42. The distance from the origin to a point \((x, y, z)\) on the surface is 
\[ d^2 = x^2 + y^2 + z^2 = f(x, z) \] 
Then \(f_x = 2x + z, f_z = x + 2z, \) and \(f_x = 0, f_z = 0 \implies x = 0, z = 0\),
so the only critical point is \((0, 0)\).
\(D(0, 0) = (2/2) - 1 = 3 > 0\) with \(f_{xx}(0, 0) = 2 > 0\),
so this is a minimum. Thus 
\[ y^2 = 9 + 0 \implies y = \pm 3 \]
and the points on the surface closest to the origin are \((0, \pm 3, 0)\).

46. Let \(x, y,\) and \(z\) be the dimensions of the box. We wish to minimize surface area 
\[ A = 2xy + 2xz + 2yz, \]
but we have

\[
\text{volume} = xyz = 1000 \implies z = \frac{1000}{xy} \text{ so we minimize}
\]

\[ f(x, y) = 2xy + 2x \left( \frac{1000}{xy} \right) + 2y \left( \frac{1000}{xy} \right) = 2xy + \frac{2000}{y} + \frac{2000}{x} \]

Then \(f_x = 2y - \frac{2000}{x^2} \) and \(f_y = 2x - \frac{2000}{y^2}\).

Setting \(f_x = 0\) implies \(y = \frac{1000}{x^2}\) and substituting into \(f_y = 0\) gives
\[ x - \frac{2000}{1000} = 0 \implies x^3 = 1000 \text{ [since } x \neq 0 \text{]} \implies x = 10.\]

The surface area has a minimum but no maximum and it must occur at a critical point,
so the minimal surface area occurs for a box with dimensions \(x = 10\) cm, \(y = 1000/10^2 = 10\) cm, \(z = 1000/10^2 = 10\) cm.

50. The cost equals \(5xy + 2(xz + yz)\) and \(xyz = V\), so \(C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})\).

Then \(C_x = 5y - 2V x^{-2}, C_y = 5x - 2V y^{-2}, f_x = 0\) implies \(y = 2V/(5x^2), f_y = 0\) implies \(x = \sqrt[3]{\frac{2}{5}} V = y\).

Thus the dimensions of the aquarium which minimize the cost are \(x = y = \sqrt[3]{\frac{2}{5}} V\) units, \(z = V^{1/3}(\frac{2}{3})^{2/3}\).

56. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point \((1, 2, 3)\). Writing the equation of the plane as \(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1\), the volume of the tetrahedron is given by

\[ V = \frac{abc}{6}. \]

But \((1, 2, 3)\) must lie on the plane, so we need \(\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1\) \((*)\) and thus can think of \(c\) as a function of \(a\) and \(b\).

Then \(V_a = \frac{b}{6} \left( c + a \frac{\partial c}{\partial a} \right) \) and \(V_b = \frac{a}{6} \left( c + b \frac{\partial c}{\partial b} \right) \).

Differentiating \((*)\) with respect to \(a\) we get \(-a^{-2} - 3c^{-2} \frac{\partial c}{\partial a} = 0 \implies \frac{\partial c}{\partial a} = -\frac{c^2}{3a^3}, \)
and differentiating \((*)\) with respect to \(b\) gives \(-2b^{-2} - 3c^{-2} \frac{\partial c}{\partial b} = 0 \implies \frac{\partial c}{\partial b} = -\frac{2c^2}{3b^3}.

Then \(V_a = \frac{b}{6} \left( c + a \frac{-c^2}{3a^3} \right) = 0 \implies c = 3a, \) and \(V_b = \frac{a}{6} \left( c + b \frac{-2c^2}{3b^3} \right) = 0 \implies c = \frac{2}{3} b. \)
Thus \(3a = \frac{2}{3} b\) or \(b = 2a\). Putting these into \((*)\) gives \(\frac{a}{3} = 1\) or \(a = 3\) and then \(b = 6, c = 9\). Thus the equation of the required plane is 
\[ \frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1 \]
or \(6x + 3y + 2z = 18.\)
6. \( f(x, y) = e^{x+y}, \ g(x, y) = x^2 + y^2 = 16, \) and \( \nabla f = \lambda \nabla g \Rightarrow \langle ye^{x+y}, xe^{x+y} \rangle = \langle 3\lambda x^2, 3\lambda y^2 \rangle, \) so \( ye^{x+y} = 3\lambda x^2 \) and \( xe^{x+y} = 3\lambda y^2. \) Note that \( x = 0 \iff y = 0 \) which contradicts \( x^2 + y^2 = 16, \) so we may assume \( x \neq 0, y \neq 0, \) and then \( \lambda = ye^{x+y}/(3x^2) = xe^{x+y}/(3y^2) \Rightarrow x^2 = y^2 \Rightarrow x = y. \) But \( x^2 + y^2 = 16, \) so \( 2x^2 = 16 \Rightarrow x = 2 = y. \) Here there is no minimum value, since we can choose points satisfying the constraint \( x^2 + y^2 = 16 \) that make \( f(x, y) = e^{x+y} \) arbitrarily close to 0 (but never equal to 0). The maximum value is \( f(2, 2) = e^4. \)

10. \( f(x, y, z) = x^2y^2z^2, \ g(x, y, z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle, \) \( \lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle. \) Then \( \nabla f = \lambda \nabla g \) implies (1) \( \lambda = y^2z^2 = x^2z^2 = x^2y^2 \) and \( \lambda \neq 0, \) or (2) \( \lambda = 0 \) and one or two (but not three) of the coordinates are 0. If (1) then \( x^2 = y^2 = z^2 = \frac{1}{3}. \) The minimum value of \( f \) on the sphere occurs in case (2) with a value of 0 and the maximum value is \( \frac{1}{27} \) which arises from all the points from (1), that is, the points \( \left( \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} \right) = \left( \pm \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right). \)

14. \( f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n, \ g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \Rightarrow \)
(1, 1, \ldots, 1) = \langle 2\lambda x_1, 2\lambda x_2, \ldots, 2\lambda x_n \rangle, \) so \( \lambda = 1/(2x_1) = 1/(2x_2) = \cdots = 1/(2x_n) \) and \( x_1 = x_2 = \cdots = x_n. \) But \( x_1^2 + x_2^2 + \cdots + x_n^2 = 1, \) so \( x_i = \pm 1/\sqrt{n} \) for \( i = 1, \ldots, n. \) Thus the maximum value of \( f \) is \( f(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n} \) and the minimum value is \( f(-1/\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}. \)

20. \( f(x, y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \Rightarrow x = 1, y = 0. \) Thus \( (1, 0) \) is the only critical point of \( f, \) and it lies in the region \( x^2 + y^2 < 16. \) On the boundary, \( g(x, y) = x^2 + y^2 = 16 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle, \) so \( 6y = 2\lambda y \) \( \Rightarrow \) either \( y = 0 \) or \( \lambda = 3. \) If \( y = 0, \) then \( x = \pm 4; \) if \( \lambda = 3, \) then \( 4x - 4 = 2\lambda x \Rightarrow x = -2 \) and \( y = \pm 2 \sqrt{3}. \) Now \( f(1, 0) = -7, f(4, 0) = 11, f(-4, 0) = 43, \) and \( f(-2, \pm 2 \sqrt{3}) = 47. \) Thus the maximum value of \( f(x, y) \) on the disk \( x^2 + y^2 \leq 16 \) is \( f(-2, \pm 2 \sqrt{3}) = 47, \) and the minimum value is \( f(1, 0) = -7. \)

32. Let \( f(x, y, z) = d^2 = x^2 + y^2 + z^2. \) Then we want to minimize \( f \) subject to the constraint \( g(x, y, z) = y^2 - xz = 9. \)
\( \nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z \rangle = \langle -x, 2\lambda y, -\lambda z \rangle, \) so \( 2x = -\lambda z, y = \lambda y, \) and \( 2z = -\lambda z. \) If \( x = 0 \) then the last equation implies \( z = 0, \) and from the constraint \( y^2 - xz = 9 \) we have \( y = \pm 3. \) If \( x \neq 0, \) then the first and third equations give \( \lambda = -2x/z = -2z/x \Rightarrow x^2 = z^2. \) From the second equation we have \( y = 0 \) or \( \lambda = 1. \) If \( y = 0 \) then \( y^2 - xz = 9 \Rightarrow z = -9/x \) and \( x^2 = z^2 \Rightarrow x^2 = 81/x^2 \Rightarrow x = \pm 3. \) Since \( z = -9/x, \)
\( x = 3 \Rightarrow z = -3 \) and \( x = -3 \Rightarrow z = 3. \) If \( \lambda = 1, \) then \( 2x = -z \) and \( 2z = -x \) which implies \( x = z = 0, \) contradicting the assumption that \( x \neq 0. \) Thus the possible points are \( (0, \pm 3, 0), (3, 0, -3), (-3, 0, 3). \) We have \( f(0, \pm 3, 0) = 9 \) and \( f(3, 0, -3) = f(-3, 0, 3) = 18, \) so the points on the surface that are closest to the origin are \( (0, \pm 3, 0). \)
HW8

36. If the dimensions of the box are \( x, y, \) and \( z \) then minimize \( f(x, y, z) = 2xy + 2xz + 2yz \) subject to \( g(x, y, z) = xyz = 1000 \) \((x > 0, y > 0, z > 0)\). Then \( \nabla f = \lambda \nabla g \Rightarrow (2y + 2z, 2x + 2z, 2x + 2y) = \lambda (yz, xz, xy) = 2y + 2z = \lambda yz, \)
\[ 2x + 2z = \lambda xz, \quad 2x + 2y = \lambda xy. \]
Solving for \( \lambda \) in each equation gives \( \frac{2}{z} + \frac{2}{y} = \frac{2}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x} \Rightarrow x = y = z. \)

From \( xyz = 1000 \) we have \( x^3 = 1000 \) \( \Rightarrow \) \( x = 10 \) and the dimensions of the box are \( x = y = z = 10 \text{ cm.} \)

42. Let the dimensions of the box be \( x, y, \) and \( z, \) so its volume is \( f(x, y, z) = xyz, \) its surface area is \( 2xy + 2yz + 2xz = 1500 \) and its total edge length is \( 4x + 4y + 4z = 200. \) We find the extreme values of \( f(x, y, z) \) subject to the constraints \( g(x, y, z) = xy + yz + xz = 750 \) and \( h(x, y, z) = x + y + z = 50. \) Then
\[ \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda (y + z), \lambda (x + z), \lambda (x + y) \rangle + \langle \mu, \mu, \mu \rangle. \]
So \( yz = \lambda (y + z) + \mu \) (1), \( xz = \lambda (x + z) + \mu \) (2), and \( xy = \lambda (x + y) + \mu \) (3). Notice that the box can’t be a cube or else \( x = y = z = \frac{100}{3} \)
but then \( xy + yz + xz = \frac{2500}{3} \neq 750. \) Assume \( x \) is the distinct side, that is, \( x \neq y, \) \( x \neq z. \) Then (1) minus (2) implies \( z (y - x) = \lambda (y - x) \) or \( \lambda = z, \) and (1) minus (3) implies \( y (z - x) = \lambda (z - x) \) or \( \lambda = y. \) So \( y = z = \lambda \) and \( x + y + z = 50 \)
implies \( x = 50 - 2\lambda; \) also \( xy + yz + xz = 750 \) implies \( x(2\lambda) + \lambda^2 = 750. \) Hence \( 50 - 2\lambda = \frac{750 - \lambda^2}{2\lambda} \) or
\[ 3\lambda^2 - 100\lambda + 750 = 0 \text{ and } \lambda = \frac{50 \pm 5 \sqrt{10}}{3}, \text{ giving the points } \left( \frac{1}{3}(50 + 10 \sqrt{10}), \frac{1}{3}(50 + 5 \sqrt{10}), \frac{1}{3}(50 + 5 \sqrt{10}) \right). \]
Thus the minimum of \( f \) is \( f \left( \frac{1}{3}(50 - 10 \sqrt{3}), \frac{1}{3}(50 + 5 \sqrt{10}), \frac{1}{3}(50 + 5 \sqrt{10}) \right) = \frac{1}{27}(87,500 - 2500 \sqrt{10}), \) and its maximum is \( f \left( \frac{1}{3}(50 + 10 \sqrt{10}), \frac{1}{3}(50 - 5 \sqrt{10}), \frac{1}{3}(50 - 5 \sqrt{10}) \right) = \frac{1}{27}(87,500 + 2500 \sqrt{10}). \)

Note: If either \( y \) or \( z \) is the distinct side, then symmetry gives the same result.
48. (a) Let \( f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i, \) \( g(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2, \) and \( h(x_1, \ldots, x_n) = \sum_{i=1}^{n} y_i^2. \) Then

\[
\nabla f = \nabla \sum_{i=1}^{n} x_i y_i = (y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n), \quad \nabla g = \nabla \sum_{i=1}^{n} x_i^2 = (2x_1, 2x_2, \ldots, 2x_n, 0, 0, \ldots, 0) \quad \nabla h = \nabla \sum_{i=1}^{n} y_i^2 = (0, 0, \ldots, 0, 2y_1, 2y_2, \ldots, 2y_n). \]

So \( \nabla f = \lambda \nabla g + \mu \nabla h \iff y_i = 2\lambda x_i \) and \( x_i = 2\mu y_i, \)

\[
1 \leq i \leq n. \text{ Then } 1 = \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} 4\lambda^2 x_i^2 = 4\lambda^2 \sum_{i=1}^{n} x_i^2 = 4\lambda^2 \Rightarrow \lambda = \pm \frac{1}{2}. \text{ If } \lambda = \frac{1}{2} \text{ then } y_i = 2\left(\frac{1}{2}\right)x_i = x_i, \]

\[
1 \leq i \leq n. \text{ Thus } \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i^2 = 1. \text{ Similarly if } \lambda = -\frac{1}{2} \text{ we get } y_i = -x_i \text{ and } \sum_{i=1}^{n} x_i y_i = -1. \text{ Similarly we get } \mu = \pm \frac{1}{2} \text{ giving } y_i = \pm x_i, 1 \leq i \leq n, \text{ and } \sum_{i=1}^{n} x_i y_i = \pm 1. \text{ Thus the maximum value of } \sum_{i=1}^{n} x_i y_i \text{ is 1.} \]

(b) Here we assume \( \sum_{i=1}^{n} a_i^2 \neq 0 \) and \( \sum_{i=1}^{n} b_i^2 \neq 0. \) (If \( \sum_{i=1}^{n} a_i^2 = 0, \) then each \( a_i = 0 \) and so the inequality is trivially true.)

\[
x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \Rightarrow \sum x_i^2 = \frac{\sum a_i^2}{\sum a_j^2} = 1, \text{ and } y_i = \frac{b_i}{\sqrt{\sum b_j^2}} \Rightarrow \sum y_i^2 = \frac{\sum b_i^2}{\sum b_j^2} = 1. \text{ Therefore, from part (a),}
\]

\[
\sum x_i y_i = \frac{1}{\sqrt{\sum a_j^2} \sqrt{\sum b_j^2}} \leq 1 \iff \sum a_i b_i \leq \sqrt{\sum a_j^2} \sqrt{\sum b_j^2}. \]