(19.0) Recall: for a function $f(x)$ we have defined

- \[ \int f(x) \, dx = \text{antiderivative of } f(x) = F(x) + C \]
  where \[ \frac{d}{dx} F(x) = f(x) \quad [\text{Indefinite integral}] \]

- \[ \int_{a}^{b} f(x) \, dx = \text{area under } y = f(x), \quad x \in [a, b] \]
  \[ = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} \right] \quad [\text{Definite integral}] \]

Fundamental Theorem of Calculus relates the three operations on functions:

- Derivatives
- Indefinite Integrals
- Definite Integrals

* Important * \[ \int f(x) \, dx \] is a function (+ constant unspecified)

\[ \int_{a}^{b} f(x) \, dx \] is a real number.
(9.1) We can use definite integrals to construct a function.

Let \( f(t) \) be a continuous function on \([a, b]\). Define a new function \( g(x) \) \((x \in [a, b])\) as:

\[
g(x) = \int_{a}^{x} f(t) \, dt \quad \text{[area under } y = f(t) \text{ from } t = a \text{ to } t = x]\n\]

Example. \( f(t) = |t| \)

Define \( g(x) = \int_{-1}^{x} f(t) \, dt \)

For \(-1 \leq x \leq 0\) we have

\[
g(x) = \frac{1}{2} - \frac{x^2}{2}
\]

For \(x > 0\) we have

\[
g(x) = \frac{1}{2} + \frac{x^2}{2}
\]
Fundamental Theorem of Calculus 1.

\[ \frac{d}{dx} g(x) = f(x) \]

equivalently \[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \]

In words, \( \int_{a}^{x} f(t) \, dt \) is an antiderivative of \( f \).

This statement makes connection between definite integral and antiderivatives.

Given a function \( f(t) \), an antiderivative of \( f \) can be found as

\[ g(x) = \text{area under } y = f(t) \text{ from } t = a \text{ to } t = x \]

and hence a general antiderivative is \( g(x) + C \).
Proof.

\[ g(x+h) - g(x) = \int_{x}^{x+h} f(t) \, dt \]

Let \( f(c_1) \) be absolute min. value of \( f \) on \([x, x+h] \)

Let \( f(c_2) \) be absolute max. value of \( f \) on \([x, x+h] \)

Then \( h \cdot f(c_1) \leq g(x+h) - g(x) \leq h \cdot f(c_2) \) \( (h > 0) \)

\[ f(c_1) \leq \frac{g(x+h) - g(x)}{h} \leq f(c_2) \]

as \( h \to 0 \), \( c_1 \) and \( c_2 \) coincide with \( x \).

\[ \Rightarrow \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x) \]

Example 1.

\[ g(x) = \int_{0}^{x} \left(1 + t^4\right)^{\frac{1}{3}} \, dt \]

\[ \Rightarrow \frac{d}{dx} g(x) = \left(1 + x^4\right)^{\frac{2}{3}} \]
2. \( g(x) = \int_{0}^{\frac{e^x}{1+x}} \frac{1}{1+t} \, dt \)

composition of \( \int_{0}^{u} \frac{1}{1+t} \, dt \) and \( e^x \)

By Chain rule

\[
\frac{d}{du} \left( \int_{0}^{u} \frac{1}{1+t} \, dt \right) \cdot \frac{du}{dx} \quad (u = e^x)
\]

\[
= \frac{1}{1+u} \cdot \frac{du}{dx}
\]

\[
= \frac{1}{1+e^x} \cdot e^x
\]

(19.2) Now a general antiderivative \( F(x) \) of \( f \) must be of the form

\[
F(x) = \int_{a}^{x} f(t) \, dt + C
\]

\[
\Rightarrow \quad F(a) = \int_{a}^{a} f(t) \, dt + C = C
\]

and

\[
F(b) = \int_{a}^{b} f(t) \, dt + C = \int_{a}^{b} f(t) \, dt + F(a)
\]

\[
\Rightarrow \quad \int_{a}^{b} f(t) \, dt = F(b) - F(a)
\]
Fundamental Theorem of Calculus 2.

Let \( F(x) \) be any antiderivative of \( f(x) \). Then

\[
\int_{a}^{b} f(t) \, dt = F(b) - F(a) \quad \text{(usually written as} \quad \left[ F(x) \right]_{a}^{b} \text{)}
\]

Thus, problem of finding area under a curve and that of finding antiderivatives are equivalent.

(19.3) Table of known antiderivatives

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)
\]

\[
\int \frac{1}{x} \, dx = \ln |x| + C
\]

\[
\int a^x \, dx = \frac{a^x}{\ln(a)} + C
\]

\[
\int \sin(x) \, dx = -\cos(x) + C
\]

\[
\int \cos(x) \, dx = \sin(x) + C
\]

\[
\int \sec^2(x) \, dx = \tan(x) + C
\]
\[ \int \sec(x) \tan(x) \, dx = \sec(x) + C \]

\[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C \]

\[ \int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + C \]

Remark. (i) One needs to be careful writing antiderivatives of a function that is not continuous everywhere. The constant \( C \) can change over different intervals of continuity.

E.g.: \( f(x) = \frac{1}{x^2} \) is continuous on \((-\infty, 0)\) and \((0, \infty)\).

Implying:

\[ \int x^{-2} \, dx = -x^{-1} + C_1 \quad x \in (-\infty, 0) \]

\[ = -x^{-1} + C_2 \quad x \in (0, \infty) \]

\( C_1 \) and \( C_2 \) can very well be different.

(ii) Fundamental Theorem of Calculus is only valid for continuous functions.

(19.4) Physical meaning of definite integral.

If \( f(t) \) represents instantaneous rate of change of \( F(t) \)

then

\[ \int_{a}^{b} f(t) \, dt = \text{net change of quantity over} \]

\[ t = a \text{ to } t = b \]
Example: If instantaneous velocity of a particle is given by \( v(t) = 1 - t^2 \), find its displacement over \( t = 0 \) to \( t = 4 \).

\[
\int_0^4 (1-t^2) \, dt = \left[ t - \frac{t^3}{3} \right]_0^4 = (4 - \frac{64}{3}) - (0 - \frac{0}{3}) = -\frac{52}{3}
\]

What is the total distance travelled?

\( 1-t^2 \) is positive on \((0, 1)\) and negative on \((1, 4)\).

\[\Rightarrow \text{Total distance travelled} = \int_0^1 (1-t^2) \, dt + \int_1^4 (t^2-1) \, dt
\]

\[= \left[ t - \frac{t^3}{3} \right]_0^1 + \left[ \frac{t^3}{3} - t \right]_1^4
\]

\[= (1 - \frac{1}{3}) - 0 + \left( \frac{64}{3} - 4 \right) - 1 = \frac{2}{3} + \frac{52}{3} = \frac{54}{3} = 18
\]

Example: If density of a metal rod at distance \( x \) from its one end point is \( p(x) = 1 + \sqrt{x} \) kg/m and rod is 5 m long, find its total mass.

\[
\text{Total mass} = \int_0^5 (1+\sqrt{x}) \, dx = \left[ x + \frac{x^{3/2}}{3/2} \right]_0^5
\]

\[= 5 + \frac{5^{3/2}}{3/2} \text{ kg}.
\]