Spectral sequences and applications

4.1 Basic definitions and results

Let $R$ be a ring and let $(C^\cdot, d)$ be a complex of $R$-modules, where $d$ has degree one. With minor modifications, we could just assume that the $C^p$ and $d$ are objects and morphisms in an abelian category; a future application will involve the abelian category of sheaves of $R$-modules on a topological space $X$. Suppose in addition that $F^\cdot$ is a decreasing filtration on $C^\cdot$, stable under $d$ (i.e. $dF^pC^k \subseteq F^pC^{k+1}$). For simplicity, we assume that, for each $k$, $F^pC^k$ is a finite filtration, i.e. that $F^rC^k = C^k$ for $r \ll 0$ and that $F^rC^k = 0$ for $r \gg 0$. We abbreviate the above by writing that $(C^\cdot, d, F^\cdot C^\cdot)$ is a filtered complex. Morphisms $f : (C^\cdot, d, F^\cdot C^\cdot) \to (D^\cdot, d, F^\cdot D^\cdot)$ are defined in the obvious way, as morphisms of complexes such that $f(F^pC^k) \subseteq F^pD^k$ for all $p, k$.

The differential $d$ induces a differential (also denoted $d$) on the subcomplexes $F^pC^\cdot$ and on the quotients $\text{gr}^p_F C^k = F^pC^k / F^{p+1}C^k$. The inclusion $F^pC^k \subseteq C^k$ also induces a decreasing filtration on $H^k(C^\cdot)$:

$$F^pH^k(C^\cdot) = \text{Im} H^k(F^pC^\cdot) \subseteq H^k(C^\cdot).$$

We can then form the associated quotients

$$\text{gr}^p_F H^k(C^\cdot) = F^pH^k(C^\cdot) / F^{p+1}H^k(C^\cdot).$$

The main goal of a spectral sequence is to compare $H^k(\text{gr}^p_F C^\cdot)$ with $\text{gr}^p_F H^k(C^\cdot)$. Note that we always have an intermediate group

$$H^k(F^pC^\cdot) / \text{Im} H^k(F^{p+1}C^\cdot) \xrightarrow{\phi_2} H^k(\text{gr}^p_F C^\cdot) \xrightarrow{\phi_1} \text{gr}^p_F H^k(C^\cdot)$$

Recall also the following definition:

**Definition 1.** In the above notation, the differential $d$ is strict with respect to the filtration $F^\cdot$ $\iff$ for all $k, p$,

$$F^pC^k \cap dC^{k-1} = dF^pC^{k-1}.$$

**Theorem 2.** The differential $d$ is strict with respect to the filtration $F^\cdot$ $\iff$ for all $k, p$, the map $\phi_1$ in the above diagram is an isomorphism $\iff$ for all $k, p$, the map $\phi_2$ in the above diagram is an isomorphism.
**Proof.** First suppose that $d$ is strict with respect to the filtration $F$. This is equivalent to saying that, for all $k, p$, the homomorphism
\[ H^k(F^pC) \to H^k(C) \]
is injective. As its image by definition is $F^pH^k(C)$, we see that $H^k(F^pC) \cong F^pH^k(C)$ under the natural map. Likewise, $H^k(F^{p+1}C) \cong F^{p+1}H^k(C)$, and the map $H^k(F^{p+1}C) \to H^k(F^pC)$ is an inclusion. Thus
\[ \phi_1 : H^k(F^pC)/\text{Im } H^k(F^{p+1}C) \to \text{gr}_F^p H^k(C) \]
is an isomorphism.

Next consider the long exact cohomology sequence $(\ast)$:
\[ \cdots \to H^k(F^{p+1}C) \to H^k(F^pC) \to H^k(\text{gr}_F^p C) \to H^{k+1}(F^{p+1}C) \to \cdots \]
Since $H^k(F^{p+1}C) \to H^k(F^pC)$ is injective for all $k$, it follows that
\[ \phi_2 : H^k(F^pC)/H^k(F^{p+1}C) \to H^k(\text{gr}_F^p C) \]
is also an isomorphism for all $p$ and $k$, proving that $\phi_1$ and $\phi_2$ are isomorphisms.

Next suppose that $\phi_1$ is an isomorphism. Let $\alpha \in F^pC^k$ be such that $\alpha = d\beta$ for some $\beta \in C^{k-1}$. Then $d\alpha = 0$, so that $\alpha$ defines a class $[\alpha] \in H^k(F^pC)$. By assumption $[\alpha] \in \text{Ker } H^k(F^pC) \to H^k(C)$. Since $\phi_1$ is injective, $[\alpha] \in \text{Im } H^k(F^{p+1}C)$, i.e. there exists $\alpha_1 \in F^{p+1}C^k$ with $d\alpha_1 = 0$ and such that $\alpha = \alpha_1 + d\beta_1$ for some $\beta_1 \in F^pC^{k-1}$. Then $\alpha_1 = d(\beta - \beta_1)$, so that $[\alpha_1] \in \text{Ker } H^k(F^{p+1}C) \to H^k(C)$. Thus, arguing as above, $\alpha_1 = \alpha_2 + d\beta_2$, where $\alpha_2 \in F^{p+2}C^k$ with $d\alpha_1 = 0$ and $\beta_2 \in F^{p+1}C^{k-1}$. Continuing in this way, we see that, for all $r \geq 1$, there exist $\alpha_{r+1} \in F^{p+r+1}C^k$, $\beta_{r+1} \in F^{p+r}C^{k-1}$, such that
\[ \alpha_r = \alpha_{r+1} + d\beta_{r+1}. \]
Choosing $r$ so large that $F^{p+r+1}C^k = 0$, we see that
\[ \alpha = d(\beta_1 + \cdots + \beta_{r+1}), \]
where $\beta_1 + \cdots + \beta_{r+1} \in F^pC^{k-1}$. Thus the filtration is strict.

Finally assume that $\phi_2$ is an isomorphism. This implies in particular that $H^k(F^pC) \to H^k(\text{gr}_F^p C)$ is surjective for all $p, k$, and hence, by looking at the long exact cohomology sequence $(\ast)$, that $H^k(F^{p+1}C) \to H^k(F^pC)$ is injective for all $k, p$. In particular, the maps
\[ H^k(F^pC) \to H^k(F^{p-1}C) \to \cdots \to H^k(F^0C) \]
are injective for all $r \leq p$. Choosing $r \ll 0$, we see that $H^k(F^pC^r) \to H^k(C^r)$ is injective, so that the filtration is strict.

**Remark 3.** The above proof shows that, in any case,

$$\phi_1 : H^k(F^pC^r) / \text{Im } H^k(F^{p+1}C^r) \to \text{gr}_F^p H^k(C^r)$$

is surjective and that

$$\phi_2 : H^k(F^pC^r) / \text{Im } H^k(F^{p+1}C^r) \to H^k(\text{gr}_F^p C^r)$$

is injective. Thus, suppose that $R$ is a field $k$ and that all of the relevant cohomology groups $H^k(C^r)$, $H^k(\text{gr}_F^p C^r)$, $H^k(F^p C^r)$ are finite dimensional. Then

$$\dim_k H^k(\text{gr}_F^p C^r) \geq \dim_k \text{gr}_F^p H^k(C^r),$$

with equality for all $p, k \iff$ the filtration is strict.

One interpretation of a spectral sequence for a filtered complex $F^rC^r$ is then that it measures how far the differential fails to be strict with respect to the filtration. The basic model is a subcomplex $F^1C^r$ of $C^r$, or equivalently a one step filtration

$$0 \subseteq F^1C^r \subseteq F^2C^r = C^r.$$

Associated to the short exact sequence

$$0 \to F^1C^r = \text{gr}_F^1 C^r \to C^r \to C^r/F^1C^r = \text{gr}_F^0 C^r \to 0,$$

we have the long exact cohomology sequence

$$\cdots \to H^k(\text{gr}_F^1 C^r) \to H^k(C^r) \to H^k(\text{gr}_F^0 C^r) \xrightarrow{d_1} H^k(\text{gr}_F^1 C^r) \to \cdots,$$

and the connecting homomorphism $d_1$ measures the discrepancy between $H^k(\text{gr}_F^p C^r)$ with $\text{gr}_F^p H^k(C^r)$. For a filtration with more than one step, we get multiple homomorphisms stitched together from multiple connecting homomorphisms (or rather subquotients defined via their kernels and images). We will give the basic definitions and results about constructing spectral sequences, but are not concerned with the (straightforward) proofs so much as with the applications.

**Definition 4.** With $(C^r, d)$ and $F^rC^r$ as above, we define:

$$Z^p_{r,q} = \{ \alpha \in F^p C^{p+q} : d\alpha \in F^{p+r} C^{p+q+1} \};$$

$$B^p_{r,q} = Z^{p+1,q-1}_{r-1} + dZ^{p+1,q+r-2}_{r-1}.$$
It is straightforward to verify that $B^p_{r} \subseteq Z^p_{r}$; we set

$$E^p_{r} = E^p_{r}(C^r) = Z^p_{r}/B^p_{r},$$

and note that $d$ induces a map $d_r: E^p_{r} \to E^{p+r,q-r+1}_{r}$ with $d^2_r = 0$.

It is often helpful to visualize the differential $d_r$ as follows:

<table>
<thead>
<tr>
<th>$E^p_{2}$</th>
<th>$E^{p+2,q-1}_{2}$</th>
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We then have the following basic facts:

**Theorem 5.** With notation as above:

(i) $E^p_{0} = \text{gr}^p_F C^{p+q}$ and $E^p_{1} = H^{p+q}(\text{gr}^p_F C)$.

(ii) The differential $d_1$ is the connecting homomorphism coming from the exact sequence of complexes

$$0 \to \text{gr}^{p+1}_F C \to F^p C / F^{p+2} C \to \text{gr}^p_F C \to 0.$$

(iii) For fixed $p + q$ and $r \gg 0$,

$$E^p_{r} = \text{gr}^p_F H^{p+q}(C^r),$$

and we set

$$E^p_{\infty} = \text{gr}^p_F H^{p+q}(C) = E^p_{r}$$

for all $r \gg 0$ (depending possibly on $p + q$). We say that the spectral sequence converges to $\text{gr}^p_F H^{p+q}(C)$ or abuts to $H^{p+q}(C)$, and write this as

$$E^p_{r} \implies H^{p+q}(C).$$

(iv) The cohomology

$$H(E^p_{r}, d_r) = \ker\{d_r: E^p_{r} \to E^{p+r,q-r+1}_{r}\}/\text{Im}\{d_r: E^{p-r,q+r-1}_{r} \to E^p_{r}\}$$

is isomorphic to $E^p_{r+1}$. 

(v) The construction is functorial: given a map of filtered complexes

\[ f : (C', d, F C') \to (D', d, F D'), \]

for every \( r \geq 0 \) there is an induced map

\[ f : E^{p,q}_r(C') \to E^{p,q}_r(D') \]

commuting with \( d_r \). In particular, if there exists an \( r \) such that

\[ f : E^{p,q}_r(C') \to E^{p,q}_r(D') \]

is an isomorphism, then \( f : E^{p,q}_s(C') \to E^{p,q}_s(D') \) is an isomorphism for all \( s \geq r \). Hence, \( f : \text{gr}_F^p H^{p+q}(C') \to \text{gr}_F^p H^{p+q}(D') \) and \( f : H^k(C') \to H^k(D') \) are isomorphisms.

\[ \square \]

Definition 6. Given \( r > 0 \), if, for all \( s \geq r \), the differentials \( d_s \) are all 0, then we say that the spectral sequence degenerates at \( E_r \). In this case, for all \( s \geq r \) \( E^{p,q}_r \cong E^{p,q}_s \cong E^{p,q}_\infty \).

Along the lines of Theorem 2, we have the following:

Proposition 7. The spectral sequence degenerates at \( E_1 \iff \) the differential \( d \) is strict with respect to the filtration.

Proof. It follows easily from the definitions that, generalizing the case \( r = 1 \) from Theorem 2, there is a map

\[ H^{p+q}(F^pC')/\text{Im}\ H^{p+q}(F^{p+1}C') \to E^{p,q}_r, \]

whose image is contained in \( \text{Ker} \ d_r \), such that following diagram is commutative:

\[
\begin{array}{ccc}
H^{p+q}(F^pC')/\text{Im}\ H^{p+q}(F^{p+1}C') & \to & \text{Ker} d_r \\
\| & & \downarrow \\
H^{p+q}(F^pC')/\text{Im}\ H^{p+q}(F^{p+1}C') & \to & E^{p,q}_{r+1}.
\end{array}
\]

First suppose that \( d \) is strict. Then \( H^{p+q}(F^pC')/\text{Im}\ H^{p+q}(F^{p+1}C') \to E^{p,q}_1 \) is surjective, but its image is contained in \( \text{Ker} d_1 \). Thus \( d_1 = 0 \), \( E^{p,q}_1 \cong E^{p,q}_1 \), and so \( H^{p+q}(F^pC')/\text{Im}\ H^{p+q}(F^{p+1}C') \to E^{p,q}_2 \) is surjective as well. Hence \( d_2 = 0 \), and by induction \( d_r = 0 \) for all \( r \geq 1 \).
Conversely suppose that the spectral sequence degenerates at $E_1$. We have a commutative diagram

$$
\begin{array}{ccc}
H^{p+q}(F^pC) / \text{Im } H^{p+q}(F^{p+1}C) & \xrightarrow{\phi_1} & E_1^{p,q} \\
\phi_2 \downarrow & & \downarrow \cong \\
E_\infty^{p,q} & \xrightarrow{\phi_2} & E_\infty^{p,q}.
\end{array}
$$

Since $\phi_1$ is injective and $\phi_2$ is surjective, it follows that both are isomorphisms and hence that $d$ is strict.

**Remark 8.** The spectral sequence does not compute $H^k(C)$ but only the associated graded $\bigoplus_{p+q=k} E_\infty^{p,q}$. Note however that if $p$ is the largest integer such that $E_\infty^{p,q} \neq 0$ for $p + q = k$, then there is an inclusion $E_\infty^{p,q} \subseteq H^k(C)$. Likewise, if $p$ is the smallest integer such that $E_\infty^{p,q} \neq 0$ for $p + q = k$, then there is a surjection $H^k(C) \rightarrow E_\infty^{p,q}$. Hence, if $E_\infty^{p,q} \neq 0$ for exactly one integer $p$ such that $p + q = k$, then $H^k(C) \cong E_\infty^{p,q}$.

### 4.2 The spectral sequence of a double complex

**Definition 9.** A double complex $(A^\cdot, d', d'')$ consists of a bigraded collection $A^{p,q}$ of $R$-modules, together with two homomorphisms $d': A^{p,q} \rightarrow A^{p+1,q}$ and $d'': A^{p,q} \rightarrow A^{p,q+1}$, such that $(d')^2 = (d'')^2 = d'd'' + d''d' = 0$. The associated single complex $(s(A^\cdot, d))$ is the complex $(C^\cdot, d)$, where

$$
C^k = \bigoplus_{p+q=k} A^{p,q}
$$

and $d = d' + d''$; note that $d^2 = (d')^2 + (d'')^2 + d'd'' + d''d' = 0$. (Sometimes we assume that $d'$ and $d''$ commute instead of anti-commuting, and then we have to define $d$ by the rule $d = d' + (-1)^p d''$ to ensure that $d^2 = 0$.) We will also assume that, for every integer $k$, there are only finitely many $(p, q)$ with $p + q = k$ and $A^{p,q} \neq 0$.

The standard example is: $M$ is a complex manifold, $A^{p,q} = A^{p,q}(M)$, $d' = \partial$, and $d'' = \bar{\partial}$. In this case, $(s(A^\cdot, d)) = (A^\cdot, d)$ is the usual de Rham complex $A(M)$.

Back to a general double complex, we have two finite filtrations defined on $s(A^\cdot)$:

$$
F_p s(A^\cdot) = \bigoplus_{r \geq p} A^r; \quad F_q s(A^\cdot) = \bigoplus_{s \geq q} A^{s^\cdot}.
$$
and both of the corresponding spectral sequences converge to $H^k(C, d)$. Focusing for example on the first filtration, we have

$$\text{gr}_F^p s(A^\cdot) = A^{p^\cdot}.$$ 

In other words, $\text{gr}_F^p s(A^\cdot)^k = A^{p^\cdot-k}$. Thus

$$E_0^{p,q} = \text{gr}_F^p s(A^\cdot)^{p+q} = A^{p,q}$$

and $d$ induces $d'' : A^{p,q} \to A^{p,q+1}$. Hence,

$$E_1^{p,q} = H_d^q(A^{p^\cdot}),$$

with the understanding that $H_d^q(A^{p^\cdot})$ denotes the cohomology of the complex $(A^{p^\cdot}, d'')$. Since $d'$ and $d''$ anti-commute, there is an induced map

$$d' : H_d^q(A^{p^\cdot}) \to H_d^q(A^{p^\cdot+1}).$$

**Lemma 10.** The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is equal to $d'$. 

**Proof.** By (ii) of Theorem 5, we have to compute the connecting homomorphism induced by the exact sequence

$$0 \to A^{p^\cdot+1} \to A^{p^\cdot+1} \oplus A^{p^\cdot} \to A^{p^\cdot} \to 0.$$ 

The recipe is as follow: given a class $\xi = [\eta] \in H_d^q(A^{p^\cdot})$, it lifts to a $d''$-closed form $\eta \in A^{p,q}$, and then $d_1[\eta]$ is the class of $d\eta = d'\eta \in A^{p+1,q}$. Thus $d_1[\eta] = [d'\eta] \in H_d^q(A^{p+1,q})$. 

**Corollary 11.** With notation as above, the $E_2^{p,q}$ term from the filtration $\text{gr}_F^p s(A^\cdot)$ on $s(A^\cdot)$ is

$$E_2^{p,q} = H_d^p H_d^q(A^{p^\cdot}).$$

Likewise, the spectral sequence arising from the filtration $\text{gr}_F^p s(A^\cdot)$ on $s(A^\cdot)$ has

$$E_2^{p,q} = H_d^p H_d^q(A^{p^\cdot}).$$

**Definition 12.** Let $(A^\cdot, d', d'')$ be a double complex. Then the $d'd''$-lemma holds for $A^\cdot$ if $\text{Ker}d' \cap \text{Ker}d'' \cap \text{Im}d = \text{Im}d'd''$ in $C^k = \bigoplus_{p+q=k} A^{p,q}$ for all $k$. In other words, for all $\alpha \in C^k$,

$$d'\alpha = d''\alpha = 0 \text{ and } \alpha = d\beta \text{ for some } \beta \in C^{k-1},$$

$$\iff \alpha = d'd''\gamma \text{ for some } \gamma \in C^{k-2}.$$ 

In the special case of $(A^\cdot(M), \partial, \bar{\partial})$ for a complex manifold $M$, we call the $d'd''$-lemma the $\partial\bar{\partial}$-lemma.
We could also define the condition that the weak $d'd''$-lemma holds for $A'$ by: Ker $d' \cap$ Ker $d'' \cap$ Im $d = \text{Im} d'd'$ in $A_{p,q}$ for all $p,q$. Note that, if the $d'd''$-lemma holds for $A'$, then clearly the weak $d'd''$-lemma holds for $A'$. However, the converse to this statement fails:

**Example 13.** Let $(A', d', d'')$ be the double complex defined by: $A^{0,0}, A^{1,0}, A^{0,1}$ are free $R$-modules with generators $x, y^{1,0}$, and $y^{0,1}$ respectively, $d'x = y^{1,0}, d''x = y^{0,1}$, and otherwise $d'$ and $d''$ are 0. Then vacuously $(A', d', d'')$ satisfies the weak $d'd''$-lemma but it does not satisfy the $d'd''$-lemma. In this example, the two spectral sequences corresponding to the filtrations $'F'$ and $''F''$ on $s(A')$ both degenerate at $E_1$ and the corresponding filtrations on $H^1(s(A'), d)$ are both equal to the trivial filtration on $H^1(s(A'), d)$, i.e. $gr^0 H^1 = H^1$ and $gr^1 H^1 = 0$.

For future reference, we will also need the following result concerning the $d'd''$-lemma:

**Lemma 14.** Let $(A', d', d'')$ be a double complex. Then the following are equivalent:

(i) The $d'd''$-lemma holds for $A'$.

(ii) For all $p,q$, in $A_{p,q}$ we have the following two equalities:

\[
\begin{align*}
\text{Ker } d' \cap \text{Im } d'' &= \text{Im } d'd'; \\
\text{Ker } d'' \cap \text{Im } d' &= \text{Im } d'd''.
\end{align*}
\]

**Proof.** (i) $\implies$ (ii): We shall just check the first equality as the proof of the second is similar. Let $\alpha \in A_{p,q}$. Clearly, if $\alpha = d'd''\gamma$, then $d'\alpha = 0$ and $\alpha = -d''(d'\gamma) \in \text{Ker } d' \cap \text{Im } d''$. Conversely suppose that $d'\alpha = 0$ and $\alpha = d''\beta$, where $\beta \in A_{p,q-1}$. Then

\[
d\beta = d'\beta + d''\beta = d'\beta + \alpha,
\]

and thus $d'd\beta = d''d\beta = 0$. By the assumption that the $d'd''$-lemma holds for $A'$, $d\beta = d'd''\gamma$, and then $\alpha = d'd''\gamma_{p-1,q-1}$, where $\gamma_{p-1,q-1}$ is the component of $\gamma$ in $A_{p-1,q-1}$. Hence $\alpha \in \text{Im } d'd''$, and the two sides are equal.

(ii) $\implies$ (i): First note that, if (ii) holds for all $\alpha \in A_{p,q}$ for all $p,q$, then it holds for all $\alpha \in C^K = \bigoplus_{p+q=K} A_{p,q}$ as well, because $\alpha = \sum_{p+q=K} \alpha_{p,q} \in \text{Ker } d' \cap \text{Im } d''$ $\iff$ $\alpha_{p,q} \in \text{Ker } d' \cap \text{Im } d''$ for all $p,q$, and similarly for
Ker $d'' \cap \text{Im } d'$. Suppose that $\eta \in \bigoplus_{p+q=k} A^{p,q}$ and that $d' \eta = d'' \eta = 0$, $\eta = d \beta = d' \beta + d'' \beta$. Then

$$d'' d' \beta = d'' (d' \beta + d'' \beta) = d'' \eta = 0,$$

hence $d' \beta \in \text{Ker } d'' \cap \text{Im } d'$ so that $d' \beta = d' d'' \gamma_1$ for some $\gamma_1 \in C^{k-2}$. Likewise $d'' \beta = d' d'' \gamma_2$, hence $\eta = d' \beta + d'' \beta = d' d'' (\gamma_1 + \gamma_2) \in \text{Im } d' d''$.

\section*{4.3 Applications to complex manifolds}

Let $M$ be a complex manifold (not necessarily compact or Kähler). We have the double complex $(A^\cdot \cdot(M), \partial, \bar{\partial})$, with associated single complex $(A^\cdot \cdot(M), d)$.

The two corresponding filtrations will be denoted by $F^p A^k(M)$ and $\overline{F^p A^k(M)}$; thus

$$F^p A^k(M) = \bigoplus_{r \geq p} A^{r,k-r}(M);$$

$$\overline{F^p A^k(M)} = \bigoplus_{r \geq k-p} A^{k-r,r}(M) = \bigoplus_{r \leq p} A^{r,k-r}(M).$$

It follows that

$$F^p A^k(M) \cap \overline{F^{k-p} A^k(M)} = A^{p,k-p}(M);$$

$$F^p A^k(M) \oplus \overline{F^{k-p+1} A^k(M)} \cong A^k(M).$$

Also note that complex conjugation exchanges $F^p A^k(M)$ and $\overline{F^p A^k(M)}$, and replaces $\bar{\partial}$ with $\partial$; finally, $d$ is real and so commutes with complex conjugation. It follows easily that the spectral sequence arising from $F^p A^k(M)$ degenerates at $E_r \iff$ the spectral sequence arising from $\overline{F^p A^k(M)}$ degenerates at $E_r$. Moreover, there are two filtrations $F^p H^k(M)$ and $\overline{F^p H^k(M)}$ on $H^k(M)$. For example,

$$F^p H^k(M) = \{ \xi \in H^k(M) : \text{ there exists } \varphi \in F^p A^k(M) \text{ with } d \varphi = 0 \text{ and } [\varphi] = \xi \}.$$

Since $d$ is real, it is easy to check that, via the real structure on $H^k(M)$,

$$\overline{F^p H^k(M)} = \overline{F^p H^k(M)}.$$

The results about double complexes show:

\textbf{Theorem 15.} There exists a spectral sequence with $E_1$ term $E_1^{p,q} = H_\partial^{p,q}(M)$ converging to $H^k(M)$, and the differential $d_1$ on $E_1$ is that induced by $\partial$. \hfill \Box
The above spectral sequence has many names: the Hodge to de Rham or Fröhlicher spectral sequence. For brevity, we will just refer to it as the Hodge spectral sequence.

**Corollary 16.** Suppose that $M$ is compact, so that the Hodge numbers $h^{p,q}(M) = \dim H^{p,q}_\partial(M)$ and the Betti numbers $b_k(M) = \dim H^k(M)$ are defined. Then

$$\sum_{p+q=k} h^{p,q}(M) \geq b_k(M).$$

Moreover, equality holds for all $k$ $\iff$ the Hodge spectral sequence degenerates at $E_1$ $\iff$ the de Rham differential $d$ is strict with respect to the filtration $F^pA(M)$.

Clearly, the de Rham differential $d$ is strict with respect to the filtration $F^pA(M)$ $\iff$ it is strict with respect to the filtration $F^pA(M)$.

Another application of the above is to Stein manifolds. A complex manifold is a **Stein manifold** if, for every coherent analytic sheaf $F$ on $M$, the higher sheaf cohomology groups $H^q(M; F) = 0$, $q > 0$. One can show that $M$ is a Stein manifold $\iff$ $M$ is biholomorphic to a closed complex submanifold of $\mathbb{C}^N$ for some $N$. In particular, every smooth affine algebraic variety is a Stein manifold. The vanishing of higher coherent sheaf cohomology implies (as we shall see) that, for all $q > 0$,

$$H^{p,q}_\partial(M) = 0.$$

Thus, an examination of the Hodge spectral sequence shows:

**Theorem 17.** Let $M$ be a Stein manifold. Then the Hodge spectral sequence for $M$ degenerates at $E_2$, and in fact

$$H^k(M) \cong H^k(H^0(M; \Omega), d).$$

In particular, $H^k(M) = 0$ for $k > \dim M$.

Here, the last statement of the theorem is a generalization of a slightly weaker form of the Lefschetz theorem on hyperplane sections to the case of Stein manifolds.

Applying Corollary 16 to the case of a Kähler manifold, we get:

**Theorem 18.** Let $M$ be a compact Kähler manifold. Then the Hodge spectral sequence for $M$ degenerates at $E_1$.  \[\Box\]
Remark 19. There exist compact complex manifolds $M$ for which the Hodge spectral sequence for $M$ does not degenerate at $E_1$ (or in fact at $E_r$ for any given value of $r$). Standard examples are Iwasawa manifolds; these are complex nilmanifolds, quotients of $\mathbb{C}^n$ by nilpotent arithmetic groups. Moreover, even if the Hodge spectral sequence for $M$ does degenerate at $E_1$, it is not necessarily the case that the Hodge symmetries hold (i.e. that $h^{p,q}(M) = h^{q,p}(M)$). For example, if $S$ is a compact complex surface, then one can show that the Hodge spectral sequence for $S$ degenerates at $E_1$. But if $S$ is a non-Kähler complex surface such as a Hopf surface, then $b_1(S)$ is odd and so the Hodge symmetries cannot hold.

Given a general double complex, we would like to compare the two filtrations \( \Gamma^* \) and \( \Gamma^\prime \) induced on the cohomology \( H^k(s(A^\cdot), d) \), especially in the case where the corresponding two spectral sequences degenerate at \( E_1 \).

Definition 20. Let $V$ be an $R$-module and let $\Gamma^*$ and $\Gamma^\prime$ be two filtrations on $V$. Then $\Gamma^*$ and $\Gamma^\prime$ are $k$-opposed if, for every $p$,

\[
\Gamma^p \oplus \Gamma^{k-p+1} \cong V.
\]

Thus, for example, for a complex manifold $M$, $F^k(M)$ and $\overline{F}^k(M)$ are $k$-opposed for every $k$, and similarly for the filtrations $\Gamma^*(s(A^\cdot))$ and $\Gamma^\prime(s(A^\cdot))$ for an arbitrary double complex $A^\cdot$. In general, if $\Gamma^*$ and $\Gamma^\prime$ are two $k$-opposed filtrations on an $R$-module $V$, set

\[
V^p,k-p = \Gamma^p \cap \Gamma^{k-p}.
\]

The arguments we gave for Hodge filtrations then show:

Lemma 21. If $\Gamma^*$ and $\Gamma^\prime$ are two finite $k$-opposed filtrations, then

\[
V \cong \bigoplus_{p+q=k} V^{p,q}.
\]

Moreover, $\Gamma^p = \bigoplus_{r \geq p} V^{r,k-r}$ and $\Gamma^\prime_p = \bigoplus_{r \leq p} V^{r,k-r}$. \qed

For compact Kähler manifolds, we have seen:

Theorem 22. If $M$ is a compact Kähler manifold, then the filtrations $F^k(M)$ and $\overline{F}^k(M)$ are $k$-opposed for every $k$. \qed

We now analyze the relation between the Hodge decomposition and the $\partial\overline{\partial}$-lemma.
Theorem 23. Let $M$ be a compact complex manifold such that the Hodge spectral sequence degenerates at $E_1$ and the resulting filtrations $F^i H^k(M)$ and $\overline{F} H^k(M)$ are $k$-opposed for every $k$. For all $p, q$, $p + q = k$, define

$$H^{p,q}(M) = F^p H^k(M) \oplus \overline{F}^q H^k(M),$$

so that $H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M)$ and $H^q(M) = \overline{H}^{p,q}(M)$. Then the following hold:

(i) $H^{p,q}(M)$ is the subspace of all $\xi \in H^k(M)$ such that there exists an $\eta \in A^{p,q}(M)$ with $d\eta = 0$ (or equivalently $\partial \eta = \bar{\partial} \eta = 0$) and $\xi = [\eta]$.

(ii) Every $\xi \in H^k(M)$ has a representative $\varphi \in A^k(M)$ such that $\partial \varphi = \bar{\partial} \varphi = 0$.

(iii) (\partial\bar{\partial}-lemma) If $\alpha \in A^k(M)$, then $\alpha$ is $\partial$- and $\bar{\partial}$-closed and $d$-exact $\iff$ there exists a $\beta \in A^{k-2}(M)$ such that $\alpha = \partial \bar{\partial} \beta$. In other words, $\text{Ker} \partial \cap \text{Ker} \bar{\partial} \cap \text{Im} d = \text{Im} \partial \bar{\partial}$.

Proof. (i) Clearly, if $\xi = [\eta]$ with $\eta \in A^{p,q}(M)$ and $d\eta = 0$, then $\xi \in F^p H^k(M) \oplus \overline{F}^q H^k(M) = H^{p,q}(M)$. Conversely, suppose that $\xi \in H^{p,q}(M)$. Then $\xi = [\alpha_1]$, where $\alpha_1 \in F^p A^{p+q}(M)$ and $d\alpha_1 = 0$. Likewise $\xi = [\alpha_2]$, where $\alpha_2 \in \overline{F}^q A^k(M)$ and $d\alpha_2 = 0$. Thus $\alpha_1 - \alpha_2 = d\beta$ for some $\beta \in A^{p+q-1}(M)$. Write $\beta = \beta_1 + \beta_2$, where $\beta_1 \in F^p A^{p+q-1}(M)$ and $\beta_2 \in \overline{F}^q A^{p+q-1}(M)$. Then

$$\alpha_1 - d\beta_1 = \alpha_2 + d\beta_2 \in F^p A^{p+q}(M) \cap \overline{F}^q A^{p+q}(M) = A^{p,q}(M),$$

with $d(\alpha_1 - d\beta_1) = 0$, and

$$[\alpha_1 - d\beta_1] = [\alpha_1] = \xi.$$

(ii) If $\xi \in H^k(M)$, then, since the filtrations $F^i H^k(M)$ and $\overline{F} H^k(M)$ are $k$-opposed, we can write $\xi = \sum_{p+q=k} \xi^{p,q}$ where $\xi^{p,q} \in H^{p,q}(M)$. By (i), there exists a representative $\varphi^{p,q} \in A^{p,q}(M)$ such that $\varphi^{p,q}$ is $\partial$, $\bar{\partial}$-, and $d$-closed. Hence $\sum_{p+q=k} \varphi^{p,q}$ is a representative for $\xi$ which is $\partial$- and $\bar{\partial}$-closed.

(iii) Clearly, if $\alpha = \partial \bar{\partial} \beta$, then $\alpha$ is $\partial$- and $\bar{\partial}$-closed and $\alpha = d\bar{\partial} \beta$ is $d$-exact.

Conversely, suppose that $\alpha$ is $\partial$- and $\bar{\partial}$-closed and $d$-exact. First, we claim that this implies that all of the $(p, q)$-components $\alpha^{p,q}$ of $\alpha$ satisfy: $\alpha^{p,q}$ is $\partial$- and $\bar{\partial}$-closed and $d$-exact. To see this, note that $\alpha^{p,q}$ is $\partial$- and $\bar{\partial}$-closed and hence $d$-closed. To see that it is $d$-exact, consider the cohomology class $[\alpha^{p,q}] \in H^{p,q}(M)$. Then $[\alpha] = 0 = \sum_{p+q=k} [\alpha^{p,q}]$ is the decomposition
of \([\alpha]\) into its components in the direct sum \(H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M)\). Since this is a direct sum, \([\alpha] = 0 \iff [\alpha^{p,q}] = 0\) for all \(p + q = k\). Hence the class \([\alpha^{p,q}] = 0\), i.e. \(\alpha^{p,q}\) is \(d\)-exact. Also, note that, if \(\alpha^{p,q} \in \text{Im} \partial \bar{\partial}\), then \(\alpha \in \text{Im} \partial \bar{\partial}\).

Thus we may assume that \(\alpha = \alpha^{p,q} \in A^{p,q}(M)\). Then \(\alpha \in F^p A^k(M) \cap dA^{k-1}(M)\). Since the Hodge spectral sequence degenerates at \(E_1\), \(d\) is strict with respect to \(F^\cdot A^\cdot\), and hence there exists a \(\beta_1 \in F^p A^{k-1}(M)\) such that \(\alpha = d\beta_1\). Likewise, \(d\) is strict with respect to \(\bar{F} A^\cdot\), and hence there exists a \(\beta_2 \in \bar{F}^q A^{k-1}(M)\) such that \(\alpha = d\beta_2\). Then \(d(\beta_1 - \beta_2) = 0\). By (ii), there exists a \(\gamma\) which is \(\partial\)- and \(\bar{\partial}\)-closed such that

\[\beta_1 - \beta_2 = \gamma + d\varphi.\]

Write \(\varphi = \sum_{r+s=k-2} \varphi^{r,s}\). By comparing types, if we set \(\varphi_1 = \sum_{r \geq p} \varphi^{r,s}\), then

\[\beta_1 - \gamma_1 = d\varphi_1 + \partial \varphi^{p-1,k-p-1}.\]

Then

\[\alpha = d\beta_1 = d(\beta_1 - \gamma_1) = d(d\varphi_1 + \partial \varphi^{p-1,k-p-1}) = d\partial \varphi^{p-1,k-p-1} = \partial \bar{\partial}(-\varphi^{p-1,k-p-1})\]

as claimed. \(\square\)

There is a converse to (iii) of Theorem 23:

**Theorem 24.** Let \(M\) be a compact complex manifold such that the \(\partial \bar{\partial}\)-lemma holds for \(M\). Then

(i) \(d\) is strict with respect to the filtration \(F^\cdot A^\cdot\), i.e. the Hodge spectral sequence degenerates at \(E_1\).

(ii) The induced filtrations \(F^\cdot H^k(M)\) and \(\bar{F}^\cdot H^k(M)\) are \(k\)-opposed for every \(k\).
Proof. (i) Strictness: Suppose that \( \eta \in F^p A^k(M) \) with \( \eta = d\varphi \). Write \( \varphi = \varphi_1 + \varphi_2 \), with \( \varphi_1 \in F^p A^{k-1}(M) \) and \( \varphi_2 \in \overline{F}^{k-p} A^{k-1}(M) \). Then, by type considerations,

\[
\eta - d\varphi_1 = d(\varphi - \varphi_1) = d\varphi_2 \in F^p A^k(M) \cap \overline{F}^{k-p} A^k(M) = A^{p,q}(M).
\]

Moreover, \( \eta - d\varphi_1 \) is \( d \)-exact. Thus it is \( d \)-closed, and so \( \partial \)- and \( \bar{\partial} \)-closed since it is of type \((p,q)\). Hence, by assumption, there exists a \((k-2)\)-form \( \gamma \) of type \((p-1,q-1)\) such that

\[
\eta - d\varphi_1 = \partial \bar{\partial} \gamma.
\]

Then

\[
\eta = d\varphi_1 + \partial \bar{\partial} \gamma = d(\varphi_1 - \partial \gamma),
\]

with \( \varphi_1 - \partial \gamma \in F^p A^{k-1}(M) \). Thus \( d \) is strict with respect to the filtration \( F^*A(M) \). (Note that this part only used the weak \( \partial \bar{\partial} \)-lemma.)

(ii) First we claim that \( F^p H^k(M) \cap \overline{F}^{k-p+1} H^k(M) = 0 \). In fact, if \( \xi \in F^p H^k(M) \cap \overline{F}^{k-p+1} H^k(M) \), then \( \xi = [\alpha] \), where \( \alpha \in F^p A^k(M) \), \( d\alpha = 0 \), and also \( \xi = [\beta] \), where \( \beta \in \overline{F}^{k-p+1} A^k(M) \), \( d\beta = 0 \). Thus

\[
\alpha - \beta = d\varphi = d(\varphi_1 + \varphi_2),
\]

where \( \varphi_1 \in F^p A^{k-1}(M) \) and \( \varphi_2 \in \overline{F}^{k-p} A^{k-1}(M) \). By considerations of type,

\[
\alpha - d\varphi_1 = \partial \varphi^{p-1,k-p} \in A^{p,q}.
\]

Since \( \alpha - d\varphi_1 \) is \( d \)-closed and of pure type \((p,q)\), it is \( \bar{\partial} \)-closed and \( \partial \)-exact. Hence, by one of the reformulations of the \( \partial \bar{\partial} \)-lemma (Lemma 14), there exists a \( \gamma \in A^{p-1,q-1} \) such that

\[
\alpha - d\varphi_1 = \partial \bar{\partial} \gamma.
\]

But then \( \alpha = d(\varphi_1 + \bar{\partial} \gamma) \), so that \( \xi = [\alpha] = 0 \).

Finally, we must show that the image of \( F^p H^k(M) \oplus \overline{F}^{k-p+1} H^k(M) \) is \( H^k(M) \). This follows more generally from:

**Proposition 25.** Suppose that \( M \) is a compact complex manifold for which the Hodge spectral sequence degenerates at \( E_1 \) and such that, for all \( k,p \),

\[
F^p H^k(M) \cap \overline{F}^{k-p+1} H^k(M) = 0.
\]

Then \( F^* H^k(M) \) and \( \overline{F}^* H^k(M) \) are \( k \)-opposed for every \( k \).
Proof. By assumption, the induced homomorphism

$$F^p H^k(M) \oplus \overline{F}^{k-p+1} \to H^k(M)$$

is injective. Thus, it suffices to prove that, for all $k$ and $p$,

$$\dim F^p H^k(M) + \dim \overline{F}^{k-p+1} = \dim H^k(M) = b_k(M).$$

First, by injectivity, we clearly have

$$\dim F^p H^k(M) + \dim \overline{F}^{k-p+1} \leq \dim H^k(M) = b_k(M),$$

with equality $\iff$ the map $F^p H^k(M) \oplus \overline{F}^{k-p+1} H^k(M) \to H^k(M)$ is an isomorphism. Next, by the degeneration of the Hodge spectral sequence,

$$\sum_{p+q=k} h^{p,q}(M) = b_k(M).$$

Also, $\dim F^p H^k(M) = \sum_{r \geq p} h^{r,k-r}$. Since $\overline{F} H^k(M)$ is the complex conjugate filtration of $F^p H^k(M)$,

$$\dim \overline{F}^{k-p+1} = \dim F^{k-p+1} = \sum_{r \geq k-p+1} h^{r,k-r}.$$

Combining these, we see that

$$\sum_{r \geq k-p+1} h^{r,k-r}(M) \leq \sum_{r \leq p-1} h^{r,k-r}(M),$$

with equality $\iff$ the map $F^p H^k(M) \oplus \overline{F}^{k-p+1} H^k(M) \to H^k(M)$ is an isomorphism.

Now apply the above to $H^{2n-k}(M)$, replacing $k$ by $2n - k$ and $p$ by $n + p - k$. We obtain:

$$\sum_{a \geq n-p+1} h^{a,2n-k-a}(M) \leq \sum_{a \leq n-k+p-1} h^{a,2n-k-a}(M).$$

Write $a = n - r$, so that $a \geq n - p + 1 \iff r \leq p - 1$, $a \leq n - k + p - 1 \iff r \geq k - p + 1$, and $h^{a,2n-k-a}(M) = h^{n-r,n-(k-r)}(M)$. By Kodaira-Serre duality, $h^{n-r,n-(k-r)}(M) = h^{r,k-r}(M)$ for all $r$. Thus we obtain

$$\sum_{r \leq p-1} h^{r,k-r}(M) \leq \sum_{r \geq k-p+1} h^{r,k-r}(M).$$

15
In particular, equality must hold, and combining the above we see that

$$F^p H^k(M) \oplus F^{k-p+1} H^k(M) \cong H^k(M)$$

as claimed. \qed

**Remark 26.** The results of Theorem 23 and Theorem 24 can be generalized to double complexes as follows (Deligne-Griffiths-Morgan-Sullivan):

**Theorem 27.** Let $$(A^\cdot, d', d'')$$ be a double complex.

(i) Suppose that the two spectral sequences corresponding to the filtrations $'$ and $''$ both degenerate at $E_1$, and the corresponding filtrations on $H^k(s(A^\cdot), d)$ are $k$-opposed for every $k$. Then the $d'd''$-lemma holds for $A^\cdot$.

(ii) Conversely, suppose that the $d'd''$-lemma holds for $A^\cdot$. Then the two spectral sequences corresponding to the filtrations $'$ and $''$ both degenerate at $E_1$, and the corresponding filtrations on $H^k(s(A^\cdot), d)$ satisfy:

$$'F^p H^k(s(A^\cdot), d) \cap ''F^{k-p+1} H^k(s(A^\cdot), d) = 0.$$

If in addition the double complex $$(A^\cdot, d', d'')$$ consists of vector spaces over a field (i.e. $k$ is a field in the notation of the previous section), then

$$'F^p H^k(s(A^\cdot), d) \oplus ''F^{k-p+1} H^k(s(A^\cdot), d) \to H^k(s(A^\cdot), d)$$

is an isomorphism for all $p, k$, i.e. the two filtrations are $k$-opposed for every $k$. \qed

Here, the methods of proof of Theorem 23 and Theorem 24 apply essentially unchanged to prove all but the last statement, which can be proved directly. We used the approach of Proposition 25 because we will apply it in the proof of Theorem 28 below.

As an application, we extend the existence of Hodge decompositions to a broader class of compact complex manifolds.

**Theorem 28.** Let $M$ and $N$ be two compact complex manifolds and $f : M \to N$ a holomorphic, surjective map. Suppose that at least one of the following holds:
(i) \( M \) is Kähler (and in particular the \( \partial \bar{\partial} \)-lemma holds for \( M \)).

(ii) \( \dim M = \dim N \) and the \( \partial \bar{\partial} \)-lemma holds for \( M \).

Then the \( \partial \bar{\partial} \)-lemma holds for \( N \).

**Proof.** First, we claim that, for all \( k, p, q \), \( f^* : H^k(N) \to H^k(M) \) and \( f^* : H^{p,q}_\partial(N) \to H^{p,q}_\partial(M) \) are injective. We begin with the case \( k = 2n, p = q = n \), where \( n = \dim N \). First consider the case \( \dim M = n \). Note that we have a commutative diagram

\[
\begin{array}{ccc}
H^{n,n}_\partial(N) & \xrightarrow{\cong} & H^{2n}(N) \\
\downarrow f^* & & \downarrow f^* \\
H^{n,n}_\partial(M) & \xrightarrow{\cong} & H^{2n}(M).
\end{array}
\]

In this case, \( f^* : H^{2n}(N) \to H^{2n}(M) \) is multiplication by the integer \( \deg f \), which is computed by looking at a general fiber \( f^{-1}(x) = \{y_1, \ldots, y_d\} \) and adding up the local degrees \( \pm 1 \) at the points \( y_i \). Here the local degrees correspond to the sign of the (real) determinant of the Jacobian of \( f \) measured against the local orientations of the tangent or cotangent spaces at \( x \) and the \( y_i \). However, since the Jacobian of \( f \) is complex linear, its real determinant is always positive, and so the local degrees are all \( +1 \). Thus \( f^* : H^{2n}(N) \to H^{2n}(M) \) is multiplication by a positive integer \( d \), and hence it is injective.

In case \( M \) is Kähler, let \( m = \dim M \geq n \). If \( x \in M \) is a regular value for \( f \), then there exist local holomorphic coordinates \( z_1, \ldots, z_n, z_{n+1}, \ldots, z_m \) at \( x \) such that \( z_1, \ldots, z_n \) are the pullbacks of local holomorphic coordinates at \( f(x) \). We can further assume after a linear change of coordinates that, if \( L \) is the Kähler form for some Kähler metric on \( M \), then its value at \( x \) is given by \( L_x = \frac{\sqrt{-1}}{2} \sum (dz_i \wedge d\bar{z}_i)_x \). Let \( \omega \) be the volume form (for some Hermitian metric) on \( M \). Thus, the class of \( \omega \) is a generator for \( H^{n,n}_\partial(N) \) and \( H^{2n}(N) \).

Locally around \( x \), there exists a positive valued \( C^\infty \) function \( g \) such that

\[
f^* \omega = \left( \frac{\sqrt{-1}}{2} \right)^n g (dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n).
\]

Then a computation shows that

\[
(f^* \omega \wedge L^{m-n})_x = \left( \frac{\sqrt{-1}}{2} \right)^m (m-n)! g (dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m)_x,
\]

17
and in particular its value at \( x \) is a positive multiple of the volume form of \( M \) (with respect to the Kähler metric or any other Hermitian metric on \( M \)). Hence the class \([f^*\omega \wedge L^{n-m}]\) is nonzero in \( H^{m,m}_\partial(M) \) and \( H^{2m}(M) \). It follows that \( f^*\omega \neq 0 \) in \( H^{n,n}_\partial(M) \) and \( H^{2n}(M) \), so that \( f^* : H^{2n}(N) \to H^{2n}(M) \) and \( f^* : H^{n,n}_\partial(N) \to H^{n,n}_\partial(M) \) are injective.

To prove that \( f^* : H^k(N) \to H^k(M) \) and \( f^* : H^{p,q}_\partial(N) \to H^{p,q}_\partial(M) \) are injective in general, it is clearly enough to consider the cases \( k \leq 2n \) and \( p \leq n, q \leq n \). Suppose for example that \( \alpha \in H^k(N) \), with \( \alpha \neq 0 \). By Poincaré duality, there exists a \( \beta \in H^{2n-k}(N) \) such that \( \alpha \wedge \beta \) is a generator of \( H^{2n}(N) \). Then \( f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta \neq 0 \), by what we have already seen, and so \( f^*\alpha \neq 0 \). It follows that \( f^* : H^k(N) \to H^k(M) \) is injective for all \( k \leq 2n \), and hence for all \( k \). The case \( f^* : H^{p,q}_\partial(N) \to H^{p,q}_\partial(M) \) is similar, using Kodaira-Serre duality instead of Poincaré duality.

Next we claim that the Hodge spectral sequence degenerates for \( N \). We have a commutative diagram where the two horizontal maps are injective:

\[
\begin{array}{ccc}
H^{p,q}_\partial(N) & \xrightarrow{f^*} & H^{p,q}_\partial(M) \\
\downarrow d_1=\partial & & \downarrow d_1=\partial \\
H^{p+1,q}_\partial(N) & \xrightarrow{f^*} & H^{p+1,q}_\partial(M).
\end{array}
\]

In other words, \( f^* \) induces an injection on the \( E_1 \) terms of the Hodge spectral sequences for \( M \) and \( N \). Since \( d_1 : H^{p,q}_\partial(M) \to H^{p+1,q}_\partial(M) \) is 0, the same is true for \( d_1 : H^{p,q}_\partial(N) \to H^{p+1,q}_\partial(N) \). Let \( E^{p,q}_r(N) \) denote the corresponding term of the Hodge spectral sequence for \( N \), and similarly for \( M \). We show inductively that \( E^{p,q}_r(N) \cong E^{p,q}_1(N) \) and that \( d_r = 0 \) for all \( r \geq 1 \). The above establishes the case \( r = 1 \). Assume the result for \( r - 1 \). Then \( d_{r-1} = 0 \) for \( E^{p,q}_{r-1}(N) \), so that \( E^{p,q}_r(N) \cong E^{p,q}_{r-1}(N) \cong E^{p,q}_1(N) \) by the inductive hypothesis. Note that \( E^{p,q}_r(M) \cong E^{p,q}_1(M) \) and \( d_r : E^{p,q}_r(M) \to E^{p+1,q-r+1}_r(M) \) is zero for all \( r, p, q \). By functoriality we have a commutative diagram

\[
\begin{array}{ccc}
E^{p,q}_r(N) & \xrightarrow{f^*} & E^{p,q}_r(M) \\
\downarrow d_r & & \downarrow d_r \\
E^{p+r,q-r+1}_r(N) & \xrightarrow{f^*} & E^{p+r,q-r+1}_r(M),
\end{array}
\]

with the horizontal maps \( f^* \) injective since the same holds for the \( E_1 \) terms, and \( d_r : E^{p,q}_r(M) \to E^{p+1,q-r+1}_r(M) \) equal to 0. Thus the same is true for \( d_r : E^{p,q}_r(N) \to E^{p+1,q-r+1}_r(N) \), establishing the inductive step. Hence the Hodge spectral sequence degenerates for \( N \).
By definition, 
\[ F^p H^k(N) = \{ \xi \in H^k(N) : \xi = [\alpha] \text{ for some } \alpha \in F^p A^k(N) \text{ with } d\alpha = 0 \}. \]
Since \( f \) is holomorphic,
\[ f^* F^p H^k(N) \subseteq F^p H^k(M) \]
for all \( p, k \). Similarly,
\[ f^* F^{k-p+1} H^k(N) \subseteq F^{k-p+1} H^k(M). \]
Hence
\[ f^* (F^p H^k(N) \cap F^{k-p+1} H^k(N)) \subseteq F^p H^k(M) \cap F^{k-p+1} H^k(M) = 0. \]
Since \( f^* \) is injective, \( F^p H^k(N) \cap F^{k-p+1} H^k(N) = 0 \) as well. Thus the Hodge spectral sequence for \( N \) degenerates at \( E_1 \) and the corresponding filtrations on \( H^k(N) \) are \( k \)-opposed for every \( k \), so the \( \partial \bar{\partial} \)-lemma holds for \( N \).

**Definition 29.** Let \( M \) and \( N \) be two compact complex manifolds. Then \( M \) and \( N \) are *bimeromorphic* if there exists an irreducible subvariety \( Z \subseteq M \times N \) such that the projections \( \pi_1 : Z \to M \) and \( \pi_2 : Z \to N \) both have degree one. By Hironaka’s theorem, it is equivalent to assume that there exists an iterated blowup \( \tilde{M} \) of \( M \) along submanifolds and a holomorphic degree one map \( f : \tilde{M} \to N \). Here, by an iterated blowup, we mean that there is a sequence of compact complex manifolds and holomorphic maps
\[ X_r = \tilde{M} \xrightarrow{f_r} X_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_1} X_0 = M, \]
such that each \( f_i : X_i \to X_{i-1} \) is the blowup of \( X_{i-1} \) along a smooth submanifold. We say that \( N \) is a *Moishezon manifold* if it is bimeromorphic to a smooth projective variety and that \( N \) is of class \( \mathcal{C} \) if it is bimeromorphic to a compact Kähler manifold. In particular, a Moishezon manifold is of class \( \mathcal{C} \). Also, by Chow’s lemma, the complex manifold associated to a smooth scheme which is proper over \( \text{Spec } \mathbb{C} \) is a Moishezon manifold.

By a result of Blanchard, if \( M \) is a Kähler manifold and \( f : M' \to M \) is the blowup of \( M \) along a smooth submanifold, then \( M' \) is Kähler as well. Combining this with Theorem 28, we obtain:

**Corollary 30.** If \( N \) is a compact complex manifold of class \( \mathcal{C} \), then the \( \partial \bar{\partial} \)-lemma holds for \( N \).