1. Three dimensional coordinate system.

Recall that a point in two dimensional space ($\mathbb{R}^2$) is completely determined by a pair of numbers.

\[ P(a, b) \]

\[ x \text{ coordinate} \]
\[ y \text{ coordinate} \]

Similarly, a point in three dimensional space ($\mathbb{R}^3$) is given by a triple of numbers.

\[ P(a, b, c) \]

\[ a = x \text{-coordinate} \]
\[ b = y \text{-coordinate} \]
\[ c = z \text{-coordinate} \]

Note: The coordinate axes are directed according to the right-hand rule.
Distance between two points in $\mathbb{R}^3$.

Let $P(a, b, c)$ be a point in $\mathbb{R}^3$.

Since $OQP$ is a right angled triangle, we have

$$|OP|^2 = |OQ|^2 + |PQ|^2 \quad \text{-- (1)}$$

Now $ORQ$ is also a right angled triangle.

$$|OQ|^2 = |OR|^2 + |RQ|^2 = a^2 + b^2$$

Substituting back in (1) gives

$$|OP|^2 = a^2 + b^2 + c^2$$

$$\Rightarrow \quad |OP| = \sqrt{a^2 + b^2 + c^2} \quad \text{-- (2)}$$

More generally, if $P(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$ are two points in $\mathbb{R}^3$, then the distance between $P$ and $Q$ is given by

$$|PQ| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2} \quad \text{-- (3)}$$
Coordinate planes.

\[ \begin{align*}
\text{xy plane} & \quad \text{(all points with z-coordinate equal to zero)} \\
\text{yz plane} & \quad (x\text{-coordinate} = 0) \\
\text{xz plane} & \quad (y\text{-coordinate} = 0)
\end{align*} \]

For a point \( P(a,b,c) \):

\[ \begin{align*}
a & = \text{x-coordinate} \\
b & = \text{y-coordinate} \\
c & = \text{z-coordinate}
\end{align*} \]

Distance between \( P \) and \( yz \) plane: \( |a| \)

Distance between \( P \) and \( xz \) plane: \( |b| \)

Distance between \( P \) and \( xy \) plane: \( |c| \)

Examples. Sketch the set in \( \mathbb{R}^3 \) given by

(a) \( y = 1 \)

(b) \( x = z \)

Sphere. Sphere of radius \( r \) centered at \( P(a,b,c) \) consists of all points \( Q(x,y,z) \) such that \( |PQ| = r \).

\[ |PQ|^2 = r^2 \]

\[ (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \]
2. Vectors.

Informally, a vector represents the data of magnitude (or length) and direction. We can represent this information by a directed line segment.

- length of the segment = magnitude
- arrow indicates direction.

(1) We do not distinguish between parallel vectors of equal magnitudes.

(2) \( \vec{0} \) is the unique vector of magnitude 0 (and no direction)

Notation. \( |\vec{v}| \) = magnitude of \( \vec{v} \).
Operations with vectors.

1. Addition.

\[ \vec{u} + \vec{v} \]  (Triangle law)

2. Scalar multiplication.

If \( \vec{v} \) is a vector and \( c \) is a number, then \( c\vec{v} \) is a vector determined by:

- magnitude of \( c\vec{v} \) = \( |c| \) magnitude of \( \vec{v} \)
- direction of \( c\vec{v} \) = \( \begin{cases} \text{direction of } \vec{v} & \text{if } c > 0 \\ \text{opposite to the direction of } \vec{v} & \text{if } c < 0 \end{cases} \)

E.g. \( \vec{3} \to 2\vec{3} \to \)

If \( \vec{u}, \vec{v} \) are two vectors then

\[ \vec{u} - \vec{v} = \vec{u} + (-\vec{v}) \]
Components of a vector.

- In order to have a consistent (algebraic) representation of vectors, we fix the initial point to be the origin.

e.g.

\[ \vec{v} = \langle 2, 1 \rangle \]

\[ \vec{v} = \langle 3, 2 \rangle \]

\[ \vec{u} = \langle 5, 1, 1 \rangle \]

There is a difference between a point \( P(a, b, c) \) and a vector \( \vec{v} = \langle a, b, c \rangle \).

(dangerous bend ahead)

\( \vec{v} = \vec{OP} = \) position vector of \( P \)

\( \vec{v} = \langle a_1, b_1, c_1 \rangle + \langle a_2, b_2, c_2 \rangle = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2 \rangle \)

Scalar multiplication

\( c \cdot \langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle \)

Magnitude or length

\[ |\langle a, b, c \rangle| = \sqrt{a^2 + b^2 + c^2} \]
Standard basis vectors

\[ \hat{i} = \langle 1, 0, 0 \rangle \quad \hat{j} = \langle 0, 1, 0 \rangle \quad \hat{k} = \langle 0, 0, 1 \rangle \]

Thus \( \vec{v} = \langle a_1, a_2, a_3 \rangle \) can also be written as

\[ a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \]

Definition.

A unit vector is a vector of length 1.

Note. Any vector can be written as a product of a scalar and a unit vector.

If \( \vec{v} \) is a vector (non-zero) and \( c = \| \vec{v} \| \), then

\[ \vec{v} = c \cdot \left( \frac{1}{c} \cdot \vec{v} \right) \]

unit vector in the direction of \( \vec{v} \).

Finding components of a vector.

- length = \( a \)
- inclined at an angle \( \theta \) with the x-axis
Recall, by definition, for a right-angled triangle

\[ P = H \sin \theta \]
\[ B = H \cos \theta \]

Thus if \( \vec{v} \) is a vector of length \( a \), inclined at an angle \( \theta \) with the \( x \)-axis:

\[ \vec{v} = \langle a \cos \theta, a \sin \theta \rangle \]

Then \( \vec{v} = \langle a \cos \theta, a \sin \theta \rangle \)

Example. Find the net force experienced by the following object

Solution.

\[ \vec{F}_1 = \langle 50 \cos 30^\circ, 50 \sin 30^\circ \rangle \]
\[ = \langle 25\sqrt{3}, 25 \rangle \]
\[ \vec{F}_2 = \langle -100 \cos 60^\circ, 100 \sin 60^\circ \rangle \]
\[ = \langle -50, 50\sqrt{3} \rangle \]

Total force \( \vec{F} = \vec{F}_1 + \vec{F}_2 = \langle 25\sqrt{3} - 50, 25 + 50\sqrt{3} \rangle \)