(1) Assume $f(x)$ is a continuous function of a real variable $x$ (defined for every $x \in \mathbb{R}$). Assume further that
- $\int_{0}^{\infty} f(x) \, dx$ exists (and is finite). Meaning: for every $\varepsilon > 0$ there exists $T > 0$ such that $\left| \int_{0}^{T} f(x) \, dx \right| < \varepsilon$ for every $Q \geq T$.
- $C = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx$ exists (and is finite). Meaning: for every $\varepsilon > 0$ there exists $T > 0$ such that $\left| \int_{-Q}^{Q} f(x) \, dx - C \right| < \varepsilon$ for every $Q \geq T$.

Prove that $\int_{-\infty}^{\infty} f(x) \, dx$ exists and is equal to $C$.

(2) In the following steps, prove that $\int_{0}^{\infty} \frac{x \cos(x)}{x^2 - 2x + 10} \, dx$ exists.

(a) Let $b_1, b_2, \cdots$ be real numbers, such that $b_1 \geq b_2 \cdots \geq 0$. Assume that $\lim_{n \to \infty} b_n = 0$.

Then prove that $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

(b) Prove that $\frac{x}{x^2 - 2x + 10}$ is a decreasing function of $x$ for $|x| > \sqrt{10}$.

(c) Recall that $\cos(x)$ for $x \in \left[ \frac{(2n + 1)\pi}{2}, \frac{(2n + 3)\pi}{2} \right]$ is positive if $n$ is odd and negative if $n$ is even. Define real numbers $c_n$ by

$$(-1)^{n-1} c_n = \int_{\frac{(2n+1)\pi/2}{2}}^{\frac{(2n+3)\pi/2}{2}} \frac{x \cos(x)}{x^2 - 2x + 10} \, dx$$

Prove that $c_1 \geq c_2 \geq \cdots \geq 0$ and that $\lim_{n \to \infty} c_n = 0$.

(d) Use part (a) to conclude that $\sum_{n=1}^{\infty} (-1)^{n-1} c_n$ exists and that it equals $\int_{3\pi/2}^{\infty} \frac{x \cos(x)}{x^2 - 2x + 10} \, dx$.

(3) Prove that the Laplace transform of $\frac{t^n}{n!}$ (where $n \geq 0$ is an integer) is $z^{-n-1}$, for $\text{Re}(z) > 0$. (recall that the Laplace transform of a function $\varphi(t)$ of a real variable $t$ is given by $\int_{0}^{\infty} \varphi(t)e^{-zt} \, dt$.)