles of rational functions } \psi_i(z), E_i(z), F_i(z) \text{ acting on } U$$

(similarly $\sigma(W)$).

$$\Delta_s(\psi_i(z)) = \psi_i(z) \otimes \psi_i(z)$$

$$\Delta_s(E_i(z)) = E_i(z) \otimes 1 + \oint_{C_2} \frac{z^{-1}}{z-w} \psi_i(z) \otimes E_i(w) \, dw$$

$$\Delta_s(F_i(z)) = \oint_{C_1} \frac{z^{-1}}{z-w} F_i(w) \otimes \psi_i(z) \, dw + 1 \otimes F_i(z)$$

- $C_1$ and $C_2$ are contours enclosing $\sigma(U)$ and $\sigma(W)$ respectively.
- $\Delta_s(E_i(z)) = \oint_{C_2} \frac{z^{-1}}{z-w} \psi_i(z) \otimes E_i(w) \, dw$ defines a rational function.
of $S$ and $z$ in the domain where $z$ is not enclosed by $C_2$
and $\pi_V(\psi_1(5w))$ is analytic (holomorphic) within $C_2$.

- $\Delta_S(F_i(z)) = \int_{C_1} \frac{z^{n-1}}{z-5w} F_i(w) \otimes \psi_i(5w) \, dw + \text{rational function of } S$ and $z$ in the domain where $z$ is not enclosed by $S, C_1$ and

$\pi_W(\psi_i(5w))$ is holomorphic within $C_1$.

Thus we obtain an algebra hom. $\Delta_S : U \rightarrow \text{End}(V \otimes W)(S)$

$\Delta_S$ has poles at $S = \beta z^k$ where $\beta \in \sigma(W)$ and $\alpha \in \sigma(V)$.

Alternately, given $V$ and $W$, pick $S \in C^x$ so $\sigma(S) \cap \sigma(W) = \emptyset$

$S C_1$ and $C_2$ are disjoint and define $U C_1 V \otimes W$ by these expressions.

Notation: $U \otimes W$ the resulting representation.

Remark: $U \otimes W = U(S) \otimes W$

Example. $U_q(L_{s12})$ acts on $L_{m_1}(\alpha)$ and $L_{m_2}(\beta)$, $m, n \in \mathbb{N}; \alpha, \beta \in C^x$.

(see section (9.1))

- $e(z) m_n(r) = \frac{[n-r+1]}{z-\alpha q^{n-2r+1}} m_n(r-1)$ (0 ≤ $r$ ≤ $n$)

- $f(z) m_n(r) = \frac{[r+1]}{z-\alpha^{n-2r-1}} m_n(r+1)$

- $\psi(z) m_n(r) = q^{n-2r} \frac{(z-\alpha q^{n+1})(z-\alpha^{n-1})}{(z-\alpha q^{n-2r-1})(z-\alpha^{n-2r+1})} m_n(r)$
\[\sigma(L_n(\alpha)) = \{ \alpha^{-n+1}, \ldots, \alpha^{-1} \}\]

\[\Delta_S(D(z)) m_{n_1}(r_1) \otimes m_{n_2}(r_2) = q^{n_1 + n_2 - 2r_1 - 2r_2} \frac{(z - q^2 \alpha_1)(z - q^2 \alpha_2)(z - q \alpha_+)(z - q \alpha_-)}{(z - q \alpha_1)(z - q \alpha_2)(z - q \alpha_+)(z - q \alpha_-)} \]

. \quad m_{n_1}(r_1) \otimes m_{n_2}(r_2)

\[\Delta_S(e^{\psi_1}) m_{n_1}(r_1) \otimes m_{n_2}(r_2) = \frac{[n_1 - r_1 + 1] z}{z - 5 \alpha_1 q^{n_2 - 2r_1 + 1}} \frac{n_1 - 2r_1}{2 - \alpha_2 q^{n_2 - 2r_1 + 1}} \frac{n_1 - 2r_1 - 1}{2 - \alpha_2 q^{n_2 - 2r_1 + 1}} \frac{n_2 - 2r_2 + 1}{2 - \alpha_1 q^{n_2 - 2r_1 + 1}} \frac{n_2 - 2r_2 + 1}{2 - \alpha_1 q^{n_2 - 2r_1 + 1}} m_{n_1}(r_1) \otimes m_{n_2}(r_2) \]

This example does not fit in the framework of naive meromorphic categories introduced last line, since we don't have a bifunctor

\[\otimes : \text{Rep}(U) \times \text{Rep}(U) \to \text{Rep}(U)\]

\[U \otimes W \text{ only exists for } S \in \{ \beta \alpha \mid \beta \in \sigma(W) \} \text{ a finite subset of } \mathbb{C}^x \]

Depending on U and W.

Aim: to introduce a more sophisticated notion of meromorphic tensor categories (due to Y. Soibelman), so that

\[(\text{Rep } U, \otimes) \text{ and } (\text{Rep } U, \otimes) \text{ are both mono tensor categories}\]

(also braided).

Braiding on (Rep U, \otimes) was constructed last line. On (Rep U, \otimes) we will give a mono braiding in future.
Schematic diagram for Sorbelman's definitions

Pseudo monoidal category  Pseudo braided category

\[ \exists \text{ (representability)} \]

Monoidal (resp. braided monoidal) categories

Introduce spaces (+ symmetric group action)

Pseudo monoidal (resp. braided monoidal) categories / a space

Meromorphic \iff spaces are irr. algebraic varieties / C (smooth)

Special case: Categories equipped with G-action

"Spaces" are defined using G

we get [pseudo meromorphic G-tensor (or braided tensor) categories]

Then we drop the adjective pseudo if representability holds.

(11.2) Let \( T_n \) = set of complete bracketings on \( n \) letters
\[ = \text{set of planar binary trees (n incoming and 1 outgoing edge)} \]

Concatenation operation

\[ T_n \times T_{k_1} \times \ldots \times T_{k_n} \rightarrow T_{k_1 + \ldots + k_n} \]

e.g. \( n = 3 \)
\[
\begin{bmatrix}
\star & \star & \star \\
\end{bmatrix}
\begin{bmatrix}
\star & \star & \star \\
\end{bmatrix}
\begin{bmatrix}
\star & \star & \star \\
\end{bmatrix}
\]

Concatenation

\[ \rightarrow \times \rightarrow \rightarrow \rightarrow \]

\( (T_2 \times T_3 \times T_3 \rightarrow T_6) \)
Concatenation is often denoted by
\[(b, b_1, \ldots, b_n) \rightarrow b(b_1 \ldots b_n)\]

Definition. Pseudo monoidal category [Beilinson-Drinfeld]

Let \( C \) be a class of objects. Assume the following data is given:

1. For all \( b \in T_n \), \( X_1, \ldots, X_n, Y \in C \) we have a \( C \)-vector space
   \[ P_b \left( X_1, \ldots, X_n; Y \right) \]

2. Composition: For all \( b \in T_n \), \( b_1 \ldots b_n \in T_{k_1} \times \ldots \times T_{k_n} \)
   \[ X_1, \ldots, X_n, \{ Y_{i,j} \}, \{ Y_{i,k_i} \}_{i=1,\ldots,n}, Z \text{ objects from } C \]
   \[ P_b \left( X_1, \ldots, X_n; Z \right) \times \prod_{i=1}^{n} P_{b_i} \left( Y_{i,j}; X_i \right) \rightarrow P_{b(b_1 \ldots b_n)} \left( \{ Y_{i,j} \}_{i=1,\ldots,n}; Z \right) \]

3. A distinguished element \( \text{Id}_X \in P_b \left( X; X \right) \)

4. Associator: A natural isomorphism of vector spaces \( \forall b, b' \in T_n \)
   \[ P_b \left( X_1, X_2, X_3; Y \right) \sim \rightarrow P_{b \cdot b'} \left( X_1, X_2, X_3; Y \right) P_{b'} \left( X, Y \right) \]

Satisfying the following axioms:

1. Composition is associative.

For every \( b \in T_n \)
\[ b_1 \ldots b_n \quad (b_i \in T_{k_i}) \]
\[ b_{i_1} \ldots b_{i_{k_i}} \quad b_{i_r} \in T_{k_{i_r}} \]
\[ Z_{i_1,1} \ldots Z_{i_r,1}; b_{i_r} \quad \text{ and } U \]
\[ (1 \leq i \leq n) \quad (1 \leq r \leq k_i) \]

Objects \( X_1 \ldots X_n \)
\[ Y_{i_1} \ldots Y_{i_{k_i}} \quad (1 \leq i \leq n) \]
\[ Z_{i_1,1} \ldots Z_{i_{k_i},1} \quad \text{ and } U \]
\[ (1 \leq r \leq k_i \text{ and } 1 \leq i \leq n) \]

The following diagram is commutative.
\[ P_b(X, U) = \prod_{i=1}^{n} P_{b_i}(Y_i, X_i) \times \prod_{r=1}^{k_r} P_{b_{ir}}(Z_{ir}, Y_{ir}) \]

For notational convenience, we abbreviate, for instance:

\[ \overline{Z_{ir}} = Z_{ir,1} \ldots Z_{ir,b_{ir}} \text{ and so on.} \]

(2) Composition with \( \text{Id}_X \) is identity:

\[ P_b(X_1 \ldots X_n; Y) \times P_b(Y, Y) \rightarrow P_b(X_1 \ldots X_n; Y) \]

\[ (\psi, \text{Id}_Y) \rightarrow \psi \]

\[ P_b(X_1, X_i) \times P_b(X_1 \ldots X_n; Y) \rightarrow P_b(X_1 \ldots X_n; Y) \]

(3) Pentagon axiom:

\[ a_{b''b'}a_{b'b} = a_{b''b} \]

\[ \text{commutes with composition} \]

\[ (((\ast \ast) \ast) \ast) \rightarrow \ast (((\ast \ast) \ast) \ast) \]

is only sufficient when representability is imposed.
Braided structure. \( \forall \sigma \in B_n, \ b \in T_n \) we have
\[
\mu_\sigma : P_b (X_1 \ldots X_n; Y) \xrightarrow{\sim} P_b (X_{\sigma(1)} \ldots X_{\sigma(n)}; Y)
\]
\[\text{s.t. } \mu_1 = \text{id}, \ \mu_\sigma \mu_\tau = \mu_{\sigma \tau}\]

\(\mu_\sigma\) is compatible with compositions and associator.

Representability means \( P_b (X_1 \ldots X_n; Y) = \text{Hom}_C (X_b; Y)\)

where \(\text{Hom}_C (X; Y) := P_\to (X, Y)\)

From now onwards \(C\) is assumed to be \(C\)-linear category with
\(\text{Hom}_C (X; Y) := P_\to (X, Y)\).

(11.2) Introducing spaces.

For each \(n \in \mathbb{N}\), we have a smooth variety \(C_n \subset C\), with an
(action of symmetric group, in braided case).

Concatenation:
\(\otimes : C_n \times C_{k_1} \times \ldots \times C_{k_n} \to C_{k_1 + \ldots + k_n}\)

(J morphism)

More generally, we can assume we are given \(C_b \forall b \in T_n\)
and morphisms
\(C_b \times C_{b_1} \times \ldots \times C_{b_n} \to C_{b(b_1 \ldots b_n)}\)

\(A_{b_1 b} : C_b \xrightarrow{\sim} C_{b'}\)

\(S_n \subset C \subset C_b\) compatible with associator.
\( E = \mathcal{C} \) linear category.

- \( \forall b \in T_n, X_1 \ldots X_n; Y \in \mathcal{C} \) we have a quasi-coherent \( \mathcal{O}_{C_n} \)-module

\[ \mathcal{P}_b(X_1, \ldots, X_n; Y) \] over \( C_n \).

- Composition is a morphism of sheaves

\[ \mathcal{P}_b(X_1 \ldots X_n; Y) \times \prod \mathcal{P}_{b_i}(U_{i_1} \ldots U_{i_k}; X_i) \rightarrow \mathcal{P}_{b(b_1 \ldots b_n)}(U, Y) \]

\[ \gamma : C_n \times C_{k_1} \times \ldots \times C_{k_n} \rightarrow C_{k_1 + \ldots + k_n} \]

- \( \text{id}_x \in \Gamma(C_n, \mathcal{P}_x(X, X)) \)

- Associator \( a_{b', b} \) is a meromorphic (function) section on \( \otimes \mathcal{C}_n \)

- \( \mu_\sigma : \mathcal{P}_b(X_1 \ldots X_n; Y) \rightarrow \sigma^* \mathcal{P}_b(X_{\sigma(1)} \ldots X_{\sigma(n)}; Y) \) mer. section \( C_n \)

Representability: \( \forall b \in T_n, X_1 \ldots X_n \in \mathcal{C} \) we have a family of objects \( \{ X_b(s) \} \) s \in U \subseteq C_n \) (dense open subset)

\[ \text{st.} \quad \mathcal{P}_b(X_1 \ldots X_n; Y)_s = \text{Hom}_E(X_b(s), Y) \quad \forall s \in U \subseteq C_n. \]

Special case: group \( G \) acting on category \( E \)

\[ C_n = \frac{G^n}{S_n \text{ act by permuting the factors}} \]

\[ G^n \times G^{k_1} \times \ldots \times G^{k_n} \rightarrow G \]

\[ g_1 \ldots g_n, (g_{i_1}, \ldots, g_{i_k}) \ldots (g_{m_1}, \ldots, g_{m_k}) \rightarrow (g_1 g_{i_1} \ldots g_1 g_{i_k} \ldots g_n g_{m_1} \ldots g_n g_{m_k}) \]