Recall that we have the following set up. $\tau \in \mathbb{C}$ is a complex number lying in the upper half plane, that is, $\text{Im}(\tau) > 0$. Let $q = e^{\pi i \tau}$ and let $\theta(z; \tau)$ be the holomorphic function defined as:

$$
\theta(z; \tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^n (n-1) e^{2\pi i nz}
$$

1. Prove the following

$$
\theta(z; \tau) = 2i e^{\pi iz} \left( \sum_{n=1}^{\infty} (-1)^n q^{n(n-1)} \sin((2n-1)\pi z) \right)
$$

2. What is the limit of $\theta(z; \tau)$ as the imaginary part of $\tau$ goes to infinity? That is, compute the following:

$$
\lim_{\text{Im}(\tau) \to \infty} \theta(z; \tau)
$$

3. Consider the system of equations for an unknown function $f(z)$:

$$
f(z + 1) = f(z) \quad \text{and} \quad f(z + \tau) = e^{2\pi ia} f(z)
$$

where $a \in \mathbb{C}$ is a complex number.

(a) Use the theta function to write a solution of these equations.

(b) Prove that if $f_1(z)$ and $f_2(z)$ are two solutions, then their ratio is an elliptic function.

(c) Combine the previous two parts to prove the following: assuming $a \neq m + n\tau$ for any $m, n \in \mathbb{Z}$, there are no holomorphic solutions to these equations: $(f(z + 1) = f(z)$ and $f(z + \tau) = e^{2\pi ia} f(z)$).

4. Recall that $\theta_2(z; \tau) = \theta \left( z + \frac{1}{2}; \tau \right)$. Carry out the computations given in sections (20.5) and (20.6) of Lecture 20, for $\theta_2$ to prove the following:

$$
\frac{1}{\pi i} \left( \frac{1}{\theta_2(0; \tau)} \frac{\partial}{\partial \tau} \theta_2(0; \tau) \right) = 2 \sum_{n=1}^{\infty} \frac{q^{3n}}{(1 + q^{2n})^2}
$$