PRACTICE FINAL

(1) There is no partial credit to the following problems. Check your answer carefully.

(a) Find the volume of the parallelopiped from by:

\[ \vec{u} = \langle 1, 0, 2 \rangle \quad \vec{v} = \langle 2, -1, 0 \rangle \quad \vec{w} = \langle 4, 1, 1 \rangle \]

The volume is given by:

\[
\begin{vmatrix}
1 & 0 & 2 \\
2 & -1 & 0 \\
4 & 1 & 1
\end{vmatrix}
\]

\[ = |1(-1) - 0(2) + 2(2 + 4)| \]

\[ = 11 \]

(b) Find the parametric equations describing the tangent line to \( \vec{r}(t) = \langle \sqrt{2}t, 4 - t^3, t^2 - 1 \rangle \) at \((2, -4, 3)\).

The parametric curve passes through \((2, -4, 3)\) at \(t = 2\).

\[ \vec{r}'(t) = \langle \frac{1}{\sqrt{2t}}, -3t^2, 2t \rangle \]

\[ \vec{r}'(2) = \langle \frac{1}{2}, -12, 4 \rangle \]

Parametric equations of the tangent line are:

\[ x = 2 + \frac{t}{2} \quad y = -4 - 12t \quad z = 3 + 4t \]

(c) Find the equation of the tangent plane to \( z = x^3 - 2 \cos(y) + x^2y \) at \( x = 2, y = \pi \).

At \( x = 2, y = \pi \) we have \( z = 2^3 - 2 \cos(\pi) + 2^2\pi = 10 + 4\pi \).

\[ \frac{\partial z}{\partial x} = 3x^2 + 2xy \quad \frac{\partial z}{\partial y} = 2\sin(y) + x^2 \]

\[ \frac{\partial z}{\partial x}(2, \pi) = 12 + 4\pi \quad \frac{\partial z}{\partial y}(2, \pi) = 4 \]

The equation of the tangent plane is:

\[ z - (10 + 4\pi) = (12 + 4\pi)(x - 2) + 4(y - \pi) \]

(d) Write \( w = \left( \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} \right)^{10} \) in the \( a + bi \) form.

We begin by writing \( \left( \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} \right) \) in polar form:
\[
\frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} \cdot \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} = \frac{-2 + 2\sqrt{3}i}{4} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i
\]

This \( \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} = \cos \left( \frac{2\pi}{3} \right) + \sin \left( \frac{2\pi}{3} \right) i. \) Hence we get

\[
\left( \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} \right)^{10} = \cos \left( \frac{20\pi}{3} \right) + \sin \left( \frac{20\pi}{3} \right) i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i
\]

(e) Find the length of the following curve

\[ \vec{r}(t) = \langle 12t, 8t^{3/2}, 3t^2 \rangle; \quad 0 \leq t \leq 1 \]

\[ \vec{r}'(t) = \langle 12, 12\sqrt{t}, 6t \rangle \]

\[ |\vec{r}'(t)| = \sqrt{144 + 144t + 36t^2} = \sqrt{(6t + 12)^2} = 6t + 12 \]

Thus the length of the parametric curve is given by

\[
\int_0^1 (6t + 12) dt = [3t^2 + 12t]_{t=0}^{t=1} = 15
\]
(2) Let \( f(x, y) = \ln(1 + x^2 + y^2) \).

(a) Compute \( f_{xx} \) and \( f_{xy} \).

\[
f_x = \frac{1}{1 + x^2 + y^2} 2x = \frac{2x}{1 + x^2 + y^2}
\]
\[
f_y = \frac{2y}{1 + x^2 + y^2}
\]
\[
f_{xx} = \frac{(1 + x^2 + y^2)2 - 2x(2x)}{(1 + x^2 + y^2)^2} = \frac{2 - 2x^2 + 2y^2}{(1 + x^2 + y^2)^2}
\]
\[
f_{xy} = \frac{-2x(2y)}{(1 + x^2 + y^2)^2} = \frac{-4xy}{(1 + x^2 + y^2)^2}
\]

(b) Write the unit vector along which \( f \) is increasing fastest at \( x = y = 1 \).

\[
f_x(1, 1) = \frac{2}{3} \quad f_y(1, 1) = \frac{2}{3}
\]

Therefore \( \vec{\nabla}(f)(1, 1) = \frac{2}{3} \langle 1, 1 \rangle \). The function \( f \) increases fastest in the direction of \( \vec{\nabla}(f) \), which is the unit vector along \( \langle 1, 1 \rangle \):

\[
\vec{u} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle
\]

(c) What is the rate of change of \( f \) at \( (1, 1) \) in the direction of \( \langle 1, -2 \rangle \)?

The rate of change of \( f \) at \( (1, 1) \) in the direction of \( \langle 1, -2 \rangle \) is given by the directional derivative \( D_{\vec{v}}(f)(1, 1) \). Here \( \vec{v} \) is the unit vector in the direction of \( \langle 1, -2 \rangle \):

\[
\vec{v} = \frac{1}{\sqrt{5}} \langle 1, -2 \rangle
\]
\[
D_{\vec{v}}(f)(1, 1) = \vec{\nabla}(f)(1, 1) \cdot \vec{v}
\]
\[
= \frac{2}{3} \cdot \frac{1}{\sqrt{5}} \langle 1, 1 \rangle \cdot \langle 1, -2 \rangle
\]
\[
= \frac{2}{3\sqrt{5}} \cdot (-1) = \frac{-2}{3\sqrt{5}}
\]
(3) Find all critical points of \( f(x, y) = 2x^3 + y^3 - 5xy \) and classify them as local minimum, local maximum, saddle points.

\[
\begin{align*}
f_x &= 6x^2 - 5y & f_y &= 3y^2 - 5x \\
f_{xx} &= 12x & f_{yy} &= 6y & f_{xy} &= -5 \\
D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 = 72xy - 25
\end{align*}
\]

Critical point: \( f_x = 0 \) and \( f_y = 0 \).

\[
\begin{align*}
6x^2 &= 5y \\
3y^2 &= 5x \\
\Rightarrow 9y^4 &= 25x^2 = 25\left(\frac{5y}{6}\right) = \frac{125}{6}y \\
\Rightarrow y \left(y^3 - \frac{125}{54}\right) &= 0
\end{align*}
\]

Therefore, \( y = 0 \) or \( y = \frac{5}{3(2)^{1/3}} \). Using \( x = \frac{3}{2}y^2 \) we get:

\[
\begin{align*}
y &= 0 & \Rightarrow & x = 0 \\
y &= \frac{5}{3(2)^{1/3}} & \Rightarrow & x = \frac{5}{3(2)^{2/3}}
\end{align*}
\]

- \((x, y) = (0, 0)\): \( D = -25 < 0 \). Therefore \((0, 0)\) is a saddle point.
- \((x, y) = \left(\frac{5}{3(2)^{2/3}}, \frac{5}{3(2)^{1/3}}\right)\):

\[
D = 72\frac{25}{9(2)} - 25 = 75 > 0 \text{ and } f_{xx} = 12 \left(\frac{5}{3(2)^{2/3}}\right) > 0
\]

Therefore, this is a local min.
(4) Find the absolute maximum and minimum values of \( f(x, y) = xy - 5x^2 + 3 \) on the finite domain \( D \) bounded by \( x \)-axis, \( x = 2 \) and \( y = x^3 \).

(Step 1) Critical points of \( f \).
\[
\begin{align*}
  f_x &= y - 10x \\
  f_y &= x
\end{align*}
\]

\( f_y = 0 \) imples that \( x = 0 \). \( f_x = 0 \) implies that \( y = 10x = 0 \). Therefore \((0, 0)\) is the only critical point and \( f(0, 0) = 3 \).

(Step 2) Extreme values of \( f \) on the boundry. The boundry of \( D \) consists of three segments: (I) \( y = 0 \) and \( 0 \leq x \leq 2 \). (II) \( x = 2 \) and \( 0 \leq y \leq 8 \). (III) \( y = x^3 \) and \( 0 \leq x \leq 2 \).

(I) \( y = 0 \) gives the function \( f(x, 0) = -5x^2 + 3 \). This function is decreasing on \( 0 \leq x \leq 2 \), and has extreme values at \( x = 0 \) and \( x = 2 \):
\[
\begin{align*}
  f(0, 0) &= 3 \\
  f(2, 0) &= -17
\end{align*}
\]

(II) \( x = 2 \) gives the function \( f(2, y) = 2y - 17 \). This function is increasing on \( 0 \leq y \leq 8 \) with extreme values at \( y = 0 \) and \( y = 8 \):
\[
\begin{align*}
  f(2, 0) &= -17 \\
  f(2, 8) &= -1
\end{align*}
\]

(III) \( y = x^3 \) gives the function \( g(x) = f(x, x^3) = x^4 - 5x^2 + 3 \).
\[
\begin{align*}
  g'(x) = 0 &\iff 4x^3 - 10x = 0 \iff x = 0 \text{ or } x = \sqrt{\frac{5}{2}}
\end{align*}
\]

At the additional critical point, we get:
\[
\begin{align*}
  g \left( \sqrt{\frac{5}{2}} \right) &= \frac{25}{4} - \frac{25}{2} + 3 = -\frac{13}{4}
\end{align*}
\]

Among the values computed above, we have the absolute maximum of 3 at the point \((0, 0)\) and an absolute minimum of \(-17\) at \((2, 0)\).
(5) Let $C$ be the curve of intersection of the following two surfaces

\[ x^2 + y^2 = 1 \]

\[ z = 3 - 2x^2 - 4y^2 \]

Find points on $C$ which are closest to and farthest from the origin.

We have to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to two constraints: $g_1(x, y, z) = x^2 + y^2 = 1$ and $g_2(x, y, z) = 2x^2 + 4y^2 + z = 3$.

\[ \vec{\nabla} f = \langle 2x, 2y, 2z \rangle \]
\[ \vec{\nabla} g_1 = \langle 2x, 2y, 0 \rangle \]
\[ \vec{\nabla} g_2 = \langle 4x, 8y, 1 \rangle \]

Thus the Lagrange equations are:

\[ 2x = 2\lambda_1 x + 4\lambda_2 x \]
\[ 2y = 2\lambda_1 y + 8\lambda_2 y \]
\[ 2z = \lambda_2 \]
\[ x^2 + y^2 = 1 \]
\[ 2x^2 + 4y^2 + z = 3 \]

From the first two equations we get

\[ 2x = 2\lambda_1 x + 4\lambda_2 x \Rightarrow x = 0 \text{ or } \lambda_1 + 2\lambda_2 = 1 \]
\[ 2y = 2\lambda_1 y + 8\lambda_2 y \Rightarrow y = 0 \text{ or } \lambda_1 + 4\lambda_2 = 1 \]

If $x = 0$ we get $y = \pm 1$ and hence $z = -1$. Similarly, if $y = 0$ we get $x = \pm 1$ and hence $z = 1$. For both these cases $f(x, y, z) = 2$.

If both $x$ and $y$ are non-zero, we get $\lambda_1 + 2\lambda_2 = 1$ and $\lambda_1 + 4\lambda_2 = 1$ which yield $\lambda_1 = 1$ and $\lambda_2 = 0$. Thus $z = \lambda_2/2 = 0$ and the two constraints give us $x^2 = y^2 = 1/2$. Hence $x = \pm 1/\sqrt{2}$ and $y = \pm 1/\sqrt{2}$ and we get $f(x, y, z) = 1$.

Thus the points on $C$ closest to the origin are $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0)$ and the points farthest from the origin are $(0, \pm 1, -1)$ and $(\pm 1, 0, 1)$. 
(6) A projectile is fired with an initial speed of 100 m/s at an angle of 60°.

(a) Write the position \( \vec{r}(t) \) and velocity \( \vec{v}(t) \) of the particle at time \( t \).

The acceleration \( \vec{a}(t) = <0, -g> \) and the initial data is: \( \vec{r}(0) = \vec{0} \), \( \vec{v}(t) = <100 \cos(60°), 100 \sin(60°)> = <50, 50\sqrt{3}> \).

Solving for \( \vec{v}(t) \) gives:

\[ \vec{v}(t) = <50, 50\sqrt{3} - gt> \]

Solving for \( \vec{r}(t) \) gives:

\[ \vec{r}(t) = <50t, 50\sqrt{3}t - \frac{1}{2}gt^2> \]

(b) At what times is the projectile at the height three quarters of its maximum height?

The maximum height is reached by the projectile at the time \( t \) such that the \( y \)-component of \( \vec{v}(t) \) is zero. That is, for \( t = \frac{50\sqrt{3}}{g} \). Hence the maximum height reached is given by

\[
50\sqrt{3} \cdot \frac{50\sqrt{3}}{g} - \frac{1}{2}g \left( \frac{50\sqrt{3}}{g} \right)^2 = \frac{(50\sqrt{3})^2}{2g}
\]

We want to find the value of \( t \) when the height reached is \( 3/4 \) times the max. height. That is, \( t \) such that

\[
50\sqrt{3}t - \frac{1}{2}gt^2 = \frac{3}{4} \cdot \frac{(50\sqrt{3})^2}{2g}
\]

\[
\equiv 4g^2t^2 - 8(50\sqrt{3})gt + 3(50\sqrt{3})^2 = 0
\]

\[
\equiv (2gt - 50\sqrt{3})(2gt - 3(50\sqrt{3})) = 0
\]

Thus \( t = \frac{50\sqrt{3}}{2g} \) or \( t = \frac{3(50\sqrt{3})}{2g} \).
(7) Assume that \( z \) is implicitly defined as a function of \( x \) and \( y \) by 
\[
\cos(yz) + x^2z = 9
\]
If at \( x = 2, y = 0 \) and \( z = 2 \), the value of \( x \) starts increasing at the rate of 1 unit per second, and the value of \( y \) starts decreasing at the rate of 2 units per second, compute the rate of change of \( z \).

By the chain rule, we have
\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]
Thus we have to compute \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \). Take the derivative with respect to \( x \) of \( \cos(yz) + x^2z = 9 \):
\[
\sin(yz) y \frac{\partial z}{\partial x} + 2xz + x^2 \frac{\partial z}{\partial x} = 0
\]
At \( x = 2, y = 0, z = 2 \) this equation gives:
\[
0 + 8 + 4 \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -2
\]
Similarly we take the derivative with respect to \( y \) of \( \cos(yz) + x^2z = 0 \) to get
\[
-\sin(yz) \left( z + y \frac{\partial z}{\partial y} \right) + x^2 \frac{\partial z}{\partial y} = 0
\]
Agian at \( x = 2, y = 0, z = 2 \) this equation becomes:
\[
0 + 4 \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = 0
\]
Thus we get
\[
\frac{dz}{dt} = (-2)(1) + (0)(-2) = -2
\]
(8) Let \( \vec{r}(t) \) be a parametric curve. Prove that

\[
\frac{d}{dt} \left( \frac{\vec{r}(t)}{|\vec{r}(t)|} \right) = \frac{1}{|\vec{r}(t)|} \left( \vec{r}'(t) - \text{Proj}_{\vec{r}(t)}(\vec{r}'(t)) \right)
\]

Use the quotient rule to find the derivative:

\[
\frac{d}{dt} \left( \frac{\vec{r}(t)}{|\vec{r}(t)|} \right) = \frac{1}{|\vec{r}(t)|^2} \left( |\vec{r}(t)| \vec{r}'(t) - \frac{d|\vec{r}(t)|}{dt} \vec{r}(t) \right)
\]

\[
= \frac{1}{|\vec{r}(t)|^2} \left( |\vec{r}(t)| \vec{r}'(t) - \left( \frac{\vec{r}(t) \cdot \vec{r}'(t)}{|\vec{r}(t)|} \right) \vec{r}(t) \right)
\]

\[
= \frac{1}{|\vec{r}(t)|} \left( \vec{r}'(t) - \left( \frac{\vec{r}(t) \cdot \vec{r}'(t)}{|\vec{r}(t)|} \right) \vec{r}(t) \right)
\]

\[
= \frac{1}{|\vec{r}(t)|} \left( \vec{r}'(t) - \text{Proj}_{\vec{r}(t)}(\vec{r}'(t)) \right)
\]
(9) Prove that the curvature $\kappa(x)$ of a curve $y = f(x)$ is given by:

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

We can write the curve $y = f(x)$ in its parametric form as:

$$\vec{r}(t) = \langle t, f(t), 0 \rangle$$

which implies

$$\vec{r}'(t) = \langle 1, f'(t), 0 \rangle$$
$$\vec{r}''(t) = \langle 0, f''(t), 0 \rangle$$
$$\vec{r}'(t) \times \vec{r}''(t) = \langle 0, 0, f''(t) \rangle$$

Using the formula for the curvature

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

we get:

$$\kappa(t) = \frac{|f''(t)|}{(1 + f'(t)^2)^{3/2}}$$

Use this formula to find the curvature of $y = x^3$ at $(2, 8)$.

For $f(x) = x^3$, we have $f'(x) = 3x^2$ and $f''(x) = 6x$. At $x = 2$ we get

$$f'(2) = 12 \quad f''(x) = 12$$

Thus

$$\kappa(2) = \frac{12}{(1 + 12^2)^{3/2}} = \frac{12}{(145)^{3/2}}$$