(11.0) Recall: we have studied two classes of differential equations

- **Separable** \[ y' = \frac{g(x)}{h(y)} \]
  \[
  \text{Solution: } \int h(y) \, dy = \int g(x) \, dx
  \]

- **Linear** \[ y' + P(x) \, y = Q(x) \]
  \[
  \text{Solution: let } I(x) = e^{\int P(x) \, dx} \quad \text{(integrating factor)}
  \]
  \[
  \text{then } I(x) \cdot y = \int I(x) \cdot Q(x) \, dx
  \]

Some examples

(1) \[ x^2 y' - y = 2x^3 e^{-\frac{1}{x}} \quad \text{Linear} \]

\[ \Rightarrow y' - \frac{x^2}{x} \, y = 2x e^{-\frac{1}{x}} \]

\[ I(x) = e^{\int -x^2 \, dx} = e^x \quad \text{(integrating factor)} \]

\[ \text{Solution: } e^x \cdot y = \int e^x \left( 2x e^{-\frac{1}{x}} \right) \, dx \]

\[ = x^2 + C \]

\[ \Rightarrow y = x^2 e^{-\frac{1}{x}} + Ce^{-\frac{1}{x}} \]
(2) Find orthogonal trajectories of the family of curves 
\[ y = e^{kx} \]

\[ y' = k \cdot e^{kx} \quad \text{Slope of the orthogonal trajectories} = \frac{-1}{k e^{kx}} \]

\[ \ln y = kx \implies k = \frac{\ln y}{x} \]

**Differential equation:** \[ y' = -\frac{1}{ky} = -\frac{x}{y \ln y} \quad \text{(Separable)} \]

\[ \int y \ln y \, dy = -\int x \, dx \]

**L.H.S.** \[ \int y \ln y \, dy = \ln(y) \frac{y^2}{2} - \int \frac{1}{y} \frac{y^2}{2} \, dy \]

\[ = \ln(y) \frac{y^2}{2} - \frac{y^2}{4} \]

**Solution:** \[ \ln(y) \frac{y^2}{2} - \frac{y^2}{4} = -\frac{x^2}{2} + C \]

(3) A tank contains 100 lt. of water. A solution with a salt concentration of 0.4 kg/ltr is added at the rate of 5 ltr/min. The solution is kept thoroughly mixed and the tank is drained at a rate of 3 ltr/min.
\[ y(t) = \text{amount of salt (in kg) in the tank at time } t. \]

Write the differential equation satisfied by \( y(t) \).

Rate in \( = (0.4)(5) \text{ kg/min} \).

Rate out \( = \frac{y(t)}{V(t)} (3) \text{ kg/min} \).

\[ V(t) = \text{amount of water (in lt.) in the tank}. \]
\[ = 100 + 2t \]

\[ \Rightarrow \text{Rate out} = \frac{3y}{100 + 2t} . \]

Hence we get

\[ \frac{dy}{dt} = 2 - \frac{3y}{100 + 2t} \quad (\text{Linear}) \]

\[ y' + \left( \frac{3}{100+2t} \right) y = 2 . \quad I(t) = e^{-\frac{3}{2} \ln(100 + 2t)} \]

\[ = e^{-\frac{3}{2} \ln(100 + 2t)} \]
\[ = (100 + 2t)^{-\frac{3}{2}} \]

Solution: \( (100 + 2t)^{\frac{3}{2}} y = \int 2 (100 + 2t)^{\frac{3}{2}} dt \)

\[ = 2 \left( \frac{(100+2t)^{\frac{1}{2}}}{\frac{1}{2}} \right) + C \]
\( t = 0, \quad y(0) = 0 \)

\[ \Rightarrow 0 = 2(10) + C \Rightarrow C = -20 \]

\[
(10^n + 2t) \ y = 2 \ (10^n + 2t)^{\frac{1}{2}} - 20
\]

\[
y = \frac{2}{10^n + 2t} - \frac{20}{(10^n + 2t)^{3/2}}
\]

(11.1) Predator - Prey systems.

\[
\frac{dR}{dt} = kR - aRW
\]

\[
\frac{dW}{dt} = -rW + bRW
\]

\[
\{\text{Predator - Prey equations.}\}
\]

\[
R = \text{population of preys (rabbits)}
\]

\[
W = \text{population of predators (wolves)}
\]

\[
a, b, k, r \text{ are positive constants.}
\]
Concrete example

\[ \frac{dR}{dt} = R - 0.02RW \]
\[ \frac{dW}{dt} = -2W + 0.01RW \]

* equilibrium solution: \( R(1 - 0.02W) = 0 \)
  \( W(-2 + 0.01R) = 0 \)

\[ \Rightarrow \quad R = W = 0 \]

or \[ W = \frac{1}{0.02} = 50 \]

\[ R = \frac{2}{0.01} = 200 \]

* Direction field

\[ \frac{dW}{dR} = \frac{dW}{dt} \frac{dt}{dR} = \frac{-2W + 0.01RW}{R - 0.02RW} \]

can be used to draw the direction field in RW plane
\[ x' = x - (0.02) \times y \]
\[ y' = -2y + (0.01) \times xy \]
Assume at $t=0$ \( R = 200 \) and \( W = 10 \)

then \( \frac{dR}{dt} = 200 \left( 1 - 0.02 (10) \right) > 0 \)

eg. \( \frac{dA}{dt} = 2A \left( 1 - 0.0001A \right) - 0.01 AL \)

\( \frac{dL}{dt} = -0.5L + 0.0001AL \)

equilibrium: \( A = \frac{0.5}{0.0001} = 5000 \)

\( L = \frac{1}{0.01} = 100 \)
\[ x' = 2x(1 - 0.0001x) - 0.01xy \]
\[ y' = -(0.5)y + 0.0001xy \]