A differential equation is an equation involving derivatives of an unknown function of $x$, usually denoted by $y$, and known functions of $x$. Most general differential equation is of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_0 w y + f(x) = 0$$

where $f(x)$, $a_0(x)$, $a_1(x)$, $\ldots$ $a_n(x)$ are functions of $x$. This is a linear differential equation of order $n$.

eq (i) $y' = f(x)$.

Solving this equation amounts to finding the integral of $f(x)$.

(ii) $y = \cos(x)$ is a solution to the differential equation of order 2, namely $y'' = -y$.

($y = \sin(x)$ is also a solution to this equation)
First order differential equations.

\( y' = f(x, y) \) is a general first order linear differential equation. If value of the unknown function \( y \) is given at a point \( x_0 \), say \( y(x_0) = y_0 \), we get a differential equation with initial condition.

Geometrically, \( y' \) gives the slope of the tangent line to the solution of the differential equation. Therefore \( y' = f(x, y) \) gives us tangent lines to the solution curve at each point.

* Also \( y'' = \frac{d}{dx} f(x, y) \) determines the concavity of the solution curve.

\( y'' > 0 \implies \text{concave up} \)

\( y'' < 0 \implies \text{concave down} \)

eg.

\[
\frac{dy}{dt} = y^4 - 6y^3 + 5y^2
\]

\( \text{RHS} = y^2(y^2 - 6y + 5) = y^2(y-5)(y-1) \) (Autonomous equation)
Clearly \( y = 0, \ y = 1, \ y = 5 \) are solutions of this equation (equilibrium).

For a general solution, \( y = f(t) \), the function will be increasing for \( y < 1 \) or \( y > 5 \) and decreasing for \( 1 < y < 5 \).

E.g., Which of the following could be a solution to \( y' = 1 + x^2 + y^2 \)?

\[ y' = 1 + x^2 + y^2 \geq 1 \] the function is always increasing, so it couldn't be I.

\[ y'' = 1 + 2x + 2y y' = 1 + 2x + 2y(1 + x^2 + y^2) > 0 \text{ for } x, y > 0 \]

it must be concave up in the first quadrant.

\[ \text{Answer: II.} \]

(8.2) Direction fields. For \( y' = f(x, y) \), we can sketch tangent lines to the solution curve at any given point \((x_0, y_0)\) with slope \( f(x_0, y_0) \), called direction field.
e.g. \( y' = x^2 \)

Direction field of \( y' = x^2 \)

\( y' = -xy \)

Direction field of \( y' = -xy \)

\( y' = 5-y \)

(autonomous equation)

Equilibrium solution
Euler's Method.

\[ y' = f(x, y) \quad y(x_0) = y_0. \]

- Choose a step size \( h \). Let \( x_1 = x_0 + h, \ x_2 = x_1 + h, \ldots, \ x_n = x_{n-1} + h \).

- Find values of \( y \) iteratively:
  
  \[ y_0 = y_0, \]
  
  \[ y_1 = y_0 + h f(x_0, y_0), \]
  
  \[ y_2 = y_1 + h f(x_1, y_1) \quad \text{and so on.} \]

- Assume we want to find the value of \( y \) at \( x = b \).
  
  Choose step size \( \frac{b-x_0}{n} \), so that \( x_n = b \). Euler's method gives an algorithm to find \( y_n \). As we take the smaller and smaller step sizes, the answer given by Euler's method becomes more and more accurate.
e.g. \( y' = xy - x^2 \) \( y(0) = 1 \)

Estimate \( y(x) \) using Euler's method with step size 0.2

\( x_0 = 0 \) \( x_1 = 0.2 \) \( x_2 = 0.4 \) \( x_3 = 0.6 \) \( x_4 = 0.8 \) \( x_5 = 1 \)

\( y_0 = 1 \)

\[ y_1 = y_0 + h f(x_0, y_0) = 1 + 0.2(0) = 1 \]

\[ y_2 = 1 + 0.2 \left( 0.2(1) - (0.2)^2 \right) = 1 + 0.2(0.16) = 1.032 \]