

Commutative Algebra II Spring 2008
Fourth Problem Set due February 25

1. Let R be a Dedekind domain and let I be a nonzero ideal in R . Show that every ideal in R/I is generated by one element. Conclude that every ideal of R can be generated by at most two elements.

2. (Chinese remainder theorem) Let R be a Dedekind domain and let I_1, \dots, I_n be proper ideals in R . Given $r_1, \dots, r_n \in R$, show that there exists an $r \in R$ such that $r \equiv r_i \pmod{I_i}$ for every i if and only if $r_i \equiv r_j \pmod{I_i + I_j}$ for all $i \neq j$. (This is the statement that

$$R \rightarrow \bigoplus_i R/I_i \rightarrow \bigoplus_{i < j} R/(I_i + I_j)$$

is exact, where the first map is the diagonal projection and the second maps $(\alpha_1, \dots, \alpha_n)$ to $(\alpha_i - \alpha_j \pmod{I_i + I_j})$. It is enough to check exactness after localizing at every maximal ideal.)

3. Let R be a Dedekind domain and let M be a nonzero finitely generated torsion R -module. Show that there exist uniquely determined prime ideals \mathfrak{p}_i and positive integers n_i such that

$$M \cong \bigoplus_i R/\mathfrak{p}_i^{n_i}.$$

First show that M is the direct sum of submodules $M_{\mathfrak{p}}$, where $\mathfrak{p}^N M_{\mathfrak{p}} = 0$ for $N \gg 0$. Then argue that $M_{\mathfrak{p}}$ is a module over the PID $R_{\mathfrak{p}}$.)

4. Let R be a Dedekind domain and let M be a finitely generated R -module. Show that M is a flat R -module if and only if M is a projective R -module if and only if M is torsion free. (Localize at each prime \mathfrak{p} and use the fact that $R_{\mathfrak{p}}$ is a PID, together with the classification of modules over a PID.) Thus every finitely generated R -module M is isomorphic to a direct sum $M_1 \oplus M_2$, where M_1 is torsion and M_2 is projective.

5. (Strong approximation for Dedekind domains) Let R be a Dedekind domain with quotient field K , and let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be distinct nonzero prime ideals in R . Show that, for all $\alpha_1, \dots, \alpha_k \in K$ and $n_1, \dots, n_k \in \mathbb{Z}$, there

exists $\beta \in K$ such that $v_{\mathfrak{p}_i}(\beta - \alpha_i) \geq n_i$ for all i and $v_{\mathfrak{q}}(\beta) \geq 0$ for all nonzero prime ideals \mathfrak{q} such that $\mathfrak{q} \neq \mathfrak{p}_i$ for all i .

(First assume that $\alpha_i \in R$ for all i and that $n_i \geq 0$, and apply the Chinese Remainder Theorem to find the appropriate $\beta \in R$. In the general case, there exists $t \in R - \{0\}$ such that $t\alpha_i \in R$; write down the inequalities to be satisfied by $t\beta$, possibly after enlarging the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ to include those primes \mathfrak{p} such that $v_{\mathfrak{p}}(t) \neq 0$.)

Conclude that, with $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ as above and for $a_1, \dots, a_k \in \mathbb{Z}$, there exists an $\alpha \in K$ such that $v_{\mathfrak{p}_i}(\alpha) = a_i$ for all i and such that $v_{\mathfrak{q}}(\alpha) \geq 0$ for all nonzero prime ideals \mathfrak{q} such that $\mathfrak{q} \neq \mathfrak{p}_i$ for all i .

6. Let R be a Dedekind domain with only finitely many prime ideals. Then R is a PID. (If $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the prime ideals of R , we must show that \mathfrak{p}_i is principal for all i . Find an element $r \in R$ such that $r \in \mathfrak{p}_i$, $r \notin \mathfrak{p}_i^2$, and $r \notin \mathfrak{p}_j$ for $i \neq j$, and factor (r) into a product of primes.)