Problem 1. Let $\Phi \subset \mathfrak{h}^* \setminus \{0\}$ be a finite root system, $\Gamma$ the Dynkin diagram and $W$ the Weyl group. Let $G \subset GL(\mathfrak{h}^*)$ be the group of automorphisms of $\Phi$ (that is, $G$ preserves $\Phi$ and the inner product on $\mathfrak{h}^*$). Prove that we have a short exact sequence $1 \to W \to G \to \text{Aut}(\Gamma) \to 1$.

Problem 2. Let $g$ be the simple Lie algebra of type $D_4$. Let $\sigma$ be the order 3 automorphism of the Dynkin diagram of $D_4$: $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ and $\sigma(4) = 4$ where

$\begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
2
\end{array}$

Let $\zeta = \exp\left(\frac{2\pi i}{3}\right)$ and let $g_j = \{x \in g : \sigma(x) = \zeta^j x\}$. Define

$L(g, \sigma) = \bigoplus_{j \in \mathbb{Z}} g_j z^j \oplus \mathbb{C}c \oplus \mathbb{C}\partial \subset \tilde{g}$

Prove that $L(g, \sigma)$ is the Kac–Moody algebra associated with the matrix

$\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -3 \\
0 & -1 & 2
\end{pmatrix}$

Problem 3. Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix and let $g(A)$ be the Kac–Moody algebra associated with it. Given $\lambda \in \mathfrak{h}(A)^*$, let $M_\lambda$ be the Verma module and let $1_\lambda \in M_\lambda$ be the highest weight vector. Prove that there exists a unique symmetric bilinear form $(.,.) : M_\lambda \times M_\lambda \to \mathbb{C}$ such that

(a) $(1_\lambda, 1_\lambda) = 1$,

(b) $(hv, v') = (v, hv')$ and

(c) $(e_i v, v') = (v, f_i v')$

for each $i \in I, h \in \mathfrak{h}(A)$ and $v, v' \in M_\lambda$. This bilinear form is called the Shapovalov form.

Problem 4. Prove that $M_\lambda$ is irreducible if, and only if the Shapovalov form is non–degenerate.

Problem 5. Recall the action of the coroot lattice $Q^\vee \subset W^a = W \ltimes Q^\vee$ on $\tilde{\mathfrak{h}}^*$:

$t_{\alpha^\vee} \cdot (\lambda + k\Lambda + l\delta) = \lambda + mk\nu^{-1}(\alpha^\vee) + k\Lambda + \left(l - k\lambda(\alpha^\vee) + \frac{km}{2}\vert\alpha^\vee\vert^2\right)\delta$

where $m = |\theta|^2/2$.

Consider the action of $W^e := W \ltimes P^\vee$ on $\tilde{\mathfrak{h}}^*$ defined by the same formula as above for $\alpha^\vee \in P^\vee$ and $W$ acts as usual.

(1) Prove that this action preserves the set of (affine) roots $\hat{\Phi}$.

(2) Define the length function

$l(w) := \vert\{\alpha \in \hat{\Phi}_+ \text{ such that } w(\alpha) \notin \hat{\Phi}_+\}\vert$
For each $j \in I$ let $\varpi^\vee_j \in P^\vee$ be the fundamental coweight, defined by $\alpha_i(\varpi^\vee_j) = \delta_{ij}$. Prove that the set of elements of $W^e$ of length 0 can be identified with the set of minuscule coweights: $\{j \in I : \theta(\varpi^\vee_j) = 1\}$ via the following

$$j \in I \mapsto t_{\varpi^\vee_j}w_0^jw_0 \in W^e$$

where $w_0 \in W$ is the longest element of the Weyl group, and $w_0^j$ is the longest element of the Weyl group of the root system obtained by omitting the $j$th node.