(17.1) Anti-derivatives. $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

Remain. (i) Anti-derivative of a function is not unique. For example both $x$ and $x+1$ are anti-derivatives of $f(x) = 1$.

(ii) If $F(x)$ and $G(x)$ are both anti-derivatives of $f(x)$ then

$$\frac{d}{dx} (F(x) - G(x)) = F'(x) - G'(x) = f(x) - f(x) = 0$$

implying that $F(x) - G(x) = C$ (some constant).

Thus, if $F(x)$ is an anti-derivative of $f(x)$, then any anti-derivative of $f(x)$ is of the form $F(x) + C$.

Examples. (i) $f(x) = x$

Since $\frac{d}{dx} (x^2) = 2x$ we get $\frac{d}{dx} \left( \frac{x^2}{2} \right) = x$

Hence anti-derivative of $f(x) = x$ is $\frac{1}{2} x^2 + C$. 

\(\square\)
Table of antiderivatives.

(i) \[ \frac{d}{dx} (x^n) = n x^{n-1}. \quad \text{Or} \quad \frac{d}{dx} (x^{n+1}) = (n+1) x^n \]

implies \[ \frac{d}{dx} \left[ \frac{x^{n+1}}{n+1} \right] = x^n \quad (\text{except when } n+1 = 0) \]

Antiderivative of \( x^n = \frac{x^{n+1}}{n+1} \).

(ii) \[ \frac{d}{dx} (a^x) = a^x \cdot \ln(a) \]

implies

Antiderivative of \( a^x = \frac{a^x}{\ln(a)} \).

(iii) \[ \frac{d}{dx} (\ln(x)) = \frac{1}{x} \quad \text{for } x > 0 \]

\[ \frac{d}{dx} (\ln(-x)) = \frac{1}{x} \quad \text{for } x < 0 \]

implies

Antiderivative of \( \frac{1}{x} = \ln(|x|) \).

(iv) \[ \frac{d}{dx} (\sin(x)) = \cos(x) \]

\[ \frac{d}{dx} (-\cos(x)) = \sin(x) \]

Antiderivative of \( \cos(x) = \sin(x) \)

Antiderivative of \( \sin(x) = -\cos(x) \)
(v) \[ \frac{d}{dx} \left( \tan(x) \right) = \sec^2(x) \]

Antiderivative of \( \sec^2(x) = \tan(x) \)

(vi) \[ \frac{d}{dx} \left( \sin^{-1}(x) \right) = \frac{1}{\sqrt{1-x^2}} \]

Antiderivative of \( \frac{1}{\sqrt{1-x^2}} = \sin^{-1}(x) \)

(vii) \[ \frac{d}{dx} \left( \tan^{-1}(x) \right) = \frac{1}{1+x^2} \]

Antiderivative of \( \frac{1}{1+x^2} = \tan^{-1}(x) \)

Example. Find the function \( f(x) \) which satisfies

\[ f'(x) = 5^x - \frac{2}{\sqrt{1-x^2}} \quad \text{and} \quad f(0) = 2 \]

\[ f(x) = \text{Antiderivative of} \ 5^x - \frac{2}{\sqrt{1-x^2}} \]

\[ = \frac{5^x}{\ln(5)} - 2 \sin^{-1}(x) + C \quad (\text{for some constant } C) \]

\[ f(0) = 2 \quad \Rightarrow \quad \frac{5^0}{\ln(5)} - 2 \sin^{-1}(0) + C = 2 \]

\[ \Rightarrow \quad \frac{1}{\ln(5)} + C = 2 \quad \Rightarrow \quad C = 2 - \frac{1}{\ln(5)} \]
Thus \( f(x) = \frac{5^x}{\ln(5)} - 2 \sin^{-1}(x) + 2 - \frac{1}{\ln(5)} \).

Example. If the velocity of a particle at time \( t \) is given by \( v(t) = 2 - t^2 \), then find its position as a function of time if at \( t = 0 \) \( s(0) = 0 \).

Since \( v(t) = s'(t) = 2 - t^2 \), \( s(t) \) is an antiderivative of \( 2 - t^2 \).

\[ \Rightarrow s(t) = 2t - \frac{t^3}{3} + C \quad \text{(for some constant } C) \]

\( s(0) = 0 \quad \Rightarrow \quad C = 0 \). Therefore \( s(t) = 2t - \frac{t^3}{3} \).

Example. If an object is thrown upwards with initial velocity \( u \) m/s, find its height as a function of \( t \).
When does the object hit the ground?

Acceleration \( a(t) = -g \quad (\approx -9.8 \text{ m/s or } 32 \text{ ft/s}) \)

\[ \Rightarrow v(t) = \text{antiderivative of } -g \]

\[ = -gt + C \quad \text{(some constant } C) \]
\[ v(0) = u \quad \text{gives} \quad v(t) = u - gt \]

\[ S(t) = \text{antiderivative of } u - gt \]
\[ = ut - \frac{1}{2} gt^2 + C' \quad (\text{constant } C') \]

\[ S(0) = 0 \quad \Rightarrow \quad C' = 0 \quad \text{and hence} \]

\[ \boxed{S(t) = ut - \frac{1}{2} gt^2} \]

Height is zero (i.e., object is at ground) if \( S(t) = 0 \)
\[ t(u - \frac{1}{2} gt) = 0 \quad \Rightarrow \quad t = 0 \quad \text{or} \]
\[ t = \frac{2u}{g} \]

(17.3) Areas.
- Rectangle
  \[ \text{Area} = l \cdot w \]
- Triangle
  \[ \text{Area} = \frac{1}{2} bh \]

Idea: area of a general region can be approximated by small squares (or rectangles)
Let \( f(x) \) be a continuous function on \([a, b]\).

- Divide \([a, b]\) into \(n\) intervals of equal length:
  \[
  \frac{b-a}{n}
  \]
- \([x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\)
- Choose "sample points"
  \[
  x_1^* \in [x_0, x_1], \quad x_2^* \in [x_1, x_2], \ldots, \quad x_n^* \in [x_{n-1}, x_n]
  \]
- \[
  \left( \frac{b-a}{n} \right) \left[ f(x_1^*) + f(x_2^*) + \ldots + f(x_n^*) \right]
  \]
  \[
  = \text{sum of rectangles of width } \frac{b-a}{n} \text{ and height given by } f(x_1^*), f(x_2^*), \ldots
  \]

Area under the curve \( y = f(x) \), over \([a, b]\), is

\[
\text{Area} = \lim_{n \to \infty} \left[ \left( \frac{b-a}{n} \right) \left( f(x_1^*) + f(x_2^*) + \ldots + f(x_n^*) \right) \right]
\]

- Sample points are chosen completely arbitrarily. Some of the most natural choices are: left endpoints, right endpoints, mid point.
Example.

\[ f(x) = x \quad [0, 1] \]

→ Choose left end point as sample points.

\( n = 2 \): \[ [0, \frac{1}{2}] \quad [\frac{1}{2}, 1] \]

\[ x_0 = 0 \quad x_1 = \frac{1}{2} \quad x_2 = 1 \]
\[ x_1^* = 0 \quad x_2^* = \frac{1}{2} \]
\[ f(x^*_1) = 0 \quad f(x^*_2) = \frac{1}{2} \]

\[ A_2 = \frac{1}{2} \left( 0 + \frac{1}{2} \right) = \frac{1}{4} \]

\( n = 3 \): \[ [0, \frac{1}{3}] \quad [\frac{1}{3}, \frac{2}{3}] \quad [\frac{2}{3}, 1] \]

\[ x_0 \quad x_1 \quad x_2 \quad x_3 \]
\[ x_1^* \quad x_2^* \quad x_3^* \]

\[ A_3 = \frac{1}{3} \left[ 0 + \frac{1}{3} + \frac{2}{3} \right] = \frac{1}{3} \]

In general \[ [0, \frac{1}{n}] \quad [\frac{1}{n}, \frac{2}{n}] \quad \ldots \quad [\frac{n-1}{n}, \frac{n}{n}] \]

\[ x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_{n-1} \quad x_n \]
\[ x_1^* \quad x_2^* \quad \ldots \quad x_{n-1}^* \quad x_n^* \]

\[ A_n = \frac{1}{n} \left( 0 + \frac{1}{n} + \frac{2}{n} + \ldots + \frac{n-1}{n} \right) \]

\[ = \frac{1}{n} \sum_{i=0}^{n-1} \frac{i}{n} \quad \text{\leftarrow sigma notation.} \]
\[
\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}
\]

\[
A_n = \frac{1}{n} \frac{n(n-1)}{2n} = \frac{n-1}{2n}
\]

\[
\text{Area} = \lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{n-1}{2n} = \frac{1}{2}
\]