The aim of this course is to study the structure and representation theory of three classes of affine (or infinite-dimensional) quantum groups: Yangians, Quantum loop algebras, and Elliptic quantum groups. These are Hopf algebras associated to a simple Lie algebra \( \mathfrak{g} \). Most importantly, they come equipped with an \( R \)-matrix which satisfies the Yang-Baxter equation. We will also study (time permitting) several difference-differential equations related to these.

Today's lecture is aimed at explaining the origins of these objects for \( \mathfrak{g} = \mathfrak{sl}_2 \). The plan is as follows:

1. Lattice models of Statistical Mechanics
2. Yang-Baxter equation = Solvability Criteria
3. Three solutions of YBE:
   - Rational
   - Trigonometric
   - Elliptic

These solutions lead to RTT algebras of Faddeev-
Reshetikhin-Takhtajan:

- \( Y_h \mathfrak{sl}_2 \)
- \( \mathcal{U}_q(\mathfrak{sl}_2) \)
- \( E_{h,c}(\mathfrak{sl}_2) \)
Finally I will explain how to obtain a new presentation (Drinfeld's) of these algebras ($\mathcal{T}_1$, $\mathcal{U}_q(\text{sl}_2)$); through Gaussian dec. 
This is the presentation that can be generalized to arbitrary $\mathcal{O}$ and will be the focus of our course.

0.1 Lattice Models. Let $M, N \in \mathbb{N}$ and let $\Lambda_{M,N}$ be a grid (rectangular) with $M$ rows and $N$ columns. We impose periodic boundary condition, so $\Lambda_{M,N}$ is in fact a grid on a torus.

A configuration $C$ is an assignment of + or - on the edges of $\Lambda_{M,N}$.

Thus around each vertex there are 16 possible assignments.

Let $a_1, \ldots, a_{16}$ be arbitrary (complex) numbers. Weight of a configuration $C$ is

$$w(C) = \prod_{i=1}^{16} a_i$$

Partition function $Z = \sum_{C: \text{config}} w(C)$.

E.g. example of a configuration on $\Lambda_{4,4}$.

In Statistical Mechanics, people are interested in computing $Z$ (and expectation values, etc.) or at least their behavior as $M, N \to \infty$.

We realize $Z$ as trace of a product of a matrix called Transfer matrix.
R-matrix and T-matrix.

We encode \(a_1 \ldots a_N\) in a matrix

\[
R = (R(\alpha \beta | \gamma \delta))_{\alpha, \beta, \gamma, \delta \in \{\pm\}}
\]

\(R(\alpha \beta | \gamma \delta)\) is the weight of \(\alpha \beta \gamma \delta\).

Notation \(\alpha, \beta, \gamma, \delta, \ldots\) Greek letters \(\epsilon \in \{\pm\}\)

\(\alpha, \beta, \ldots \epsilon \in \{\pm\}^N\) \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)\).

Transfer matrix \(T\). For \(\alpha, \beta \in \{\pm\}^N\), define

\[
T(\alpha, \beta) = \sum_{\gamma \in \{\pm\}^N} R(\gamma_1 \alpha_1 | \gamma_2 \beta_1) R(\gamma_2 \alpha_2 | \gamma_3 \beta_2) \ldots R(\gamma_N \alpha_N | \gamma_1 \beta_1)
\]

We can think of \(R\) as an operator on \(\mathbb{C}^2 \otimes \mathbb{C}^2\) and \(T\) as an operator on \((\mathbb{C}^2)^{\otimes N}\).

**Lemma.** \(Z = \text{Tr} (T^M)\)

**Proof.** \(\text{Tr} (T^M) = \sum_{\phi^{(1)}, \ldots, \phi^{(n)}} T(\phi^{(1)} | \phi^{(2)}) \ldots T(\phi^{(m)} | \phi^{(1)})\)

\[
= \sum_{\phi^{(1)}, \ldots, \phi^{(n)}} \sum_{\psi^{(1)}, \ldots, \psi^{(n)}} \prod_{i=1}^M R(\psi^{(i)}_k \phi^{(i)}_k | \psi^{(i)}_{k+1} \phi^{(i)}_k)
\]

\[
= \sum_C \omega(C) = Z
\]

In fact, \(T\) can be thought of as trace of another matrix, called the monodromy matrix \(\Phi\).
\[ \mathcal{T} : C^2 \otimes (C^2)^{\otimes N} \rightarrow C \] is defined analogously.

\[ \mathcal{T}(\gamma \alpha | \gamma' \beta) = \sum_{\gamma_1, \ldots, \gamma_N \in \{\pm\}} R(\gamma \alpha_1 | \gamma_1 \beta_1) \ldots R(\gamma_N \alpha_N | \gamma'_N \beta_N) \]

\[ \mathcal{T} = R_{\alpha_1} R_{\alpha_2} \ldots R_{\alpha_N} \]

Then \[ T = \text{Tr}_{C^2} (\mathcal{T}). \]

(0.8) Commuting Transfer matrices.

\[ T = \text{transfer matrix corresponding to} \ (a_1, \ldots, a_{16}) \] similarly \[ R, R', R'' \].

\[ T' = \ldots (a_1', \ldots, a_{16}') \]

Let us obtain (necessary and) sufficient conditions for \[ TT' = T'T \]

\[ TT' = \text{Tr}_{C^2 \otimes C^2} (T \otimes T') \]

\[ T'T = \text{Tr}_{C^2 \otimes C^2} (T' \otimes T) \]

If \[ R'' \] conjugates \[ T_0 \otimes T'_0 \] to \[ T'_0 \otimes T_0 \], we will have \[ TT' = T'T \].

\[ R''_0 \otimes T_0 T'_0 = T'_0 \otimes T_0 R''_0 \in \text{End} (C^2 \otimes C^2 \otimes (C^2)^{\otimes N}) \]

For \( N = 1 \) we get

\[ R''_0 \otimes R_{\alpha_1} R_{\alpha_1}' = R'_{\alpha_1} R_{\alpha_1} R''_{\alpha_1} \]

Yang-Baxter equation (YBE)

\[ \text{Thm.} \quad \text{(YBE)} \Rightarrow TT' = T'T \]
Proof. \( R''_0 T_0 T''_0 = R''_0 R_{01} \cdots R_{0n} R'_{12} \cdots R'_{n1} \)
\[ = R'_{01} R_{01} R''_{00} R_{02} \cdots R_{0n} R'_{02} \cdots R'_{0n} \]
\[ \cdots = T''_0 T_0 R''_{00} \)

(0.4) 6 vertex model.

Allowed configurations

\[ + \quad + \quad + \quad + \quad + \quad - \quad - \quad - \quad - \quad - \quad - \]

\[ \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\end{array} \]

For 6v model YBE \[ \Rightarrow \frac{\alpha^2 + b^2 - c^2}{2ab} = \frac{d'^2 + b'^2 - c'^2}{2d'b'} \] \( (= \Delta \text{ say}) \)

Proof. YBE can be unfolded to following 64 equations

\[ \sum_{\delta, \mu, \nu} R''_\beta R''_\gamma R''_\delta = \sum_{\delta, \mu, \nu} R'_{\beta \nu} R'_{\gamma \mu} R''_\delta \]

\[ \forall \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \{+\} \]

\[ \sum_{\delta, \mu, \nu} R''(\alpha \beta / \delta \mu) R(\delta \gamma / \alpha' \nu) R'(\mu \nu / \beta' \gamma') = \sum_{\delta, \mu, \nu} R'(\beta \gamma / \mu \nu) R(\alpha \nu / \delta \gamma') R''(\delta \mu / \alpha' \beta') \]

6v case \[ \Rightarrow \text{both sides are zero unless } \alpha + \beta + \gamma = \alpha' + \beta' + \gamma' \]

\[ \Rightarrow 20 \text{ equations} \quad \text{sign symmetry} \quad \rightarrow \quad 10 \text{ equations} \]
If $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, the equation holds trivially since $R$ is symmetric.

We are left with 6 equations which come in pairs.

1. \[ R'' = \begin{cases} + & R' = + \\ - & R' = - \end{cases} \]

\[ a'' b' c' = c'a'' b'' + b' c' c'' \]

2. \[ R'' = \begin{cases} + & R' = + \\ - & R' = - \end{cases} \]

\[ a'' c' a' = c'a'' c'' + b' c' b'' \]

3. \[ R'' = \begin{cases} + & R' = + \\ - & R' = - \end{cases} \]

\[ c'' b' a' = b'a'' c'' + c' c' b'' \]

Eliminate $a'' b''$ and $c''$ to get \[ \frac{a^2 + b^2 - c^2}{2ab} = \frac{a'^2 + b'^2 - c'^2}{2a'b'} \]

Use 1 & 2 to eliminate $a''$:
\[ a'(c'a'' b'' + b' c' c'') = b'(c'a'' c'' + b' c' b'') \]

\[ b'(aa'' c' - bb'' c') = c''(ab'(c')^2 - a'b' c^2) \]

Use 3 \[ b' c' = a'b''c'' - ab''c'' \]

\[ (aa' - bb')(a'b' - ab') = ab'(c')^2 - a'b' c^2 \]

\[ a''^2 b - a''^2 a' b' - a'^2 b' b' + a b'^2 = ab (c^2 - a'b' c^2) \]

\[ (a'^2 + b'^2 - c'^2) a'b' = (a^2 + b^2 - c^2) ab \]

\[ \square \]
Rational solution. Let $\Delta = 1$ so that

$$(a-b)^2 = c^2 \quad a = b + c$$

Let $c = \tan \theta$, $b = u$ and hence $a = u + \tan \theta$. We get

(fixed)

$$R(u) = \begin{bmatrix} u + \tan \theta & 0 & 0 & 0 \\ 0 & u + \tan \theta & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u + \tan \theta \end{bmatrix}$$

satisfying $R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u)$

Trigonometric solution

$$\frac{a^2 + b^2 - c^2}{2ab} = \Delta$$

$$\Rightarrow 1 + \left(\frac{b}{a}\right)^2 - \left(\frac{c}{a}\right)^2 = 2 \left(\frac{b}{a}\right) \Delta$$

Let $x = \frac{b}{a}$ and $y = \frac{c}{a}$

$$y^2 = x^2 - 2x\Delta + 1 = (x - q)(x - q') \quad \text{when } \Delta = \frac{q + q'}{2}$$

Let $t = \left(\frac{x - q'}{x - q}\right)^{\frac{1}{2}} \Rightarrow x = \frac{qt - q't'}{t - t'}$

$$y = (x - q)t \Rightarrow y = \frac{q - q'}{t - t'}$$

$a = t - t'$

$b = qt - q't'$

$c = q - q'$

$a = \frac{-1}{5} - \frac{-1}{5'} \quad \text{when } t = \frac{-1}{5}$

$$R(5) R_{13}(55') R_{23}(5') = R_{23}(5') R_{13}(55') R_{12}(5)$$
(0.6) RTT algebra (rational case)

Let $Y$ be an algebra generated by $\{t_{ij}^{(r)}\}_{r \in \mathbb{N}, i,j \in S2}$. 
\[ t_{ij}^{(r)} = \delta_{ij} + \hbar \sum_{r \geq 0} t_{ij}^{(r)} z^{-1}. \]
We impose the relations
\[ R(u-v) T^{(1)}(v) T^{(2)}(v) = T^{(2)}(v) T^{(1)}(u) R(u-v) \]
These relations can be unfolded to
\[ [t_{ij}(u), t_{k\ell}(v)] = \frac{\hbar}{u-v} \left( t_{kj}(u) t_{\ell\ell}(v) - t_{kj}(u) t_{\ell\ell}(u) \right) \]
Natural Hopf structure $\Delta(t_{ij}) = \sum_{k=1}^{2} t_{ik} \otimes t_{kj}$.

This algebra turns out to be the Yangian of $g_{\mathbb{C}2}$. To get $sl_2$, we must set $qDet(T(u)) := t_{11}(u) t_{22}(u-k) - t_{21}(u) t_{12}(u-k) = 1$.

Drinfeld's new presentation can be obtained as follows.
\[ T(u) = \begin{bmatrix} 1 & 0 \\ \bar{x}(u) & 1 \end{bmatrix} \begin{bmatrix} \bar{x}(u) & 0 \\ 0 & \bar{x}(u) \end{bmatrix} \begin{bmatrix} 1 & x^+(u) \\ 0 & 1 \end{bmatrix} \]
Thus $Y$ can be presented on generators $\{x_1, x_2, x_3, x^\pm \}_{r \in \mathbb{N}}$ which is closer to the presentation of Kac-Moody algebras and can be generalized.

(0.7) RTT algebra (trigonometric case)

Let $U$ be an algebra generated by $\{l_{ij}^{(r)}, \bar{l}_{ij}^{(r)}\}_{r \in \mathbb{N}}$.
\[ l_{ij}^{(r)}(z) = \sum_{r \geq 0} l_{ij}^{(r)} z^{-r}, \quad \bar{l}_{ij}^{(r)}(z) = \sum_{r \geq 0} \bar{l}_{ij}^{(r)} z^{-r}. \]
Relations: \( l_{ij}^{(0)} = \overline{l}_{ji}^{(0)} \quad (i = 0) \quad \forall \; i < j \)

\[
l_{ii}^{(0)} = (\overline{l}_{ii}^{(0)})^{-1}
\]

\[
R(\overline{z} \overline{w}) L^{(z)} L^{(w)} = L^{(w)} L^{(z)} R(\overline{z} \overline{w})
\]

for \( \varepsilon_1, \varepsilon_2 \in \{\pm\} \).

This turns out to be quantum loop algebra of \( \mathfrak{gl}_k \). One can similarly obtain its presentation on \( \{H_1, \kappa, H_2, \kappa, \kappa^\pm \}_{k \in \mathbb{Z}} \) (Drinfeld's new presentation).