MODERN ALGEBRA I FALL 2018:
SIXTH PROBLEM SET

1. Let $G$ be an abelian group (but with the group operation written multiplicatively) and let $n \in \mathbb{Z}$. Show that the subset
\[ \{ g \in G : g^n = 1 \} \]
is a subgroup of $G$ (called the $n$-torsion subgroup). (Notes: (1) the subgroup $\mu_n$ of $n$th roots of unity in $\mathbb{C}^*$ is a special case of this construction. (2) It is easy to give examples where this result fails if $G$ is not abelian; see the previous HW.)

2. Let $G$ be an abelian group. By the previous problem, for a fixed $n \in \mathbb{N}$, the set $\{ g \in G : g^n = 1 \}$ is a subgroup of $G$. Show that
\[ \{ g \in G : \text{there exists some } N \in \mathbb{N} \text{ such that } g^N = 1 \} \]
is a subgroup of $G$. (An element $g \in G$ such that there exists some $N \in \mathbb{N}$ such that $g^N = 1$, in other words an element of finite order, is called a torsion element of $G$, and the set of all elements of finite order in an abelian group $G$ is called the torsion subgroup of $G$.)

3. What is the torsion subgroup of $\mathbb{Z}/n\mathbb{Z}$? Of $\mathbb{Z}$? Of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$?

4. Define $\mu_\infty = \{ z \in \mathbb{C} : \text{there exists some } n \in \mathbb{N} \text{ such that } z^n = 1 \}$
\[ = \bigcup_{n \in \mathbb{N}} \mu_n. \]
Thus $\mu_\infty$ is the torsion subgroup of $\mathbb{C}^*$ and of $U(1)$. Is $\mu_\infty$ cyclic? Why or why not?

5. Let $G_1$ and $G_2$ be two groups (written multiplicatively) and let $f : G_1 \to G_2$ be an isomorphism.
   (i) Show that $f(1) = 1$ (here the first 1 is the identity in $G_1$ and the second 1 is the identity in $G_2$).
   (ii) Show that, for all $g \in G_1$, $f(g^{-1}) = (f(g))^{-1}$.
   (iii) Let $H$ be a subset of $G_1$. Show that, if $H$ is a subgroup of $G_1$, then $f(H)$ is a subgroup of $G_2$. By applying this result to the inverse isomorphism $f^{-1} : G_2 \to G_1$, show that $H$ is a subgroup of $G_1 \iff f(H)$ is a subgroup of $G_2$. 

6. Let \((G, \cdot)\) be a group. An isomorphism \(f: (G, \cdot) \rightarrow (G, \cdot)\) is called an automorphism of \(G\). (More generally, one can define an automorphism from a binary structure to itself.) Using freely Problem 7 from HW 3, show that the set of all automorphisms of \(G\) is itself a group under function composition, and hence is a subgroup of the group \(S_G\) of bijections from \(G\) to itself. This group is denoted \(\text{Aut} G\).

7. Let \(G\) be a cyclic group, say \(G = \langle g \rangle\), so that \(g\) is a generator of \(G\), and that \(f: G \rightarrow G\) is an isomorphism from \(G\) to itself. From Problem 6 of the midterm, we have seen that \(f(g)\) is also a generator of \(G\) (i.e. \(G = \langle f(g) \rangle\)). Using this, show that every isomorphism from \(\mathbb{Z}\) to itself is either the identity function \(\text{Id}\) or the function \(-\text{Id}\) which maps \(n \in \mathbb{Z}\) to \(-n\).