MID TERM I
CALCULUS I (18546 SEC 5)

Problem 1 (20 points). Evaluate the following limits.

(1) \[ \lim_{x \to \infty} \frac{e^x + 1}{3e^x - 1} = \lim_{x \to \infty} \frac{e^x(1 + e^{-x})}{e^x(3 - e^{-x})} = \lim_{x \to \infty} \frac{1 + e^{-x}}{3 - e^{-x}} = \frac{1 + 0}{3 - 1(0)} = \frac{1}{3} \]

Since \( \lim_{x \to \infty} e^{-x} = 0 \).

(2) \[ \lim_{h \to 0} \frac{(1 + h)^{\frac{1}{3}} - 1}{h} = \lim_{h \to 0} \left( \frac{(1 + h)^{\frac{1}{3}} - 1}{h} \right) \left( \frac{(1 + h)^{2/3} + (1 + h)^{1/3} + 1}{(1 + h)^{2/3} + (1 + h)^{1/3} + 1} \right) = \lim_{h \to 0} \frac{1}{h((1 + h)^{2/3} + (1 + h)^{1/3} + 1)} = \frac{1}{3} \]

Since we have \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \).

(3) \[ \lim_{y \to \infty} \left( \sqrt{1 + y^2} - y \right) = \lim_{y \to \infty} \left( \left( \sqrt{1 + y^2} - y \right) \left( \frac{\sqrt{1 + y^2} + y}{\sqrt{1 + y^2} + y} \right) \right) = \lim_{y \to \infty} \frac{1 + y^2 - y^2}{\sqrt{1 + y^2} + y} = \lim_{y \to \infty} \frac{1}{\sqrt{1 + y^2} + y} = 0 \]

Since the denominator \( \sqrt{1 + y^2} + y \) tends to \( +\infty \) as \( y \to \infty \).

(4) \[ \lim_{h \to 0} \frac{e^{\tan(h)} - 1}{\sin(h)} = \lim_{h \to 0} \left( \frac{e^{\tan(h)} - 1}{\tan(h)} \right) \left( \frac{1}{\cos(h)} \right) \]

Since \( \tan(h) = \frac{\sin(h)}{\cos(h)} \). Now both the limits on the right-hand side exist:

\[ \lim_{h \to 0} \frac{e^{\tan(h)} - 1}{\tan(h)} = \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \]
\[ \lim_{h \to 0} \frac{1}{\cos(h)} = \frac{1}{\cos(0)} = 1 \]

where in the first computation, we have used the substitution \( x = \tan(h) \) and the fact that \( \lim_{h \to 0} \tan(h) = 0 \); which allows us to use the composition property of limits. Thus we have:

\[ \lim_{h \to 0} \left( \frac{e^{\tan(h)} - 1}{\tan(h)} \right) \left( \frac{1}{\cos(h)} \right) = \left( \lim_{h \to 0} \frac{e^{\tan(h)} - 1}{\tan(h)} \right) \left( \lim_{h \to 0} \frac{1}{\cos(h)} \right) = (1)(1) = 1 \]
Problem 2 (20 points). Let \( f(x) = x^2 \).

(1) Compute \( f'(x) \) directly from the definition of the derivative.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \]
\[
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} 2x + h
\]
\[
= 2x
\]

(2) What is the equation of the tangent line \( L \) to \( y = f(x) \) at the point \( P = (a, a^2) \) on the graph?

The equation of the tangent line \( L \) at the point \( P = (a, a^2) \) has slope given by \( f'(a) = 2a \) (from the previous part).

Therefore the equation is

\[
y - a^2 = f'(a)(x - a)
\]
\[
\equiv y - a^2 = 2a(x - a)
\]
\[
\equiv y = 2ax - a^2
\]

(3) Let \( Q \) be the point where \( L \) intersects \( y \)-axis and \( R \) be the point where \( L \) intersects \( x \)-axis. Verify that \( R \) is the mid point of the line segment joining \( P \) and \( Q \).

The point \( Q \) has \( x \)-coordinate equal to 0. Substituting \( x = 0 \) in the equation \( y = 2ax - a^2 \) of line \( L \), we get \( y = -a^2 \). Therefore \( Q = (0, -a^2) \).

The point \( R \) has \( y \)-coordinate equal to 0. Substituting \( y = 0 \) in the equation \( y = 2ax - a^2 \) of line \( L \), we get \( x = a/2 \). Therefore \( R = (a/2, 0) \).

The mid point of \( P \) and \( Q \) has coordinates equal to

\[
\left( \frac{a + 0}{2}, \frac{a^2 + (-a^2)}{2} \right) = (a/2, 0) = R
\]
Problem 3 (10 points). Find a degree two polynomial \( P(x) \) so that we have
\[
P(2) = 6 \quad P'(2) = 3 \quad P''(2) = 1
\]
A general degree two polynomial is of the form \( P(x) = ax^2 + bx + c \). Thus we have
\[
P'(x) = 2ax + b \quad P''(x) = 2a
\]
The given constraints \( P(2) = 5 \), \( P'(2) = 3 \) and \( P''(2) = 2 \) translate to the following
\[
4a + 2b + c = 6 \\
4a + b = 3 \\
2a = 1
\]
The last equation implies that \( a = 1/2 \). Substituting this value of \( a \) in the second equation gives:
\[
4(1/2) + b = 3 \Rightarrow b = 1
\]
Substituting \( a = 1/2 \) and \( b = 1 \) in the first equation gives:
\[
4(1/2) + 2(1) + c = 6 \Rightarrow c = 2
\]
Therefore, we get \( P(x) = \frac{1}{2}x^2 + x + 2 \).
Problem 4 (20 points). If a particle is thrown upwards at time \( t = 0 \) from a height of 100\( m \), with an initial velocity of 40\( m/s \), its height at time \( t \) is given by

\[
f(t) = 40t - 5t^2 + 100 \text{ in meters}
\]

(1) At what time does the particle hit the ground?

The particle hits the ground when \( f(t) = 0 \). That is,

\[
40t - 5t^2 + 100 = 0 \iff -5(t^2 - 8t - 20) = 0 \\
\iff t^2 - 8t - 20 = 0 \iff (t - 10)(t + 2) = 0
\]

Thus at \( t = 10s \) the height is zero and hence the particle is at the ground.

(2) What will its velocity be when it hits the ground?

The velocity of the particle is given by \( f'(t) = 40 - 10t \). At \( t = 10 \), thus, we have

\[
f'(10) = 40 - 10(10) = -60m/s
\]
Problem 5 (20 points). True/False. Give proof for the true statements and counterexample for the false ones.

(1) For any two functions \( f(x) \) and \( g(x) \),
\[
\lim_{x \to a} (f(x)g(x)) = \left( \lim_{x \to a} f(x) \right) \left( \lim_{x \to a} g(x) \right)
\]
False. (The given equation is only true when both the limits on the right-hand side exist)

**Example:** Let \( f(x) = x \) and \( g(x) = \frac{1}{x} \). For \( a = 0 \) we get
\[
\lim_{x \to 0} (f(x)g(x)) = \lim_{x \to 0} \left( x \frac{1}{x} \right) = \lim_{x \to 0} 1 = 1
\]
\[
\left( \lim_{x \to 0} f(x) \right) \left( \lim_{x \to 0} g(x) \right) = \left( \lim_{x \to 0} x \right) \left( \lim_{x \to 0} \frac{1}{x} \right) \text{ does not exist}
\]

(2) If \( f(x) \) has a horizontal asymptote at \( y = 3 \), then \( f(x-2) \) has a horizontal asymptote at \( y = 5 \).

False. (The graph of \( f(x-2) \) is obtained by shifting the graph of \( f(x) \) to the right by 2 units, which does not change the horizontal asymptotes)

**Example:** Let \( f(x) = 3 \). Then \( \lim_{x \to \pm \infty} f(x) = 3 \). Thus \( y = 3 \) is the horizontal asymptote of \( f(x) \). Now \( f(x-2) = 3 \) and \( \lim_{x \to \pm \infty} f(x-2) = 3 \) as well. So \( y = 3 \) (NOT \( y = 5 \)) is the horizontal asymptote of \( f(x-2) \).

(3) The equation \( \sin(x) + x = 9 \) has a solution.

True.

**Proof:** Let \( f(x) = \sin(x) + x \). This function is continuous on \( \mathbb{R} \). Now
\[
f(0) = \sin(0) + 0 = 0 < 9
\]
\[
f(3\pi) = \sin(3\pi) + 3\pi = 3\pi > 9
\]
Therefore, by the Intermediate Value Theorem, there exists \( c \in [0, 3\pi] \) such that \( \sin(c) + c = 9 \).

(4) For any function \( f(x) \), if \( |f(x)| \) is continuous, then so \( f(x) \).

False.

**Example:** Let \( f(x) \) be given by
\[
f(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
-1 & \text{if } x < 0
\end{cases}
\]
Then \( |f(x)| = 1 \) for every \( x \) and is continuous on \( \mathbb{R} \). However, \( f(x) \) is not continuous at \( x = 0 \).
Problem 6 (10 points). Consider the function
\[ f(x) = \frac{2x^3 - 16}{x^2 - 7x + 6} \]

(1) Identify the points \( a \) where the function has a vertical asymptote \( x = a \).

The function \( f(x) \) will have vertical asymptote at the points \( a \) where the denominator vanishes (and numerator does not). We write the denominator as
\[ x^2 - 7x + 6 = (x - 1)(x - 6) \]
which is zero at \( x = 1 \) and \( x = 6 \). Furthermore, the numerator takes values
\[ 2(1)^3 - 16 = 2 - 16 = -14 \]
\[ 2(6)^3 - 16 = 432 - 16 = 416 \]

Therefore, \( x = 1 \) and \( x = 6 \) are the vertical asymptotes of \( f(x) \).

(2) Compute \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \) for each \( a \) from the previous part.

In order to compute these limits, we only need to determine the sign of \( f(x) \) near \( x = 1 \) and \( x = 6 \).

\( (x - 1)(x - 6) \) is:
- \( < 0 \) for \( 1 < x < 6 \)
- \( > 0 \) for \( x > 6 \)
- \( < 0 \) for \( x < 1 \)

and the numerator is negative at \( x = 1 \) and positive at \( x = 6 \). Combining these we have
- \( f(x) = \frac{\text{negative number}}{\text{positive number}} < 0 \) for \( x < 1 \) near 1.
- \( f(x) = \frac{\text{negative number}}{\text{negative number}} > 0 \) for \( x > 1 \) near 1.
- \( f(x) = \frac{\text{positive number}}{\text{negative number}} < 0 \) for \( x < 6 \) near 6.
- \( f(x) = \frac{\text{positive number}}{\text{positive number}} < 0 \) for \( x > 6 \) near 6.

Thus we get
\[
\lim_{x \to 1^-} f(x) = -\infty \quad \lim_{x \to 1^+} f(x) = +\infty \\
\lim_{x \to 6^-} f(x) = -\infty \quad \lim_{x \to 6^+} f(x) = +\infty
\]

Bonus (5 points). Find the equation of the line \( y = mx + c \), which is an asymptote for \( f(x) \).

By the long division we have
\[
\frac{2x^3 - 16}{x^2 - 7x + 6} = 2x + \frac{14x^2 - 12x - 16}{x^2 - 7x + 6} = 2x + 14 + \frac{86x - 100}{x^2 - 7x + 6}
\]

Therefore we have \( \lim_{x \to \pm\infty} (f(x) - 2x - 14) = 0 \). Hence \( f(x) \) has an asymptote \( y = 2x + 14 \).