COMPLEX VARIABLES: HOMEWORK 4

(1) Prove that the following series converge.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)

**Solution.** We can compute the definite integral \( \int_1^{\infty} \frac{1}{x^2} \, dx = 1 \). The given series is bounded above by this integral, since the integral computes the area under the curve \( y = \frac{1}{x^2} \) from \( x = 1 \) to \( \infty \), and
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} \, dx
\]

(b) \( \sum_{n=1}^{\infty} \frac{n}{2^n} \)

**Solution.** Since \( \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \), for \( |z| < 1 \), taking derivative of this gives
\[
\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1}
\]
Hence we get \( \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2} \) for \( |z| < 1 \). Setting \( z = 1/2 \) gives
\[
\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2
\]
therefore, convergent.

(2) Find the radius of convergence of the following power series.

(a) \( \sum_{n=1}^{\infty} \frac{n}{2^n} z^n \)

**Solution.** Either look at the solution of the previous problem part (b), to conclude that \( \sum_{n=1}^{\infty} \frac{n}{2^n} z^n \) converges for \( |z/2| < 1 \), and diverges for \( |z/2| > 1 \), hence the radius of convergence is 2. Or, we can do it directly by taking the limit of the ratio of successive terms:
\[
\lim_{n \to \infty} \left| \frac{n+1}{n} \frac{2^n}{2^{n+1}} \frac{z^{n+1}}{z^n} \right| = \frac{|z|}{2}
\]
Thus the radius of convergence is 2.

(b) \( \sum_{n=1}^{\infty} \frac{1}{n} z^n \)

**Solution.** By ratio test, we have
\[
\lim_{n \to \infty} \left| \frac{n}{n+1} \frac{z^{n+1}}{z^n} \right| = |z|
\]
Hence the radius of convergence is 1.

(3) Let \( \sum_{n=1}^{\infty} c_n z^n \) be a power series with non-zero radius of convergence. Prove that the power series \( \sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!} \) has infinite radius of convergence.

**Solution.** There are several ways to do this. Assume \( R \) is the radius of convergence of \( \sum_{n=1}^{\infty} c_n z^n \). If we assume that all \( c_n \)'s are non-zero, then by ratio test we get that, for \( 0 < r < R \):

\[
\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| r = m < 1
\]

Then applying ratio test to \( \sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!} \) gives

\[
\lim_{n \to \infty} \frac{c_n}{c_{n-1}} \left| \frac{z}{n} \right| = m \lim_{n \to \infty} \left| \frac{z}{n} \right| = 0
\]

for any value of \( |z| \). Hence the radius of convergence of \( \sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!} \) is \( \infty \).

(In order to remove the assumption that all \( c_n \)'s are non-zero, we need to introduce a bit more notation. Namely, assume \( 1 \leq i_1 < i_2 < \cdots \) are the indices where \( c_{i_1}, c_{i_2}, \cdots \) are the only non-zero terms. Then the given series is \( \sum_{n=1}^{\infty} c_{i_n} z^{i_n} \). The same argument as before will prove that the radius of convergence of \( \sum_{n=1}^{\infty} c_{i_n} \frac{z^{i_n-1}}{(i_n-1)!} \) is infinity.)

Alternately, (see the proof of Abel’s Theorem (8.4) of Lecture 8), for every \( 0 < r < R \) we know that the numbers \( |c_n| r^n \) are bounded, by say \( M > 0 \). That is, for every \( n = 1, 2, 3, \cdots, \) we have \( |c_n| r^n \leq M \). Then the series \( \sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!} \) is dominated by the exponential series, since \( |c_n| \leq \frac{M}{r^n} \):

\[
\sum_{n=1}^{\infty} \frac{|c_n| |z|^{n-1}}{(n-1)!} \leq \frac{M}{r} \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{r^{n-1}(n-1)!} = \frac{M}{r} e^{|z|/r}
\]

which converges for all values of \( |z| \). Hence we get that \( \sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!} \) has infinite radius of convergence.

(4) Find the Taylor series expansion of \( \frac{z}{z^2 - 2z - 3} \) near \( z = 0 \). What is its radius of convergence?

**Solution.** Write the partial fraction decomposition using \( z^2 - 2z - 3 = (z-3)(z+1) \):

\[
\frac{z}{z^2 - 2z - 3} = \frac{1}{4} \left( \frac{3}{z-3} + \frac{1}{z+1} \right)
\]
Now we have
\[
\frac{3}{z-3} = -\frac{1}{1-(z/3)} = -\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n
\]
\[
\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n
\]

Hence we get
\[
\frac{z}{z^2-2z-3} = \frac{1}{4} \sum_{n=0}^{\infty} \left(-1\right)^n - \frac{1}{3^n} z^n
\]

To compute the radius of convergence, we take ratio of successive terms
\[
\left|\frac{(-1)^{n+1} - 3^{-n-1}}{(-1)^n - 3^{-n}} \right| z \to |z| \text{ as } n \to \infty
\]

Hence the radius of convergence is 1.

(5) Prove the following equation holds for $|z| < 1$, and any $l = 0, 1, 2, \ldots$:
\[
\frac{1}{(1-z)^{l+1}} = \sum_{n=0}^{\infty} \frac{(n+l)!}{n!!} z^n
\]

Solution. We have already checked this equation for $l = 0$. Namely, this was shown in class that \(\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n\), for $|z| < 1$. Now \(\frac{1}{(1-z)^{l+1}}\) is obtained from \(\frac{1}{1-z}\) by taking derivative $l$ times, and dividing by $l!$:
\[
\frac{1}{(1-z)^{l+1}} = \frac{1}{l!} \left( \frac{d}{dz} \right)^l \frac{1}{1-z}
\]

By Theorem (7.3) proved in the class, power series can be termwise differentiated, keeping the same radius of convergence. Hence we get (for $|z| < 1$):
\[
\frac{1}{(1-z)^{l+1}} = \frac{1}{l!} \left( \frac{d}{dz} \right)^l \frac{1}{1-z} = \frac{1}{l!} \left( \frac{d}{dz} \right)^l \sum_{n=0}^{\infty} z^n = \sum_{n=l}^{\infty} \frac{n(n-1) \cdots (n-l+1)}{l!} z^{n-l}
\]
\[
= \sum_{n=l}^{\infty} \frac{n!}{(n-l)!!} z^{n-l} = \sum_{m=0}^{\infty} \frac{(m+l)!}{m!!} z^m
\]

where in the last line, we set $m = n - l$.

(6) Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series with radius of convergence $R > 0$.

(a) Prove that for every $r$, with $0 < r < R$ we have
\[
\frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta \geq \sum_{n=0}^{\infty} |c_n|^2 r^{2n}
\]
**Solution.** Within the radius of convergence, the power series can be multiplied as polynomials. So, we get:

\[ |f(z)|^2 = f(z) \overline{f(z)} = \left( \sum_{n=0}^{\infty} c_n z^n \right) \left( \sum_{m=0}^{\infty} \overline{c_m} (\overline{z})^m \right) = \sum_{n,m} c_n \overline{c_m} z^n (\overline{z})^m \]

Uniform convergence implies that integral of \( |f(z)|^2 \) is same as sum of integrals of \( c_n \overline{c_m} z^n (\overline{z})^m \). Let us compute this.

\[
\frac{1}{2\pi} \int_0^{2\pi} c_n \overline{c_m} r^{n+m} e^{i\theta (n-m)} d\theta = \begin{cases} 
  c_n \overline{c_m} r^{2n} & \text{if } n = m \\
  0 & \text{if } n \neq m
\end{cases}
\]

(this is clear since we can easily check that for a non–zero integer \( l \) the integral \( \int_0^{2\pi} e^{il\theta} d\theta = 0 \).

Hence the integral to compute is

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}
\]

as required.

(b) Let \( M(r) \) be the absolute maximum of the function \(|f(re^{i\theta})|\) for \( 0 \leq \theta \leq 2\pi \). Recall that we have the following inequality (Lecture 8, section (8.7)):

\[ |c_n| \leq \frac{M(r)}{r^n} \text{ for every } n = 0, 1, 2, \ldots \text{ and } 0 < r < R \]

Use the previous part to prove that if there is \( n \) and \( r \) such that \(|c_n| = \frac{M(r)}{r^n} \), then \( f(z) = c_n z^n \).

**Solution.** Since \( M(r) \) is the largest value \(|f(z)|\) can take for \( z = re^{i\theta} \), we get

\[
\left| \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right| \leq \frac{1}{2\pi} M(r)^2 2\pi = M(r)^2
\]

Therefore, we obtain the following inequality:

\[
\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq M(r)^2
\]

Now if for some \( n \) and \( r \), \(|c_n| = \frac{M(r)}{r^n} \), then the corresponding term in the left–hand side of the inequality above will be

\[
|c_n|^2 r^{2n} = \frac{M(r)^2}{r^{2n}} r^{2n} = M(r)^2
\]

Since the other terms of the series are non–negative, in order to preserve the inequality, all other terms must be 0. Hence \( c_m = 0 \) for \( m \neq n \) and the function \( f(z) \) is just \( c_n z^n \).