(20.0) Recall from last time:

Let \( f(x) \) be a continuous function on \([a, b]\). Then

I. (Fundamental Theorem of Calculus Part 1)

\[
g(x) := \int_{a}^{x} f(t) \, dt \\
\text{for } x \in [a, b]
\]

\( g(x) \) is differentiable function on \((a, b)\) and \( g'(x) = f(x) \)

II. (Fundamental Theorem of Calculus Part 2)

Let \( F(x) \) be any antiderivative of \( f(x) \). Then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

Ex. Find area under \( y = \sin(x) \), \( x \in [0, \pi] \)

\[
\int_{0}^{\pi} \sin(x) \, dx = \left[ -\cos(x) \right]_{0}^{\pi}
\]

\[
= (-(-1)) - (-1) = 1 + 1 = 2.
\]

If we are able to find antiderivative (any!), then area under the curve is computed using FTC 2.
(20.1) Substitution rule for integrals.

- This is an analogue of the Chain rule for derivatives.

Let \( F(x) \) be an antiderivative of \( f(x) \), i.e.

\[
\frac{d}{dx} F(x) = f(x)
\]

Chain rule implies, for a differentiable function \( g(x) \),

\[
\frac{d}{dx} (F(g(x))) = f(g(x)) \cdot g'(x)
\]

i.e., \( F(g(x)) \) is an antiderivative of \( f(g(x)) \cdot g'(x) \).

This observation is written more compactly as

\[
\int f(u) \, du = \int f(g(x)) \cdot g'(x) \, dx
\]

when we used the substitution \( u = g(x) \).

Ex. Compute \( \int (2 + t)^5 \, dt \)

Substitute \( u = 2 + t \), \( g(t) \mapsto g'(t) \, dt = 1 \)

\[
\int (g(t))^5 g'(t) \, dt = \int u^5 \, du = \frac{u^6}{6} + C
\]

\[
\int (2 + t)^5 \, dt = \frac{(2 + t)^6}{6} + C
\]
Alternate point of view:

\[ \int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du \]

(substitution in function)

Recall: differentials. If \( u = g(x) \) then

\[ du = g'(x) \, dx \]

This is perhaps the easiest way to remember the substitution rule.

- If we replace \( g(x) \) by \( u \) then \( g'(x) \, dx \) gets replaced by \( du \)
  
  (NOT \( dx \) getting replaced by \( du \))

Ex.

\[ \int \sin^2 x \cos x \, dx \]

Set \( u = \sin(x) \) so that \( du = \cos(x) \, dx \)

\[ \int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C \]

\[ = \frac{\sin^3(x)}{3} + C \]
\[ \int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx \]

Set \( u = \cos(x) \) so that \( du = -\sin(x) \, dx \)

\[ \int \frac{\sin(x)}{\cos(x)} \, dx = \int \frac{-1}{u} \, du = -\ln |u| + C \]

\[ = -\ln |\cos(x)| + C \]

\[ = \ln |\sec(x)| + C \]

\[ \int \tan(x) \, dx = \ln |\sec(x)| + C \]

Ex.

\[ \int \frac{\ln(x)}{x} \, dx \quad u = \ln(x) \text{ implies } du = \frac{1}{x} \, dx \]

\[ \int \frac{\ln(x)}{x} \, dx = \int u \, du = \frac{u^2}{2} + C \]

\[ = \frac{(\ln(x))^2}{2} + C \]

Some more examples:

\[ \int \sin \left(1 + 2x^2\right) x \, dx \quad \int \frac{2x \, e^{x^2}}{1 + e^{2x^2}} \, dx \quad \int \frac{1 + x}{1 + x^2} \, dx \]
(20.2) For definite integrals:

- making substitution \( u = g(x) \) changes limits of integration
  \( x \in [a, b] \) and \( u \in [g(a), g(b)] \)

Thus substitution rule takes the following form:

\[
\int_{a}^{b} f(g(x)) \, g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

Examples:

(i) \[ \int_{e}^{e} \frac{dx}{x \sqrt{\ln(x)}} \]

(ii) \[ \int_{0}^{\pi/2} \cos(x) \sin(\sin(x)) \, dx \]

(20.3) An application:

If \( f(x) \) is even function then \( \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \)

If \( f(x) \) is odd function then \( \int_{-a}^{a} f(x) \, dx = 0 \)