Absolute Convergence

(20.1) A series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent if the series \( \sum_{n=1}^{\infty} |a_n| \) is convergent.

**Theorem** Absolute convergence implies convergence.

**Proof.** Let \( \sum_{n=1}^{\infty} a_n \) be a series such that \( \sum_{n=1}^{\infty} |a_n| \) is convergent.

\[
0 \leq a_n + |a_n| \leq 2|a_n|
\]

Thus \( \sum_{n=1}^{\infty} a_n + |a_n| \) is a series with positive terms which are smaller than terms of a convergent series \( \sum_{n=1}^{\infty} 2|a_n| \). By the comparison test, \( \sum_{n=1}^{\infty} a_n + |a_n| \) is convergent.

Now term-wise sum or difference of convergent series is again convergent.

\[
\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - |a_n|
\]

is therefore convergent. \( \square \)
e.g. \[ \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \] is absolutely convergent since
\[ \left| \frac{\cos(n)}{n^2} \right| \leq \frac{1}{n^2} \quad \text{for every } n \geq 1. \]

Note: the converse of this theorem is not true. For instance,
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \] is convergent but \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent.

A series \( \sum_{n=1}^{\infty} a_n \) is called **conditionally convergent** if it is convergent but not absolutely convergent.

(20.2) **Ratio test for absolute convergence:**

- If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \) then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent (and hence convergent).
- If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \) or equals \( \infty \), then \( \sum_{n=1}^{\infty} a_n \) is divergent.
- If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \) then the test is inconclusive.
Proof. Assume \( L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1. \)

Let \( r \) be a positive number \( L < r < 1. \) Then there exists \( N > 0 \) such that

\[
\left| \frac{a_{n+1}}{a_n} \right| < r \quad \text{for} \quad n \geq N.
\]

\[
\Rightarrow |a_{N+1}| < r |a_N| \quad |a_{N+2}| < r^2 |a_N| \\
\ldots |a_{N+k}| < r^k |a_N|.
\]

Hence \( \sum_{n=N}^{\infty} |a_n| \) is a series with positive terms which are smaller than terms of the series \( \sum_{k=1}^{\infty} |a_N| r^k. \) Since \( r < 1, \)

the geometric series \( \sum_{k=1}^{\infty} |a_N| r^k \) converges, and hence by the geometric series \( \sum_{k=1}^{\infty} |a_N| r^k \) converges. Therefore, \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

Now assume that \( L > 1. \) Then there exists \( N > 0 \) such that

\[
\left| \frac{a_{n+1}}{a_n} \right| > 1 \quad \text{for} \quad n \geq N
\]

\[
\Rightarrow |a_{n+1}| > |a_n| \quad \text{for} \quad n \geq N \quad \text{and hence} \quad \lim_{n \to \infty} a_n \neq 0.
\]
Thus \( \sum_{n=1}^{\infty} a_n \) is divergent.

Example (i) \( \sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!} \)

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-3)^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-3)^n} \right| = \frac{3}{(2n+3)(2n+2)} \quad \text{as} \quad n \to \infty
\]

Hence by ratio test: \( \sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!} \) is absolutely convergent.

(ii) \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \)

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n^n}
\]

\[
= \left(1 + \frac{1}{n}\right)^n \quad \to \quad e \quad \text{as} \quad n \to \infty
\]

Since \( e > 1 \), \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \) is divergent.
(iii) For $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ the ratio test is inconclusive.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{n^p}{(n+1)^p} = \frac{1}{\left(1 + \frac{1}{n}\right)^p} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty.
\]

(20.3) Root test.

. \quad \text{if } \lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1 \quad \text{then } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent}.

. \quad \text{if } \lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1 \text{ or infinity} \quad \text{then } \sum_{n=1}^{\infty} a_n \text{ is divergent}.

. \quad \text{if } \lim_{n \to \infty} \sqrt[n]{|a_n|} = 1 \quad \text{then the root test is inconclusive.}

Remark. Ratio test is effective when terms of the series involve factorials or exponents (e.g. $\frac{n^5}{2^n!}$).

Root test is effective when terms of the series involve $n^{th}$ powers.
\[ \sum_{n=2}^{\infty} \left( \frac{-2n}{n+1} \right)^{5n} \]

\[ 2^{n/2} |a_n| = \left( \left( \frac{2n}{n+1} \right)^{5n} \right)^{1/n} = \left( \frac{2n}{n+1} \right)^{5} \rightarrow 32 \quad \text{as } n \rightarrow \infty \]

\[ \Rightarrow \sum_{n=2}^{\infty} \left( \frac{-2n}{n+1} \right)^{5n} \text{ is divergent}. \]

\[ \text{eg. Show that } \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converges for any value of } x \]

\[ \text{Ratio test: } \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty \]

\[ \text{Hence } \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is convergent}. \]

(20.4) Summary of convergence tests.

(i) \[ \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent for } p > 1 \text{ and divergent for } p \leq 1. \]

(ii) \[ \sum_{n=1}^{\infty} a \cdot r^n \text{ is convergent for } |r| < 1 \text{ and divergent for } |r| \geq 1. \]
(ii) Comparison test

(iii) Test for divergence: if \( \lim_{n \to \infty} a_n \neq 0 \) then \( \sum_{n=1}^{\infty} a_n \) diverges.

(iv) Alternating Series test

(v) Ratio and root tests

(vi) Integral tests.

e.g. \[ \sum_{n=1}^{\infty} \frac{3^n n^2}{n!} \]

Use ratio test: \[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{3^{n+1} (n+1)^2}{(n+1)!}}{\frac{3^n n^2}{n!}} = \frac{3(n+1)^2}{(n+1)n^2} = \frac{3(n+1)}{n^2} \rightarrow 0 \text{ as } n \rightarrow \infty \]

\[ \Rightarrow \sum_{n=1}^{\infty} \frac{3^n n^2}{n!} \text{ is absolutely convergent.} \]
$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n^2}}}{n^2}$

Integral test: $f(x) = \frac{e^x}{x^2}$

Check: decreasing function.

$\int_{1}^{\infty} f(x) \, dx = \int_{1}^{\infty} \frac{e^{\frac{1}{x}}}{x^2} \, dx$

Let $u = \frac{1}{x}$, $du = \frac{-1}{x^2} \, dx$

$= \int_{1}^{0} e^u (-du) = [e^u]_0^1 = e - 1$