(13.0) Recall: we studied how to compute derivatives of a function and the following applications:

1) Related rates of change: an application of the chain rule.

2) Linear approximation of a function: an application of the equation of the tangent line.

3) Differentials: \( y = f(x) \) implies \( dy = f'(x) \, dx \).
   (to estimate errors)

(13.1) Maximum and minimum values

Let \( f(x) \) be a function and \( c \) be a real number in the domain of \( f \).

Definition. We say \( f(c) \) is the absolute maximum value of \( f(x) \) (on its domain) if \( f(c) \geq f(x) \) for every \( x \) in the domain of \( f \).

Similarly, \( f(c) \) is the absolute minimum value of \( f \) if \( f(c) \leq f(x) \) for every \( x \) in the domain of \( f \).
\[ f(c) \text{ is a local maximum value of } f \text{ if } f(c) \geq f(x) \text{ for every } x \text{ near } c. \]

\[ f(c) \text{ is a local minimum value of } f \text{ if } f(c) \leq f(x) \text{ for every } x \text{ near } c. \]

Examples:

- Absolute minimum value

If \( f(x) \) is continuous on \([a,b]\) then \( f \) has an absolute maximum and an absolute minimum value on \([a,b]\).

(3.2) Critical points.

Assume \( f(x) \) is continuous on \([a,b] \), \( c \in (a,b) \) is a local max. or min. of \( f(x) \) and \( f'(c) \) exists. Then \( f'(c) = 0 \).
Reason. \[ f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \]

If \( f(c) \) is local max., then for \( h \) close to 0, we have
\[ f(c+h) \leq f(c) \implies f(c+h) - f(c) \leq 0. \]

Therefore for \( h > 0 \)
\[ \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0 \]

for \( h < 0 \)
\[ \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0 \]

Since the two quantities are equal to \( f'(c) \), we get
\[ f'(c) = 0. \]

Therefore at local extreme (max or min) point \( c \), we have the following possibilities
- \( f'(c) \) does not exist
- \( f'(c) \) exist and equals zero.

Definition. \( c \) is a real number in the domain of \( f \) is said to be critical point if either \( f \) is not differentiable at \( c \), or \( f'(c) = 0 \).
Examples. (1) $f(x) = |x|$. Recall $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

The only critical point of $f(x)$ is $0$.

(2) $f(x) = x^2 - 3x + 1$

$f'(x) = 2x - 3$. is defined for any $x$.

$f'(x) = 0 \equiv 2x - 3 = 0 \equiv x = \frac{3}{2}$ critical point.

(3) $f(x) = \sqrt{x} (x^2 - 1)$

$f'(x) = \frac{1}{2\sqrt{x}} (x^2 - 1) + \sqrt{x} \cdot 2x$

$= \frac{x^2 - 1 + 2x^2}{2\sqrt{x}} = \frac{3x^2 - 1}{2\sqrt{x}}$

$f'(x)$ not defined at $x=0$

$f'(x) = 0 \equiv 3x^2 - 1 = 0 \equiv x^2 = \frac{1}{3} \equiv x = \pm \sqrt{\frac{1}{3}}$

Critical points $0, \pm \sqrt{\frac{1}{3}}$. 
(13.3) Absolute max./min. values on a closed interval.

Let \( f(x) \) be a continuous function on \([a, b]\).

→ Absolute max./min. values of \( f \) occur at the critical points or the end points \( a, b \).

Example. Find absolute max. and min. values of

\[
f(x) = x^3 - 6x^2 + 5 \quad \text{on } [-3, 5]
\]

(i) Find the critical points of \( f \):

\[
f'(x) = 3x^2 - 12x \quad \text{defined for every } x
\]

\[
f'(x) = 0 \quad \Leftrightarrow \quad 3x^2 - 12x = 0 \quad \Leftrightarrow \quad 3x^2 = 12x
\]

\[x = 0 \quad \text{or} \quad x = \frac{12}{3} = 4\]

\[
\begin{align*}
\left[f(-3) = -27 - 54 + 5 = -76 & \quad \text{absolute min.} \\
\left[f(5) = 125 - 150 + 5 = -20 & \quad \text{absolute max.} \\
\left[f(0) = 0 - 0 + 5 = 5 & \\
\left[f(4) = 64 - 96 + 5 = -27
\end{align*}
\]
Example. \( f(x) = x - 2 \tan^{-1}(x) \) on \([0, 4]\)

\[
f'(x) = 1 - \frac{2}{1 + x^2} = \frac{x^2 + 1 - 2}{x^2 + 1} = \frac{x^2 - 1}{x^2 + 1}
\]

defined everywhere.

\[
f'(x) = 0 \equiv x^2 - 1 = 0 \equiv x = \pm 1
\]

Only critical point in \([0, 4]\) is \(x = 1\).

\[
f(0) = 0 - 2(0) = 0
\]

Absolute min. \(f(1) = 1 - 2 \tan^{-1}(1) = 1 - 2 \frac{\pi}{4} = 1 - \frac{\pi}{2} \approx -0.57\)

Absolute max. \(f(4) = 4 - 2 \tan^{-1}(4) = 1.35\)

Example. \( f(x) = xe^{-x^2/8} \) on \([-1, 4]\)

\[
f'(x) = (x)'e^{-x^2/8} + x(e^{-x^2/8})'
\]

\[
= e^{-x^2/8} + x\left(\frac{d}{dx}(\frac{-x^2}{8})\right)
\]

\[
= e^{-x^2/8} + xe^{-x^2/8} \left(-\frac{2x}{8}\right) \text{ defined everywhere}
\]

\[
f'(x) = 0 \equiv e^{-x^2/8} \left[1 - \frac{x^2}{4}\right] = 0 \equiv 1 - \frac{x^2}{4} = 0
\]

\[
\equiv x = \pm 2
\]
Only critical point in \([-1, 4]\) is \(x = 2\).

\[
f(-1) = (-1)^{\frac{1}{8}} e^{\frac{1}{2}} = -e^{\frac{1}{8}} \quad \text{← Absolute min.}
\]

\[
f(4) = 4 \cdot e^{2}
\]

\[
f(2) = 2 \cdot e^{\frac{1}{2}} \quad \text{← Absolute max.}
\]

(13.4) Mean Value Theorem.

Assume \(f(x)\) is continuous on \([a, b]\) and \(f'(x)\) exists for every \(x \in (a, b)\). Then there exists \(c \in (a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Special case. when \(f(a) = f(b)\) the theorem states existence of some point \(c\) in the interval \((a, b)\) such that

\[
f'(c) = 0
\]

(called Rolle's Theorem).

Examples: \((1)\) \(f(x) = 2x^2 - 3x + 1\) on \([0, 2]\)

\[
f(0) = 1 \quad f(2) = 8 - 6 + 1 = 3
\]

By MVT, there must be \(c \in (0, 2)\) such that
\[ f'(c) = \frac{f(2) - f(0)}{2-0} = \frac{3-1}{2} = 1 \]

Now \[ f'(x) = 4x - 3 \implies 4x - 3 = 1 \implies x = 1 \]

Some interesting examples.

(2) Assume \( f'(x) = 0 \) for every \( x \in (a,b) \). Then \( f(x) \) is a constant function.

(3) If \( f'(x) \geq 0 \) for every \( x \in (a,b) \), then \( f(x) \geq f(a) \) for every \( x \).

Examples. (1) Prove that \( x^3 - 15x + 1 = 0 \) has exactly one root in \([-2, 2]\).

\[ f(x) = x^3 - 15x + 1 \]
\[ f(-2) = -8 + 30 + 1 = 23 \]
\[ f(2) = 8 - 30 + 1 = -21 \]

By I.V.T. there is some \( c \) such that \( f(c) = 0 \).

\(-2 < c < 2\)

\[ f'(x) = 3x^2 - 15 = 3(x^2 - 5) \]. For \(-2 \leq x \leq 2\), \( x^2 - 5 < 0 \). If \( c \neq d \) are two roots of \( f(x) = 0 \) then there must be some number \( e \in (c,d) \) such that \( f'(e) = 0 \).

But there is no such number in \((-2,2)\). \( \blacksquare \)
(ii) Prove that 
\[
\sin^{-1}\left(\frac{x-1}{x+1}\right) = 2\tan^{-1}(\sqrt{x}) - \frac{\pi}{2} ; \quad x \geq 0
\]

Let 
\[
f(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) - 2\tan^{-1}(\sqrt{x})
\]

\[
f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right) - 2 \cdot \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})
\]

\[
= \frac{x+1}{2\sqrt{(x+1)^2 - (x-1)^2}} \cdot \frac{(x+1)-(x-1)}{(x+1)^2} - \frac{2}{1 + x} \cdot \frac{1}{2\sqrt{x}}
\]

\[
= \frac{2}{\sqrt{4x}(x+1)} - \frac{1}{\sqrt{x}(x+1)} = 0
\]

Therefore, \( f(x) \) is constant function.

\[
f'(0) = \sin^{-1}(-1) - 2\tan^{-1}(0) = -\frac{\pi}{2} - 0 = -\frac{\pi}{2}
\]

(iii) Assume \( f'(x) \geq 3 \) for every \( x \). Prove that:

\[
f(8) - f(2) \geq 18
\]

By MVT, there exists \( c \in (2,8) \) such that:

\[
f'(c) = \frac{f(8) - f(2)}{8-2}
\]

\[
3 \leq f'(c) \implies 3 \leq \frac{f(8) - f(2)}{6} \implies 18 \leq f(8) - f(2)
\]