Recall that we have the following set up. \( \tau \in \mathbb{C} \) is a complex number lying in the upper half plane, that is, \( \text{Im}(\tau) > 0 \). Let \( q = e^{\pi i \tau} \) and let \( \theta(z; \tau) \) be the holomorphic function defined as:

\[
\theta(z; \tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i nz}
\]

(1) Prove the following

\[
\theta(z; \tau) = 2ie^{\pi iz} \left( \sum_{n=1}^{\infty} (-1)^n q^n(n-1) \sin((2n-1)\pi z) \right)
\]

**Solution.** From the definition of \( \theta(z; \tau) \) collect the terms which give same exponent of \( q \):

\[
q^{n(n-1)} = q^{-n(-n+1)} = q^{m(m-1)} \text{ where } m = -n + 1
\]

Therefore we get

\[
\theta(z; \tau) = \sum_{n \geq 0} (-1)^n q^{n(n-1)} \left( e^{2\pi inz} - e^{2\pi i(-n+1)z} \right)
\]

\[
= e^{\pi iz} \sum_{n \geq 0} (-1)^n q^{n(n-1)} \left( e^{\pi i(2n-1)z} - e^{-\pi i(2n-1)z} \right)
\]

\[
= 2ie^{\pi iz} \sum_{n \geq 0} (-1)^n q^{n(n-1)} \sin((2n-1)\pi z)
\]

as required.

(2) What is the limit of \( \theta(z; \tau) \) as the imaginary part of \( \tau \) goes to infinity? That is, compute the following:

\[
\lim_{\text{Im}(\tau) \to \infty} \theta(z; \tau)
\]

**Solution.** After setting \( q = 0 \) the only remaining terms in the formula of \( \theta(z; \tau) \) are the ones corresponding to \( n = 0, 1 \). Consequently, we get

\[
\lim_{\text{Im}(\tau) \to \infty} \theta(z; \tau) = 1 - e^{2\pi iz}
\]

(3) Consider the system of equations for an unknown function \( f(z) \):

\[
f(z+1) = f(z) \quad \text{and} \quad f(z+\tau) = e^{2\pi ia} f(z)
\]

where \( a \in \mathbb{C} \) is a complex number.

(a) Use the theta function to write a solution of these equations.

**Solution.** \[
\frac{\theta(z-a; \tau)}{\theta(z; \tau)}
\]

(b) Prove that if \( f_1(z) \) and \( f_2(z) \) are two solutions, then their ratio is an elliptic function.
Solution. Since both \( f_1 \) and \( f_2 \) satisfy the system of equations given above, we get:

\[
\frac{f_1(z + 1)}{f_2(z + 1)} = \frac{f_1(z)}{f_2(z)} \quad \text{and} \quad \frac{f_1(z + \tau)}{f_2(z + \tau)} = \frac{e^{2\pi ia} f_1(z)}{e^{2\pi ia} f_2(z)} = \frac{f_1(z)}{f_2(z)}
\]

Hence the ratio \( f_1/f_2 \) is elliptic.

(c) Combine the previous two parts to prove the following: assuming \( a \neq m + n\tau \) for any \( m, n \in \mathbb{Z} \), there are no holomorphic solutions to these equations: \( (f(z + 1) = f(z) \) and \( f(z + \tau) = e^{2\pi ia} f(z) \).

Solution. Assume that \( f(z) \) is a holomorphic solution. Then from the previous two parts we see that \( f(z) \frac{\theta(z; \tau)}{\theta(z - a; \tau)} \) is an elliptic function with at most one simple pole in any fundamental parallelogram. Thus it must be a constant (i.e, no poles at all). But that means

\[
f(z) = \text{Constant} \cdot \frac{\theta(z - a; \tau)}{\theta(z; \tau)}
\]

is holomorphic function. For the ratio of theta functions on the right–hand side, this is only possible when \( a = m + n\tau \) for some integers \( m, n \in \mathbb{Z} \).

(4) Recall that \( \theta_2(z; \tau) = \theta \left( z + \frac{1}{2}; \tau \right) \). Carry out the computations given in sections (20.5) and (20.6) of Lecture 20, for \( \theta_2 \) to prove the following:

\[
\frac{1}{\pi i} \left( \frac{1}{\theta_2(0; \tau)} \frac{\partial}{\partial \tau} \theta_2(0; \tau) \right) = 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 + q^{2n})^2}
\]

Solution. We use the heat equation:

\[
\frac{1}{\pi i} \partial_\tau = \frac{1}{(2\pi i)^2} \partial_z^2 - \frac{1}{2\pi i} \partial_z
\]

to reduce the problem to computing

\[
\frac{1}{\pi i} \left( \frac{1}{\theta_2(0; \tau)} \frac{\partial}{\partial \tau} \theta_2(0; \tau) \right) = \frac{\theta'_2(0)}{(2\pi i)^2 \theta_2(0)} - \frac{1}{2\pi i} \frac{\theta'_2(0)}{\theta_2(0)}
\]

Now \( \theta_2(z; \tau) = G \prod_{n \geq 0} (1 + q^{2n}e^{2\pi iz}) \prod_{n \geq 1} (1 + q^{2n}e^{-2\pi iz}) \). So taking logarithmic derivative gives:

\[
\frac{\theta'_2(z)}{\theta_2(z)} = \sum_{n \geq 0} \frac{q^{2n}2\pi i e^{2\pi iz}}{1 + q^{2n}e^{2\pi iz}} + \sum_{n \geq 0} \frac{q^{2n}(-2\pi i) e^{-2\pi iz}}{1 + q^{2n}e^{-2\pi iz}}
\]

Take derivative again to get:

\[
\frac{\theta''_2(z)}{\theta_2(z)} - \left( \frac{\theta'_2(z)}{\theta_2(z)} \right)^2 = \sum_{n \geq 0} \frac{q^{2n}(2\pi i)^2 e^{2\pi iz}}{(1 + q^{2n}e^{2\pi iz})^2} + \sum_{n \geq 0} \frac{q^{2n}(-2\pi i)^2 e^{-2\pi iz}}{(1 + q^{2n}e^{-2\pi iz})^2}
\]

Setting \( z = 0 \) in these two equations implies:

\[
\frac{\theta'_2(0)}{\theta_2(0)} = \pi i \quad \Rightarrow \quad \frac{\theta''_2(0)}{\theta_2(0)} = 2(\pi i)^2 + 2 \sum_{n \geq 1} \frac{q^{2n}(2\pi i)^2}{(1 + q^{2n})^2}
\]
Substitute it back in the original equation

\[ \frac{1}{(2\pi i)^2} \frac{\theta_2''(0)}{\theta_2(0)} - \frac{1}{2\pi i} \frac{\theta_2'(0)}{\theta_2(0)} = \frac{1}{2} + 2 \sum_{n \geq 1} \frac{q^{2n}}{(1 + q^{2n})^2} - \frac{1}{2} \]

to get the desired result.