(1.0) A Lie algebra \( \mathfrak{g} \) is a vector space (over \( \mathbb{C} \)) together with a bilinear pairing \([\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) satisfying:

\[
[x, y] = -[y, x] \quad \text{skew-symmetry}
\]

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{Jacobi identity}
\]

A representation of \( \mathfrak{g} \) is a linear map \( \rho : \mathfrak{g} \to \text{End}(V) \) sat.:

\[
\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)
\]

Note: Jacobi identity is equivalent to saying that \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) is a representation, where \( \text{ad}(x) \cdot y = [x, y] \).

Universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \) is defined on the following unital associative algebra:

\[
U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\langle x \otimes y - y \otimes x = [x, y] \rangle} \quad (\text{tensor algebra})
\]

\[
T(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}
\]

(1.1) We are interested in a class of Lie algebras called Kac-Moody algebras. These are algebras associated to an integer matrix:

Cartan Matrix \( A = (a_{ij}) \) \( \forall i, j \in I \)

\[
\cdot a_{ii} = 2 \quad \cdot a_{ij} \in \mathbb{Z} \leq 0 \quad \forall i \neq j
\]

\( \text{there exists a diagonal matrix } D = (d_i) \) \( \forall i \in I \)

\( \text{such that } d_i a_{ij} = d_j a_{ij} \forall i, j \)

(\text{additionally assume } \gcd(d_i : i \in I) = 1).

Further let \( A \) be indecomposable, i.e. \( A \) non-trivial \( I = I_1 \cup I_2 \) s.t.

\[
a_{ij} = 0 \quad \forall i \in I_1, j \in I_2.
\]
A realization of $A$ is a vector space $\mathfrak{g}$ of dim 211 and rank($A$) together with linearly independent sets $\Delta = \{ \alpha_i \}_{i \in I}, \Delta' = \{ h_i \}_{i \in I}$ such that $\alpha_i(h_j) = a_{ij} \quad \forall \ i, j \in I$.

$\alpha_i$'s are called simple roots, $h_i$'s are called simple coroots.

$Q = \sum_{i \in I} \mathbb{Z} \alpha_i \subset \mathfrak{h}^*$ (root lattice) \quad $Q' = \sum_{i \in I} \mathbb{Z} h_i \subset \mathfrak{h}$ (coroot lattice)

Define $\tilde{\mathfrak{g}}(A)$ to be Lie algebra generated by $h \in \mathfrak{h}$, $e_i$,$f_i$ $(i \in I)$ subject to $[h,e_i] = \alpha_i(h)e_i \quad [h,f_i] = -\alpha_i(h)f_i \quad [e_i,f_j] = \delta_{ij} h_i$

Basic facts: (i) $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$ as vector spaces when $\tilde{\mathfrak{n}}_{\pm}$ are free Lie algebras generated by $e_i$ and $f_i$ $(i \in I)$ resp.

(ii) $\tilde{\mathfrak{n}}_{\pm} = \bigoplus_{\alpha \in Q_{\pm}} \tilde{\mathfrak{g}}_{\alpha}$ when for $\gamma \in \mathfrak{h}^*, \tilde{\mathfrak{g}}_{\gamma} = \{ x \in \tilde{\mathfrak{g}} | [h,x] = \gamma(h)x \}$

$\dim \tilde{\mathfrak{g}}_{\alpha} < \infty \quad \forall \alpha \in Q \setminus \{0\}$

(iii) There is an involution $\tilde{\omega}$ on $\tilde{\mathfrak{g}}$ defined by $e_i \mapsto -f_i, f_i \mapsto -e_i, h \mapsto -h$

(iv) If $i \in \tilde{\mathfrak{g}}$ is an ideal then $i = \bigoplus_{\alpha \in Q_{\pm} \setminus \{0\}} (\mathfrak{g}_{\alpha})$

Let $\Gamma$ be unique maximal ideal of $\tilde{\mathfrak{g}}$ not intersecting $\mathfrak{h}$

Finally Kac-Moody algebra associated to $A$ is defined to be $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{g}}(A)/\Gamma$

Remark: We will see later that $\Gamma$ is generated by $ad(e_i) e_j$ (ad $f_i$) $f_j$ \quad $\forall \ i \neq j \in I$.

Again we have $\tilde{\mathfrak{g}}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \bigoplus_{\alpha \in R_{+}} \mathfrak{g}_{\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{+}} \mathfrak{g}_{\alpha}$

$R_{+} = Q_{+} \setminus \{0\}$ set of positive roots.
Lemma. If $a \in \mathbb{N}_+$ is such that $[f_i, a] = 0 \forall i \in I$, then $a = 0$.

Similarly for $n_+$. Center of $g(A)$ is given by

$$Z(g(A)) = \left\{ h \in \mathfrak{h} \mid \alpha_i(h) = 0 \forall i \in I \right\} \text{ of dim } |I| - \text{rk}(A).$$

In particular $g(A)$ has no non-trivial ideals if and only if $\det A \neq 0$.

Proof. Let $i = \text{span of elements of the form } (ade_i)(adh)'a$

$$\left( i \in I, \quad k, l \geq 0, \quad h \in \mathfrak{g} \right).$$

Then $i \subset \mathbb{N}_+$ is an ideal. This contradicts the definition of $g(A)$.

For $\alpha \in \mathbb{Q}_+$ define $\text{ht}(\alpha) = \sum_{i \in I} n_i \in \mathbb{Z}$ if $\alpha = \sum_{i \in I} n_i \alpha_i$.

Let $c \in g(A)$ be in the center. $c = \sum_{k \in \mathbb{Z}} c_k$. $[f_i, c] = 0 \forall i$

$\Rightarrow c_k = 0 \forall k > 0$ and similarly for $k < 0$. Thus $c = c_0 \in \mathfrak{h}$.

$0 = [c, e_i] = \alpha_i(c)e_i \Rightarrow \alpha_i(c) = 0 \forall i$. Last part follows since $c \subset \mathfrak{g}$.

Ideals are graded.

(1.1) Bilinear form. Let $\mathfrak{g}' = \text{span of } h_i$'s $\subset \mathfrak{g}$. Pick a complementary subspace $\mathfrak{g}''$ and define $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$$\langle h_i, h \rangle = \delta_i \alpha_i(h) \quad (h_i, h_2) = 0 \forall h_i, h_2 \in \mathfrak{g}''.

Check: kernel of $\langle \cdot, \cdot \rangle$ restricted to $\mathfrak{g}' = Z(g(A))$. Hence $\langle \cdot, \cdot \rangle$ is non-degenerate.

$\nu : \mathfrak{g} \to \mathfrak{g}^*$ isom. induced by $\langle \cdot, \cdot \rangle$.

We extend $\langle \cdot, \cdot \rangle$ to $g$ by $\langle e_i, f_j \rangle = \delta_{ij} d_i$ and $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$.

Radical of $\langle \cdot, \cdot \rangle \subset \mathfrak{g}$ (since $\langle \cdot, \cdot \rangle|_\mathfrak{g}$ is non-deg.).

$\Rightarrow \langle \cdot, \cdot \rangle$ descends to a symmetric, non-degenerate invariant bilinear

form on $g$.  

(1.4) Weyl group. For each $i \in I$ define $S_i : \text{Aut}(\mathfrak{g}^*)$ by

$$S_i(\lambda) = \lambda - \lambda(h_i) \alpha_i$$

$W \subset \text{GL}(\mathfrak{g}^*)$ be the group generated by reflections $\{S_i \mid i \in I\}$. It also acts on $\mathfrak{g}$ through the iso. $\nu : \mathfrak{g} \to \mathfrak{g}^*$. $S_i(h) = h - \alpha_i(h) h_i \quad \forall h \in \mathfrak{g}$.

Note: $W$ preserves the lattice $Q$ and hence is discrete. $W$ also preserves $(\cdot, \cdot)$ on $\mathfrak{g}$ and $\mathfrak{g}^*$.

Partial order on $\mathfrak{g}^*$: $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

(1.5) $\mathfrak{sl}_2$ representations.

$\mathfrak{sl}_2$ = Lie algebra of $2 \times 2$ traceless matrices. It has basis given by

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[e, f^k] = -k(k+1)f^k + k(k-1)f^{k-1}h.$$ For each $n \in \mathbb{N}$, there is a unique irr $\mathfrak{sl}_2$ representation $L(n)$ with basis $V_0, \ldots, V_n$ and

$$e V_i = (n+i+1) V_{i+1}, \quad f V_i = (i+1) V_{i+1}, \quad h V_i = (n-2i) V_i.$$

(1.6) The following elements are in $\mathfrak{g}$

$$1 - \alpha_j, \quad e_j, \quad f_j, \quad \text{ad} e_i, \quad \text{ad} f_i.$$

- $(\text{ad} e_i) f_j = 0 \implies (\text{ad} e_i) (\text{ad} f_i) f_j = 0$.

$$(ad h_i) f_j = -a_{ij} f_j$$

Similarly one can show that $[e_k, (\text{ad} f_i)^{-a_{ij}} f_j] = 0$. Hence $(\text{ad} f_i) f_j = 0$.

Hence adjoint representation is locally nilpotent/ integrable.
(1.7) Let $V$ be a repn. of $g$ which is $g$-diagonalizable, i.e.

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \quad V_\lambda = \{ v \mid h \cdot v = \lambda(h) v \quad \forall h \in \mathfrak{h} \} \quad \lambda$-weight space.

(assume $\dim V_\lambda < \infty$).

We say $V$ is integrable if $\forall \ i \in I$, $e_i$ & $f_i$ are locally nilpotent.

(i.e. $\forall v \in V$, $\exists N > 0$ s.t. $e_i^N v = 0$).

(An extension of) $W$ acts on an integrable representation via

$$\tilde{S}_i = \exp(e_i) \exp(-f_i) \exp(e_i)$$

$\tilde{V}_\lambda = V_{\lambda i}$

and hence also on $g$ via ($x \mapsto \tilde{S}_ix \tilde{S}_i^{-1}$). In particular, we obtain

Prop. (1) The root system is $W$-invariant. Moreover $\dim g_\alpha = \dim g_{-\alpha}$

$\forall \alpha \in \mathfrak{R}$, $w \in W$.

(2) If $\alpha \in \mathfrak{R}_+$ and $S_i(\alpha) < 0$ then $\alpha = \alpha_i$.

(3) Let $V$ be an integrable $g$-module. $\lambda \in \mathfrak{h}^*$ a weight of $V$, $i \in I$

$\lambda = \{ t \in \mathbb{Z} \mid \lambda + t\alpha_i \text{ is a weight of } V \}$. Then

(assume $\dim V_\lambda < \infty$)

$M = \{ t \in \mathbb{Z} \mid \lambda + t\alpha_i \}$

$p, q \in \mathbb{N}, p - q = \lambda(h_i)$.

$e_i : V_{\lambda + t\alpha_i} \to V_{\lambda + (t+1)\alpha_i}$ is injective for $t < -\frac{1}{2} \lambda(h_i)$

(1.8) Structure of Weyl group.

Lemma. Let $w = S_{i_1} \cdots S_{i_t} \in W$ be a reduced expression.

(i) $l(ws_i) < l(w) \iff \lambda(\alpha_i) < 0$

(ii) $w(\alpha_i) < 0$

(iii) If $l(ws_i) < l(w)$ then $\exists 1 \leq j \leq t$ s.t.

$S_{i_j} S_{i_{j+1}} \cdots S_{i_t} = S_{i_{j+1}} \cdots S_{i_t} S_{i_j}$

Hence $W$ is a Coxeter group: $W = \langle S_i \mid i, S_i^2 = 1, \frac{S_i S_j \cdots}{m_{ij}} \cdots = \frac{S_j S_i \cdots}{m_{ij}} \rangle$

where $m_{ij} = 2, 3, 4, 6, \infty$ if $a_{ij}, a_{ji} = 0, 1, 2, 3, \geq 4$ resp.
\textbf{Proof. \ } \begin{align*}
\omega(\alpha_i) < 0 \Rightarrow \exists j \text{ s.t. } S_j \ldots S_t = S_{t+1} \ldots S_t S_i:
\end{align*}

Let $\beta_i = S_i \ldots S_t$, then $\beta_t > 0$ and $\beta_0 < 0 \Rightarrow \exists j \text{ s.t. }$

$\beta_{j-1} < 0 \& \beta_j > 0$. Then $S_j(\beta_i)$ and hence $\beta_j = \alpha_j$ and we get

$\alpha_j = \omega(\alpha_i)$ when $\omega = S_{j+1} \ldots S_t$. Therefore $S_j = \omega S_i \omega^{-1}$.

This proves that $\omega(\alpha_i) < 0 \Rightarrow \ell(\omega S_i) < \ell(\omega)$. Conversely if

$\omega(\alpha_i) > 0$ we get $\omega S_i(\alpha_i) < 0$ and hence $\ell(\omega S_i S_i) < \ell(\omega S_i)$. \hfill \square

(1.9) \quad C = \{ h \in \mathfrak{h}_R \mid \alpha_i(h) \geq 0 \ \forall i \in I \} \text{ fundamental chamber}

\[ X = \bigcup_{\omega \in W} \omega(C) \]

\[ \text{Prop. (a) Let } h \in C; \quad W_h = \{ \omega \in W \mid \omega(h) = h \} \text{. Then } W_h \text{ is generated by the fundamental reflections } S_i \in W_h. \]

(b) \quad C \text{ is the fundamental domain for action of } W \text{ on } X.

(c) \quad X = \{ h \in \mathfrak{h}_R \mid \alpha(h) < 0 \text{ only for a finite number of } \alpha \in \mathfrak{R}_+ \}.

(d) \quad C = \{ h \in \mathfrak{h}_R \mid \forall \omega \in W \quad h - \omega(h) = \sum_{i \in I} c_i \alpha_i \text{ with } c_i \geq 0 \}.

(e) \quad \text{TFAE: (i) } |W| < \infty \quad \text{(ii) } X = \mathfrak{h}_R \quad \text{(iii) } |\mathfrak{R}| < \infty.

\textbf{Proof. \ (a) and (b) by induction on } \ell(\omega) \text{ as follows. Let } h \in C \text{ and}

\[ \omega(h) = h' \in C, \quad \omega = S_i \ldots S_t \text{ reduced expression for } \omega. \text{ Then}
\]

$\alpha_i(h') > 0$ and $(\omega \alpha_i)(h') > 0$ but $\omega \alpha_i < 0 \Rightarrow \alpha_i(h') = 0 = \omega \alpha_i(h')$.

So $S_i(h') = h$.

(c) \quad \text{Let } X' = \{ h \in \mathfrak{h}_R \mid \alpha(h) < 0 \text{ only for a finite number of } \alpha \in \mathfrak{R}_+ \}.

\text{Then } C \subset X' \text{ and } X' \text{ is } W \text{-invariant } \Rightarrow X \subset X'.

Conversely let $h \in X'$ and let $M_h = \{ \alpha \mid \alpha(h) < 0 \}$. We argue by induction on $|M_h|$ since $\alpha_i \in M_h$ for some $i$ and hence $|M_{\alpha_i h}| < |M_h|$.
(d) It is clear. We prove the converse by induction. Let \( h \in \mathbb{R} \) be an element of \( C \) and \( w = s_{i_1} \ldots s_{i_k} \) a reduced expression.

If \( k = 1 \), \( h = w(h) \in \sum_{R_{\geq 0}} h_i \) by definition. Otherwise

\[
- w(h) = (h - s_{i_1} \ldots s_{i_{k-1}} h) + s_{i_1} \ldots s_{i_{k-1}} (h - s_{i_k}(h))
\]

(c) \( (i) \Rightarrow (ii) \) Let \( h \in \mathbb{R} \). Pick the maximal element of \( W h \). It must lie in \( C \). \( (ii) \Rightarrow (iii) \) Let \( h \in \int \text{erior of } C \), then \( \alpha(-h) < 0 \forall \alpha \in R_+ \).

\(-h \in X \Rightarrow |R_+| < \infty \). \( (iii) \Rightarrow (i) \) We claim that \( w(\alpha) = \alpha \forall \alpha \in R \) implies \( w = 1 \). Hence \( W \subset \text{Permutation group of } R_+ \). The proof is easy consequence of the exchange property (Lemma (1.8) (iii)).

(1.10) The numbers \( m_{ij} \):

\[
|I| = 2 \quad a_{12} = -a \quad a_{21} = -b
\]

\[
S_1 = \begin{bmatrix} -1 & -a \\ 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & 0 \\ -b & -1 \end{bmatrix} \Rightarrow S_1 S_2 = \begin{bmatrix} -1+ab & a \\ -b & -1 \end{bmatrix}
\]

\[
\det(\lambda I - S_1 S_2) = (\lambda + 1 - ab)(\lambda + 1) + ab = \lambda^2 + (2 - ab)\lambda + 1
\]

\( S_2 \) is of finite order \( \tau \) iff \( ab \leq 3 \) with \( m = 2, 3, 4, 6 \) if \( ab = 0, 1, 2, 3 \) resp.

(1.11) Classification of Cartan matrices.

Let us work in more generality. \( A \in M_{n \times n}(\mathbb{R}) \) indec., symmetric.

\( DA \) is symmetric, \( D = (d_{ij}; i \neq j, d_{ii} \geq 0) \) and \( a_{ij} \leq 0 \forall i \neq j \).

We write \( u \succ 0 \) for a real column vector \( u \) if all its entries \( u_i > 0 \).

Similarly \( u \succeq 0 \) if \( u_i \geq 0 \).

Thm. \( A \) is one of the following.

(Fin) \( \det A \neq 0 \). \( \exists u \succeq 0 \text{ st. } Au > 0 \). \( Au \succeq 0 \Rightarrow Au > 0 \) or \( u = 0 \).

(Aff) \( \text{rk}(A) = |I| - 1 \). \( \exists u \succeq 0 \text{ st. } Au = 0 \). \( Au \succeq 0 \Rightarrow Au = 0 \).

(Ind) \( \exists u \succeq 0 \text{ st. } Au < 0 \). \( Au \succeq 0 \text{ and } u \succeq 0 \Rightarrow u = 0 \).

In particular, \( A \) is of finite, affine or indefinite type \( \Leftrightarrow \exists x \succeq 0 \text{ st. } Ax > 0, \, A x = 0, \, A x < 0 \text{ resp. } A \) is finite or affine \( \Leftrightarrow DA \) is positive definite (positive semi-definite).
Proof. Let us symmetrize \( A \), so it suffices to work with the symmetric case.

\[
K_A := \{ u \mid Au \geq 0 \} \quad C := \{ u \geq 0 \}.
\]

Consider the intersection \( K_A \cap C \).

Indecomposability of \( A \) implies that \( Au \geq 0 \, \& \, u \geq 0 \Rightarrow u \geq 0 \) or \( u = 0 \).

Moreover, \( K_A \) meets boundary of \( C \) exactly at \( 0 \). Thus are three possibilities:

(i) \( K_A \cap C = \{ u > 0 \} \cup \{ 0 \} \)

(ii) \( K_A = \text{Kernel of } A \) is 1-diml subspace spanned by \( u > 0 \)

(iii) \( K_A \cap C = \{ 0 \} \).

(i) is equivalent to (Fin). (ii) is equivalent to (Aff) and (iii) is equivalent to (Ind). For the last part of (Ind) we have to use the fact that a system of linear inequalities \( \{ \lambda_1 > 0 \ldots \lambda_p > 0 \} \) has a soln if and only if there is no linear reln b/w \( \lambda_1 \ldots \lambda_p \) with non-negative coefficients.

Now one can classify all finite type Cartan matrices (see below).

1.12 Thm. TFAE:

(i) \( g \) is f.d.

(ii) \( A \) is of finite type

(iii) \( |W| < \infty \)

(iv) \( |R| < \infty \).

Proof.

(2) \( \Rightarrow \) (3) \( W \) is a discrete subgroup of \( O(g_{\mathbb{R}} , (\cdot , \cdot )) \).

(3) \( \Rightarrow \) (4) earlier

(4) \( \Rightarrow \) (1) clear.

(1) \( \Rightarrow \) (2) \( g \) of finite dim \( \Rightarrow \) \( \exists \alpha \in R_+ \) s.t. \( \alpha + \alpha_i \in R_+ \) \( (\forall i) \). Hence \( \alpha(h_i) > 0 \) \( (\forall i) \). So \( \alpha \geq 0 \Rightarrow A \) is either finite or affine.

In the affine case \( \alpha(h_i) = 0 \) \( (\forall i) \). Moreover \( \exists i \) s.t. \( \alpha - \alpha_i \in R_+ \)

By \( sl_2 \)-repn theory as \( \{ f_i , g_i , h_i \} \)-module \( \mathfrak{g}_\alpha \neq 0 \) \( (\alpha \text{-weight space}) \)

\( \Rightarrow \) \( g_{\alpha + \alpha_i} \neq 0 \) which is contradiction. \( \square \)
List of finite type Cartan matrices.

Dynkin diagram is a graph on vertex set $I$.

\[ a_{ij} = a_{ji} = -1 \quad a_{ij} = -1 \quad a_{ji} = -2 \quad a_{ij} = -1 \quad a_{ji} = -3 \]

A is of finite type if its Dynkin diagram is one of the following:

\[ A_n \]
\[ B_n \]
\[ C_n \]
\[ D_n \]
\[ E_6, 7, 8 \]
\[ G_2 \]
\[ F_4 \]

For the rest of the course $\mathfrak{g} = \mathfrak{sl}$, simple Lie algebra.

$A = (a_{ij})_{i,j \in I}$, Cartan matrix, $\mathfrak{h} = $ Cartan subalgebra.

$D = (d_{ij})_{i,j \in I}$, symmetrizing matrix.

$R = R_+ \cup R_-$, root system.

$\Delta = \{ \alpha_i \}_{i \in I}$, simple roots.

$W = \langle s_i \mid s_i^2 = 1, s_i s_j s_i = s_i s_j \ldots \rangle$, Weyl group.

$\Theta \in R_+$, highest root.

$\rho \in \mathfrak{h}^*$ st. $\rho(h_i) = 1$ (for $i \in I$).