1. Surfaces continued. Recall from last lecture

Trace of a surface \( S = \) intersection curves of \( S \) and planes parallel to coordinate planes

Example. \[ x^2 + \frac{y^2}{4} = 1 + \frac{z^2}{q} \]

- \( x \)-trace: \( x = k \) gives \[ \frac{y^2}{4} - \frac{z^2}{q} = 1 - k^2 \]
  is equation of a hyperbola

- \( y \)-trace: \( x^2 - \frac{z^2}{q} = 1 - \frac{k^2}{4} \) again hyperbola

- \( z \)-trace: \( x^2 + \frac{y^2}{4} = 1 + \frac{k^2}{q} \) ellipse
and the surface $S$ can be sketched as

Example 2. $z = 4 - x^2 - y^2$
2. Cylindrical Coordinates.

Recall that in a plane we have an alternate coordinate system (polar coordinates) \((r, \theta)\).

\[
r = \sqrt{x^2 + y^2}
\]

\[
\theta = \tan^{-1}\left(\frac{y}{x}\right)
\]

Conversely,

\[
x = r \cos \theta
\]
\[
y = r \sin \theta
\]
Note \( r \) is a non-negative real number.

\( \theta \) ranges between 0 and 2\( \pi \) \((0 \leq \theta < 2\pi)\)

This gives an alternate coordinate system in \( \mathbb{R}^3 \), cylindrical coordinates.

\[(x, y, z) \xrightarrow{\text{~}} (r, \theta, z)\]

**Ex. 1.** What are cylindrical coordinates of 

\[(1, 0, 0) \quad (1, 0, 2) \quad (1, 1, 2)\]

2. Sketch the surface

\[\theta = \frac{\pi}{4}, \quad r = 2, \quad z = r\]

Write their equations in cartesian coordinates.
3. Compute rotation around $z$-axis by $30^\circ$, in cylindrical coordinates.

3. Spherical coordinates $(p, \theta, \phi)$

Given a point $P$ in $\mathbb{R}^3$, the spherical coordinates are defined by

\[ p = \text{distance between the origin and } P, \]
\[ \theta = \text{angle between } x\text{-axis and projection of } OP \text{ onto xy-plane} \]
\[ \phi = \text{angle between } z\text{-axis and } \overrightarrow{OP}. \]

Note $p$ is non-negative number

\[ 0 \leq \theta < 2\pi \]
\[ 0 \leq \phi \leq \pi \]

Ex. (a) Change to spherical coordinates

\[ (x=1, y=1, z=1) \quad \Rightarrow \quad (r=\sqrt{3}, \theta = \frac{\pi}{4}, z=\sqrt{6}) \]

(b) Sketch the surface $|\text{div} \mathbf{F}| = C$ obtain its equation

$p = 2 \quad \theta = \frac{5\pi}{6} \quad \text{and} \quad \phi = \frac{\pi}{4}$ in cartesian system.
(c) \( \phi = \frac{\pi}{4} \)

(d) \( \phi = \frac{\pi}{4} \) and \( \rho = 2 \)

Equation of a circle in the plane \( z = 2 \cos \frac{\pi}{4} = \sqrt{2} \).

Equation of the circle:

\[
x^2 + y^2 + 2 = 4 \quad \equiv \quad x^2 + y^2 = 2.
\]

4.* Intersection of surfaces.

Describe the intersection of \( x^2 + 2y^2 = 4 - z \) with \( x^2 + y^2 = 1 \).

Note: this intersection curve does not lie in a plane.

\[\rightarrow \quad x = \cos \theta \quad y = \sin \theta \quad (0 \leq \theta < 2\pi)\]

Substitute in the equation \( x^2 + 2y^2 = 4 - z \):

\[
\equiv \quad 1 + \sin^2 \theta = 4 - z \quad \equiv \quad z = 3 - \sin^2 \theta.
\]