1. It suffices to show that $X^\text{sing} \cap D_+(x_i) = V_+(F, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}) \cap D_+(x_i)$ for all $i$, $0 \leq i \leq n$. (Note that $\frac{\partial F}{\partial x_i}$ is homogeneous of degree $d - 1$.) Assume without loss of generality that $i = 0$. Let $f_0(y_1, \ldots, y_n) = F^\text{inh}(y_1, \ldots, y_n) = F(1, y_1, \ldots, y_n)$. Then $X \cap D_+(x_0) = V(f_0)$. Clearly

$$\frac{\partial f_0}{\partial y_i}(y_1, \ldots, y_n) = \left(\frac{\partial F}{\partial x_i}\right)^\text{inh}(y_1, \ldots, y_n).$$

It follows by the discussion in class of the singular locus of a hypersurface in $\mathbb{A}^n_k$ that

$$X^\text{sing} \cap D_+(x_0) = V(f_0, \frac{\partial f_0}{\partial y_1}, \ldots, \frac{\partial f_0}{\partial y_n}) = V_+(F, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}) \cap D_+(x_0).$$

where we have used Euler’s lemma $\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = dF$ for the last equality. Finally, if the characteristic of $k$ does not divide $d$, then

$$F = \frac{1}{d} \sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} \in \left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}\right),$$

and hence $V_+(F, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}) = V_+(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})$. However, if the characteristic of $k$ is $p > 0$, taking $F = G^p$ for some nonzero $G$ which is homogeneous of degree $e > 0$, we have $\frac{\partial F}{\partial x_i} = 0$ for all $i$, but

$$V_+(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}) = V_+(0) \neq V_+(F, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}) = V_+(G).$$

2. (i) We can assume that $X = \text{Spec} R$ is affine and that $I$ is an ideal of $R$. We can define $I$ to be nilpotent if either every element of $I$ is nilpotent or if $I^N = (0)$ for some $N > 0$. The second condition implies the first (and they are equivalent if $R$ is Noetherian). We show that $\text{Proj}(\bigoplus_n I^n) = \emptyset$ under the first (and weaker) definition. If $p$ is a homogeneous prime ideal in the graded ring $R = \bigoplus_n I^n$, then for all $f \in R_+$, homogeneous of degree $d$, say $f \in I^d$, ...
then \( f^N = 0 \) for some \( N \) and hence \( f \in \mathfrak{p} \). Thus every homogeneous prime ideal in \( R \) contains \( R_+ \), so \( \text{Proj} R = \emptyset \).

(ii) Here we should assume that \( Z \) is a closed subscheme (or the conclusion should be that \( Z' \) is a subscheme of \( \tilde{X} \), not necessarily closed, and the parenthetical comments don’t necessarily hold). By the universal property of blowing up, there is a morphism from \( Z' \) to \( \tilde{X} \) (since it is easy to check that \( \mathcal{I}_Y \mathcal{O}_{Z'} = (\mathcal{I}_Y \mathcal{O}_Z) \mathcal{O}_{Z'} \) and we must check that it is a closed embedding. Let \( \tilde{T} = \mathcal{I}_Z \mathcal{O}_{\tilde{X}} \). Then \( \tilde{X} = \text{Proj}_X (\bigoplus_n \mathcal{I}_Z^n) \) and \( Z' = \text{Proj}_Z (\bigoplus_n \mathcal{I}^n) \), and the surjection \( \mathcal{O}_X \to \mathcal{O}_Z \) induces surjections \( \mathcal{I}_Z^n \to \mathcal{I}^n \) for every \( n \), clearly corresponding to the morphism from \( Z' \to \tilde{X} \). Thus \( Z' \to \tilde{X} \) is a closed embedding. The parenthetical statements then follow from the fact that \( Z' \) is integral since \( Z \) is integral (the only place where this is used) and the assumption that \( Z \) is not contained in \( Y \), so that \( Z' - E \) is a nonempty open subset of \( Z' \) which is integral as well.

(iii) Let \( S = k[x_1, \ldots, x_n], \; \mathfrak{m} = (x_1, \ldots, x_n), \) and \( E = \text{Proj}(\bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}) \). If \( n = \mathfrak{m}(S/I_Z), \) then \( Z' = \text{Proj}(\bigoplus_n \mathfrak{n}^n), \) and the exceptional divisor \( Z' \cap E = \text{Proj}(\bigoplus_n \mathfrak{n}^n/\mathfrak{n}^{n+1}). \) Now

\[
\mathfrak{n}^n/\mathfrak{n}^{n+1} = (\mathfrak{m}^n + I_Z)/(\mathfrak{m}^{n+1} + I_Z) \cong \mathfrak{m}^n/(\mathfrak{m}^n \cap I_Z + \mathfrak{m}^{n+1}).
\]

If \( f \in \mathfrak{m}^n - \mathfrak{m}^{n+1} \), then \( \text{deg} \text{in}(f) = n \) and \( f \equiv \text{in}(f) \text{ mod } \mathfrak{m}^{n+1} \). Thus \( \mathfrak{n}^n/\mathfrak{n}^{n+1} \) is the vector space quotient of \( \mathfrak{m}^n/\mathfrak{m}^{n+1} \) by the vector subspace of all in(\( \mathfrak{m} \)), \( f \in I_Z \) and \( \text{deg} \text{in}(f) = n \). Hence \( Z' \cap E \) is defined as a subscheme of \( E \cong \mathbb{P}^{n-1}_k \) by the homogeneous ideal in(\( I_Z \)). (Warning: it is not true that, if \( I_Z = (g_1, \ldots, g_r) \), then \( \text{in}(I_Z) = (\text{in}(g_1), \ldots, \text{in}(g_r)) \).)

(iv) One way to do this is as follows: Clearly \( \{0\} \in Z_{\text{sing}} \) since \( F(0) = 0 \), \( \partial F/\partial x_i(0) = 0 \) for all \( i \), since every homogeneous polynomial of degree \( > 1 \) vanishes at \( 0 \). If \( p \neq 0 \) and \( F(p) = 0 \), then it is easy to see directly that \( \partial F/\partial x_i(p) \neq 0 \) for at least one \( i \) (this will also follow from the calculations below). Another method is to argue that \( \langle F, \partial F/\partial x_1, \ldots, \partial F/\partial x_n \rangle \subseteq \mathfrak{m} \), where as in (iii) \( \mathfrak{m} = (x_1, \ldots, x_n) \), since \( F \) is homogeneous of degree \( \geq 2 \), and the condition that \( V_+(F, \partial F/\partial x_1, \ldots, \partial F/\partial x_n) = \emptyset \) implies that \( \sqrt{(F, \partial F/\partial x_1, \ldots, \partial F/\partial x_n)} \) is either \( \mathfrak{m} \) or \( k[x_1, \ldots, x_n] \), hence is \( \mathfrak{m} \). Thus \( Z_{\text{sing}} = V(F, \partial F/\partial x_1, \ldots, \partial F/\partial x_n) = \mathfrak{m} \), corresponding to the closed point \( \{0\} \).

By (iii), \( Z' \cap E = V_+(\text{in}(F)) \), where \( (F) \) is the principal ideal generated by \( F \), and it is easy to check that \( \text{in}(gF) = \text{in}(g)\text{in}(F) = \text{in}(g)F \), so that
\(V_+(\text{in}(F)) = V_+(F)\). To see that \(Z'\) is smooth, one can argue that its intersection with the Cartier divisor \(E\) is smooth as a scheme and use a little local algebra. (Let \((R, \mathfrak{m})\) be a Noetherian local ring and \(x \in \mathfrak{m}\) not a zero divisor. If \(R/xR\) is regular, then \(R\) is regular.) A direct argument is as follows: As we saw in class, \(\text{Bl}_{(0)} \mathbb{A}^n\) is covered by the the \(n\) open sets \(U_i \cong \mathbb{A}^n\), where say \(U_1 \subseteq \mathbb{A}^n \times \mathbb{A}^{n-1} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}\) is defined by

\[
\{(x_1, \ldots, x_n, y_2, \ldots, y_n) : x_i = x_1 y_i, i > 1\},
\]

with coordinates \(x_1, y_2, \ldots, y_n\). Then

\[
F(x_1, \ldots, x_n) = F(x_1, x_1 y_2, \ldots, x_1 y_n) = x_1^d F(1, y_2, \ldots, y_n) = x_1^d f_1,
\]

say, where \(f_1(y_2, \ldots, y_n) = F(1, y_2, \ldots, y_n)\). Thus \(Z' \cap U_1\) is contained in \(V(x_1^d f_1) = V(x_1) \cup V(f_1)\). Since \(V(x_1) = E \cap U_1\), where \(E\) is the exceptional divisor, it follows by the parenthetical remark in (ii) that \(Z' \cap U_1 = V(f_1)\).

By the argument of Problem 1, the fact that \(V_+(F)\) is smooth implies that \(V(f_1)\) is smooth. By symmetry, \(Z' \cap U_i\) is smooth for every \(i\), hence \(Z'\) is smooth.

3. If \(I = (x, y^d)\) and \(I/I^2\) is the conormal bundle, then \(I\) is a free rank two module over \(\mathcal{O}_Y = \text{Spec } k[y]/(y^d)\) and \(Y\) is a local complete intersection. Hence as schemes the exceptional divisor is \(\text{Proj}(\text{Sym}^* I/I^2) \cong \mathbb{P}^1_{\text{Spec } k[y]/(y^d)} \cong \mathbb{P}^1_k \times_{\text{Spec } k} \text{Spec } k[y]/(y^d)\). We will also check this directly. Thus as reduced schemes \(\tilde{E}_\text{red} \cong \mathbb{P}^1_k\).

Next, \(\tilde{X} = \text{Proj}(\bigoplus_i I^n)\). If \(R\) is the graded ring \(\bigoplus_i I^n\) and \(S = k[x, y]\), then \(R\) is an integral domain and there is a surjection of graded rings

\[
\varphi: S[s, t] \to R
\]

\((S\text{ in degree 0, } s\text{ and } t\text{ have degree 1})\) which is the identity on \(S\) and sends \(s\) to \(x\) and \(t\) to \(y^d\) in \(I\). In particular \(\tilde{X}\) is isomorphic to a closed subscheme of \(\text{Proj } S[s, t] = \mathbb{P}^1_S = \mathbb{A}^2_k \times_{\text{Spec } k} \mathbb{P}^1_k\). Clearly \(y^d s - xt \in \text{Ker } \varphi\) and hence \(\tilde{X} \subseteq V_+(y^d s - xt)\). It is easy to check directly that \(y^d s - xt\) is irreducible in \(k[x, y, s, t]\), since its degree in \(x\) is 1, say, so it would have to factor as

\[
A(y, s, t) \cdot (B(y, s, t)x + C(y, s, t)),
\]

thus \(AB = t\). Since \(t\) is relatively prime to \(y^d s\), it follows that \(A \in k^*\) and \(B = A^{-1} t\). By counting dimensions, \(\tilde{X} = V_+(y^d s - xt)\). We will give
a direct argument below as well. The scheme \( \text{Proj} S[s, t] \) is covered by the two affine open pieces \( U_1 = D_+(s) \) and \( U_2 = D_+(t) \), both \( \cong A^3_k \). In \( U_1 \) there are coordinates \( x, y, z \), with \( z = t/s \). Thus \( \widetilde{X} \cap U_1 = V(y^d - xz) \).

Hence, on \( \widetilde{X} \cap U_1 \), the exceptional divisor \( E \cap U_1 \) is \( V(x) \cap \widetilde{X} \cap U_1 \) since the ideal \((x, y^2)\) pulls back to \((x, y^d) \equiv (x, xz) = (x) \). Hence \( E \cap U_1 = \text{Spec} k[x, y, z]/(x, y^d - xz) = k[y, z]/(y^d) \). Moreover,

\[
(\widetilde{X} \cap U_1)_{\text{sing}} = V(y^d - xz, -x, -z, dy^{d-1}) = V(x, y, z) = \{0\}
\]

provided that \( d > 1 \). Note that, on the complement of the exceptional divisor, \( V(y^d - xz) - V(x) \cong (A^1_k - \{0\}) \times_{\text{Spec} k} A^1_k \) and hence is irreducible because then \( x, y \) are coordinates with \( z = y^d/x \). In particular we see directly that \( V_+(y^d s - x t) \cap U_1 = \widetilde{X} \cap U_1 \). The case \( \widetilde{X} \cap U_2 \) is similar: here there are coordinates \( x, y, w \), with \( w = s/t \). Thus \( \widetilde{X} \cap U_1 = V(y^d w - x) \), and the exceptional divisor is \( V(y^d) \). It follows that \( \widetilde{X} \cap U_1 \) is smooth since \( \partial x/\partial x = 1 \), or directly because, as \( x = y^d w \), \( V(y^d w - x) = \text{Spec} k[y, w] \). Moreover, \( E \cap U_2 = \text{Spec} k[x, y, w]/(y^d, y^d w - x) = \text{Spec} k[y, w]/(y^d) \). (Note also that on \( V(y^d w - x) - V(y^d) \), we have \( w = x/y^d \).) Also, \( V_+(y^d s - x t) \cap U_2 = \widetilde{X} \cap U_2 \), since both are irreducible, hence \( V_+(y^d s - x t) = \widetilde{X} \). Finally, \( E \) is two copies of \( A^1_k \times_{\text{Spec} k} \text{Spec} k[y]/(y^d) \) glued along \((A^1_k - \{0\}) \times_{\text{Spec} k} \text{Spec} k[y]/(y^d) \), and it follows from the explicit formulas that \( E \cong P^1_k \times_{\text{Spec} k} \text{Spec} k[y]/(y^d) \) since \( w = z^{-1} \).

4. (i) Using the exact sequence

\[
0 \to K_X \otimes \mathcal{O}_X(p - q) \to K_X \otimes \mathcal{O}_X(p) \to k(q) \to 0,
\]

we get the long exact sequence

\[
H^0(X; K_X \otimes \mathcal{O}_X(p)) \to k \to H^1(X; K_X \otimes \mathcal{O}_X(p-q)) \to H^1(X; K_X \otimes \mathcal{O}_X(p)) \to 0.
\]

By Serre duality, \((H^1(X; K_X \otimes \mathcal{O}_X(p-q)))^\vee = H^0(X; \mathcal{O}_X(q-p))\), and \((H^1(X; K_X \otimes \mathcal{O}_X(p)))^\vee = H^0(X; \mathcal{O}_X(-p)) = 0\) since \( \deg p < 0 \). As \( \deg(q - p) = 0 \), \( H^0(X; \mathcal{O}_X(q-p)) \neq 0 \iff p \equiv q \iff \dim H^0(X; \mathcal{O}_X(q-p)) = 1 \).

Thus \( q \) is a base point \( \iff p \equiv q \). If \( X \cong P^1_k \), then, for all \( q \in X \), \( p \equiv q \) and hence all points \( q \) are base points. (Alternatively, \( \deg(K_X \otimes \mathcal{O}_X(p)) = -1 \) and so \( H^0(X; K_X \otimes \mathcal{O}_X(p)) = 0 \).) Conversely, if \( q = p \), then \( p \equiv q \), and if \( q \neq p \) and \( g(x) \neq 0 \), then we have seen that \( \dim H^0(X; \mathcal{O}_X(p)) = 1 \), hence \( H^0(X; \mathcal{O}_X(q-p)) = 0 \). Thus for \( g \geq 1 \), \( p \) is the unique base point.
(ii) It is base point free because it has degree $2g$. By RR,

$$\chi(X; K_X \otimes \mathcal{O}_X(p+q)) = 2g - g + 1 = g + 1.$$ 

Since $H^1(X; K_X \otimes \mathcal{O}_X(p+q))$ is Serre dual to $H^0(X; \mathcal{O}_X(-p-q))$, it is zero, hence

$$h^0(X; K_X \otimes \mathcal{O}_X(p+q)) = g + 1.$$

(iii) Using the exact sequence

$$0 \to K_X \otimes \mathcal{O}_X \to K_X \otimes \mathcal{O}_X(p+q) \to \mathbb{k}(p) \oplus \mathbb{k}(q) \to 0,$$

we get the long exact sequence

$$H^0(X; K_X \otimes \mathcal{O}_X(p+q)) \to k \oplus k \to H^1(X; K_X) \to H^1(X; K_X \otimes \mathcal{O}_X(p+q)).$$

The usual Serre duality argument shows that $H^1(X; K_X \otimes \mathcal{O}_X(p+q)) = 0$. Since $\dim H^1(X; K_X) = 1$, the evaluation map $H^0(X; K_X \otimes \mathcal{O}_X(p+q)) \to k \oplus k$ is not surjective, and since $K_X \otimes \mathcal{O}_X(p+q)$ is base point free it fails to separate the points $p$ and $q$.

(iv) Given $x, y \in X$, begin with the short exact sequence

$$0 \to K_X \otimes \mathcal{O}_X(p+q-x-y) \to K_X \otimes \mathcal{O}_X(p+q) \to \mathbb{k}(x) \oplus \mathbb{k}(y) \to 0.$$

The usual arguments show that $H^0(X; K_X \otimes \mathcal{O}_X(p+q)) \to k \oplus k$ is not surjective $\iff H^1(X; K_X \otimes \mathcal{O}_X(p+q-x-y)) \neq 0 \iff H^0(X; \mathcal{O}_X(x + y - p - q)) \neq 0 \iff x + y \equiv p + q$. We saw in class that $\dim |p+q| = 0$ unless $X$ is hyperelliptic and $p + q$ is the unique divisor up to linear equivalence such that $\dim |p+q| = 1$. In this last case, $\varphi(x) = \varphi(y)$ for all $x, y$ such that $x + y \in |p+q|$. In this case, the degree $e$ of the morphism from $X$ to $\varphi(X)$ is at least 2, so that $\varphi(X)$ is a nondegenerate curve in $\mathbb{P}^g$ of degree $2g/e \leq g$. It follows that $e = 2$, $\varphi(X)$ has normalization $\mathbb{P}^1_k$ and is embedded by the complete linear system, hence is smooth. Thus the morphism $X \to \varphi(X)$ is a degree 2 morphism to $\mathbb{P}^1$, unique up to automorphisms of $\mathbb{P}^1$, and the image of $\varphi$ is a degree $g$ canonical curve.