Recall that we defined the following two functions:

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad \text{Re}(z) > 0 \]

\[ \Gamma_1(z) = \frac{1}{z e^{\gamma z} \prod_{n=1}^\infty ((1 + \frac{z}{n}) e^{-\frac{n}{z}})} \]

We have not yet proved that these two are the same. In the following problems, we are only concerned with \( \Gamma_1(z) \).

1. In Lecture 16, page 5, the value of \( \Gamma_1'(1) \) is computed (it is \( -\gamma \)). What is the value of \( \Gamma_1'(n) \) where \( n = 2, 3, 4, \ldots \)?

2. Find the residue \( \text{Res}_{z=-n}(\Gamma_1(z)) \) where \( n = 0, 1, 2, 3, \ldots \).

3. Prove that \( \Gamma_1(z) = \lim_{n \to \infty} \frac{(n-1)!}{z(z+1) \cdots (z+n-1)} n^z \) where \( n^z = e^{z \ln(n)} \).

4. Let \( \Psi(z) = \frac{\Gamma_1'(z)}{\Gamma_1(z)} \). Prove that \( \Psi(z+1) = \Psi(z) + \frac{1}{z} \). Use this to solve the following difference equation:

\[ f(z+1) = f(z) + \frac{1}{z^2 - 3z + 2} \]

5. Let \( \Psi(z) \) be as in the previous problem. Prove that \( \Psi(z) - \Psi(1-z) = -\pi \cot(\pi z) \).

6. Let \( A(z) = \prod_{n=1}^N \left( \frac{z-a_n}{z-b_n} \right) \), where \( a_1, b_1, a_2, b_2, \ldots, a_N, b_N \) are complex numbers. Find a solution of \( F(z+1) = A(z)F(z) \). Where are its zeroes and poles?