1. (a) \[ \frac{d}{dx} \left( \tan^{-1} \left( \sqrt{1-x^2} \right) \right) = \frac{1}{1 + \left( \sqrt{1-x^2} \right)^2} \cdot \frac{d}{dx} \left( \sqrt{1-x^2} \right) \]

\[ = \frac{1}{1 + 1 - x^2} \cdot \frac{1}{2 \sqrt{1-x^2}} \cdot \frac{d}{dx} \left( 1-x^2 \right) = \frac{1}{2-x^2} \cdot \frac{1}{2 \sqrt{1-x^2}} \cdot (-2x) \]

\[ = \frac{1}{(2-x^2)} \cdot \frac{(-x)}{\sqrt{1-x^2}} = \frac{-x}{(2-x^2) \sqrt{1-x^2}} \]

(b) \[ \frac{d}{dx} \left( \log_5 (2x+1) \right) = \frac{1}{(2x+1) \cdot \ln(5)} \cdot \frac{d}{dx} (2x+1) \]

\[ = \frac{2}{(2x+1) \ln(5)} \]

(c) \[ \lim_{x \to 0} (1+3x)^{\cot(x)} : \text{ indeterminable form} \]

\[ y = (1+3x)^{\cot(x)} \Rightarrow \ln(y) = \cot(x) \cdot \ln(1+3x) \]

\[ \lim_{x \to 0} \ln(y) = \lim_{x \to 0} \frac{\ln(1+3x)}{\tan(x)} : \frac{0}{0} \text{ indeterminable form} \]

\[ = \lim_{x \to 0} \frac{\frac{d}{dx} \ln(1+3x)}{\frac{d}{dx} \tan(x)} \text{ by L'Hopital rule} \]
\[
\lim_{x \to 0} \frac{1}{1+3x} \cdot 3 \cdot \sec^2(x) = \frac{3}{1} = 3
\]

\Rightarrow \lim_{x \to 0} \cot(x) = 3.

(d) \quad \frac{d}{dx} \ln(\sec(x)) = \frac{1}{\sec(x)} \cdot \frac{d}{dx} (\sec(x))

= \frac{1}{\sec(x)} \cdot \sec(x) \tan(x) = \tan(x)
2. Let $m =$ slope of the line

Note: $m < 0$ for the line to cut a triangle in the first quadrant.

Equation of the line: $y - 5 = m(x - 3)$

Point $P$: is given by $y = 0$. $\Rightarrow -5 = m(x - 3)
\implies x = 3 - \frac{5}{m}$

Point $Q$: is given by $x = 0$. $\Rightarrow y - 5 = m(-3)
\implies y = 5 - 3m$

Area of the triangle $OPQ$: $A(m) = \frac{1}{2} \left( 3 - \frac{5}{m} \right) (5 - 3m)$

$m \in (-\infty, 0)$

$$\frac{dA}{dm} = \frac{1}{2} \left[ \frac{5}{m^2} (5 - 3m) + (3 - \frac{5}{m}) (-3) \right] = \frac{1}{2} \left[ \frac{25}{m^2} - 9 \right] = \frac{1}{2} \left[ \frac{25 - 9m^2}{m^2} \right]$$

$$\frac{dA}{dm} = 0 \equiv 25 - 9m^2 = 0 \equiv m^2 = \frac{25}{9} \equiv m = \frac{-5}{3} \quad \text{(since } m < 0)$$

$$\frac{dA}{dm} > 0 \text{ for } m \in \left( -\frac{5}{3}, 0 \right) \text{ and } \frac{dA}{dm} < 0 \text{ for } m \in (-\infty, -\frac{5}{3})$$

Hence $m = -\frac{5}{3}$ is the absolute minimum.

Equation of the line: $y - 5 = -\frac{5}{3} (x - 3)$
3. \( f(x) = \tan^(-1)(x) \)  
\[ f'(x) = \frac{1}{1+x^2} \]

\[ f(1) = \frac{\pi}{4} \]  
\[ f'(1) = \frac{1}{2} \]

**Linear approx. of** \( f(x) \) **at** \( x = 1 \):

\[ L(x) = f(1) + \frac{1}{2} (x - 1) = \frac{\pi}{4} + \frac{1}{2} (x-1) \]

\[ \tan^(-1)(1.01) \approx L(1.01) = \frac{\pi}{4} + \frac{1}{2} (0.01) \]

\[ = \frac{\pi}{4} + 0.005 \]
4. (a) False. **Counterexample:** let \( f(x) = x^3 \). Then \( f'(x) = 3x^2 \).
   \[
   f'(0) = 0 \quad \text{but} \quad 0 \text{ is neither a local max nor a local min.}
   \]

   (b) True. \( f(x) = x^3 + x - 1 \)
   \[
   f(0) = -1 \quad \text{and} \quad f(1) = 1. \quad \text{Thus by I.V.T. } f(x) = 0 \text{ has at least one solution in } [0,1].
   \]

   \( f'(x) = 3x^2 + 1 \quad \text{is always} \quad \geq 1 \)
   
   If there were two solutions of \( f(x) = 0 \) in \([0,1]\) then by M.V.T. \( f'(c) = 0 \) for some \( c \) in \((0,1)\). Since this is not the case, \( f(x) = 0 \) has exactly one solution.

   (c) True. Let \( \theta = \cos^{-1}(x) \). Then \( \cos(\theta) = x \)
   \[
   \sin(\theta) = \pm \sqrt{1 - \cos^2 \theta} = \pm \sqrt{1 - x^2}
   \]
   Since \( \sin(\theta) \geq 0 \) for \( \theta \in [0,\pi] \), we get
   \[
   \sin(\cos^{-1}(x)) = \sqrt{1 - x^2}
   \]

   (d) False. Let \( c_1 < c_2 \) be the two local min. of \( f \).
   Then \( f' \) is positive to the right of \( c_1 \) (near \( c_1 \))
   and negative to the left of \( c_2 \) (near \( c_2 \)).

   Since \( f'(x) \) exists for every \( x \), it must change sign to \(- \) somewhere
   in \((c_1, c_2)\). Thus \( f \) must have a local max.
\[ f(x) = \frac{\sin(x)}{1 + \cos(x)} \]

(a) \[ 1 + \cos(x) = 0 \quad \equiv \quad \cos(x) = -1 \]
\[ \equiv x = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \]

\[ \text{Domain of } f = \{ x \neq (2n+1)\pi \text{ for any integer } n \} \]

(b) \[ f'(x) = \frac{(1 + \cos(x))\cos(x) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} = \frac{\cos^2 x + \cos^2 x + \sin^2 x}{(1 + \cos(x))^2} \]
\[ = \frac{1 + 2\cos(x)}{(1 + \cos(x))^2} = \frac{1}{1 + \cos(x)} > 0 \quad (\text{since } -1 \leq \cos(x) \leq 1) \]

Since \( f'(x) > 0 \) for every \( x \) in the domain of \( f \), \( f(x) \) has no critical points, and \( f \) is always increasing.

(c) \[ f''(x) = \frac{-1}{(1 + \cos(x))^2} \cdot (-\sin(x)) = \frac{\sin(x)}{(1 + \cos(x))^2} \]

\[ f''(x) = 0 \quad \equiv \quad \sin(x) = 0 \quad \equiv \quad x = 0, \pm \pi, \pm 2\pi, \ldots \]
and
\[ \cos(x) \neq -1 \]
\[ x \neq (2n+1)\pi \]

Thus \( f''(x) = 0 \) for \( x = 0, \pm 2\pi, \pm 4\pi, \ldots = 2n\pi \text{ for integer } n \).

Sign of \( f'' = \text{sign of } \sin(x) \)
\[
\Rightarrow \quad \text{Sign of } f'' \quad \Rightarrow \\
\quad -\infty \quad -\pi \quad 0 \quad \pi \quad 2\pi \quad 3\pi \quad 4\pi \\
\end{align*}

\( f(x) \) is concave up on \((0, \pi), (2\pi, 3\pi), (4\pi, 5\pi)\) \\
concave down on \((\pi, 2\pi), (3\pi, 4\pi)\) \\

Equivalently, \( f(x) \) is concave up on \( (2n\pi, (2n+1)\pi) \) for \( n \): integer \\
concave down on \( (2n+1)\pi, (2n+2)\pi) \)

Thus \( x = 2n\pi \) is an inflection point for every integer \( n \).

\[(d) \quad \lim_{x \to (2n+1)\pi^-} f(x) = \lim_{x \to (2n+1)\pi^-} \frac{\sin(x)}{1 + \cos(x)} = \frac{0}{0} \text{ indeterminate form} \]

\[
= \lim_{x \to (2n+1)\pi^-} \frac{\cos(x)}{-\sin(x)} = +\infty
\]

Similarly \( \lim_{x \to (2n+1)\pi^+} f(x) = -\infty \)

\[
\begin{align*}
(\text{e})
\end{align*}
\]

\[
\begin{align*}
\therefore f(0) = 0 \quad \therefore \quad f(2\pi) = 0 \\
\end{align*}
\]

\[
\text{concave up} \quad \text{concave down}
\]
Bonus: Graph of $f(x)$.

Note: $\sin(x + 2\pi) = \sin(x)$ \{ implies $f(x + 2\pi) = f(x)$ \}
$\cos(x + 2\pi) = \cos(x)$
Given: \( \frac{dx}{dt} = 10 \text{ m/min} \).

Now \( \frac{x}{15} = \frac{y}{50} \) implies \( y = \frac{50}{15} x \).

Take derivative with respect to \( t \):

\[
\frac{dy}{dt} = \frac{50}{15} \cdot \frac{dx}{dt}
\]

\[
= \frac{50}{15} \times 10 \text{ m/min}
\]

\[
= \frac{100}{3} \text{ m/min} \approx 33.33 \text{ m/min}
\]
7. To compute $a^{\frac{1}{3}}$, we need to find solution of

$$f(x) = x^3 - a = 0$$

$$f'(x) = 3x^2$$

Newton's Method gives iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - a}{3x_n^2}$$

$$= x_n - \frac{1}{3}x_n + \frac{a}{3x_n^2}$$

$$= \frac{a}{3}x_n + \frac{a}{3x_n^2}$$