

MODERN ALGEBRA II: FOURTH PROBLEM SET

Reading: Dummit & Foote, §7.4, §9.1, §9.2.

1. Which of the following are ideals in the given ring? Why or why not? If the subset is an ideal, is it a principal ideal?
 - (a) The subset \mathbb{Q} of the ring \mathbb{R} .
 - (b) The subset \mathbb{Z} of the polynomial ring $\mathbb{Z}[x]$.
 - (c) The subset of $\mathbb{Q}[x]$ consisting of all polynomials whose first five terms are 0 (i.e. the set of all polynomials of the form $\sum_{i=0}^n a_i x^i$ with $a_0 = \cdots = a_4 = 0$). (Note: for this definition to make sense, we assume that if $\deg f \leq 4$, then we add the missing terms $0 \cdot x^i$ through $i = 5$.)
 - (d) The subset of $\mathbb{Q}[x]$ consisting of all polynomials whose linear term is 0 (i.e. the set of all polynomials of the form $\sum_{i=0}^n a_i x^i$ with $a_1 = 0$).
 - (e) The subset of $\mathbb{Z}[x]$ consisting of all polynomials whose leading coefficient is divisible by 2.
2. By Part (i) of Problem 5 from the last problem set, we know that, if R is a subring of S and J is an ideal in S , then $I = R \cap J$ is an ideal in R . Show that, in this case, there is an injective homomorphism f from R/I to S/J , defined by $f(r + I) = r + J$. (Define f first as a homomorphism from R to S/J and then determine its kernel and use the fundamental theorem for homomorphisms.) Show that f is surjective \iff for every $s \in S$, there exists an $r \in R$ such that $s \equiv r \pmod{J}$, i.e. $s - r \in J$.
3. Let R be the subring $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ of \mathbb{C} . Let $I = (2 + 3i)$ be the principal ideal in $\mathbb{Z}[i]$ generated by $2 + 3i$.
 - (i) Show that I contains $2 + 3i$ and $-3 + 2i$, and in fact that the additive subgroup $(I, +)$ of the group $(\mathbb{Z}[i], +)$ is generated by $2 + 3i$ and $-3 + 2i$.
 - (ii) Show that $i \equiv -5 \pmod{I}$, i.e. that $i + 5 \in I$.
 - (iii) Show that the homomorphism $f: \mathbb{Z} \rightarrow \mathbb{Z}[i]/I$ defined by $f(n) = n + I$ is surjective.
 - (iv) Show that $13 \in \mathbb{Z} \cap I$.

- (v) Show that $\mathbb{Z} \cap I = 13\mathbb{Z}$. (This may take some calculation.)
- (vi) Conclude by Problem 2 that $\mathbb{Z}[i]/I \cong \mathbb{Z}/13\mathbb{Z}$ as rings. Is I a maximal ideal? A prime ideal?

(Note: for those who took Modern Algebra I last semester, $\mathbb{Z}[i] \cong \mathbb{Z} \times \mathbb{Z}$ as additive groups and the ideal I , viewed as an additive subgroup of $\mathbb{Z} \times \mathbb{Z}$, is the subgroup generated by $(2, 3)$ and $(-3, 2)$. A homework problem from last semester then shows that the additive group $\mathbb{Z}[i]/I$ is isomorphic to $\mathbb{Z}/(\det A)\mathbb{Z}$, where A is the matrix $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ and hence $\det A = 13$.)

4. Let R be the subring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ of \mathbb{R} . Let $I = (6 + \sqrt{2})$ be the principal ideal generated by $6 + \sqrt{2}$. Follow the outline of the preceding problem to show that $R/I \cong \mathbb{Z}/34\mathbb{Z}$. Is I a maximal ideal? A prime ideal?
5. Let F be a field, and consider the ring $F[x]$.
 - (i) Let I be the principal ideal $(x - r)$ in $F[x]$. Using the fact that every coset $p(x) + I$ contains a unique representative of the form a , where $a \in F$, conclude that $F[x]/I \cong F$. Note that this agrees with the fact that $I = \text{Ker } \text{ev}_r : F[x] \rightarrow F$, so that $F[x]/I \cong \text{Im } \text{ev}_r = F$. Is I a prime ideal? A maximal ideal?
 - (ii) Let I be the principal ideal (x^2) in $F[x]$. Show that every coset $p(x) + I$ contains a unique representative of the form $a_0 + a_1x$, where $a \in F$. Write the coset $a_0 + a_1x + I$ in abbreviated form as $a_0 + a_1\alpha$, where $\alpha = x + I$, and describe addition and multiplication in $F[x]/I$ using this form. In particular, what is α^2 in the ring R/I when written in the form $a_0 + a_1\alpha$? Is I a prime ideal? A maximal ideal?
 - (iii) Let I be the principal ideal $(x^2 - 1)$ in $F[x]$. Show again that every coset $p(x) + I$ contains a unique representative of the form $a_0 + a_1x$, where $a \in F$. Again write the coset $a_0 + a_1x + I$ in abbreviated form as $a_0 + a_1\alpha$, where $\alpha = x + I$, and describe addition and multiplication in $F[x]/I$ using this form. In particular, what is α^2 when written in the form $a_0 + a_1\alpha$? Is I a prime ideal? A maximal ideal? Exhibit two nonzero elements of $F[x]/I$ whose product is 0.
 - (iv) Continuing with (iii), still with $I = (x^2 - 1)$ and now assuming that F is not of characteristic 2, consider the ring homomorphism

$F[x] \rightarrow F \times F$ defined by $p(x) \mapsto (p(1), p(-1))$, in other words the homomorphism $(\text{ev}_1, \text{ev}_{-1})$. Show that I is in the kernel of both ev_1 and ev_{-1} , and that there is an induced ring homomorphism $\varphi: F[x]/I \rightarrow F \times F$, where $F \times F$ is viewed as a product ring. (In other words, $\varphi(p(x) + I) = (p(1), p(-1))$.) What is $\varphi(\alpha)$ (where as before $\alpha = x + I$)? Find elements $a_0 + a_1\alpha$ and $b_0 + b_1\alpha$ such that $\varphi(a_0 + a_1\alpha) = (1, 0)$ and $\varphi(b_0 + b_1\alpha) = (0, 1)$. (Where do we need to assume that characteristic $F \neq 2$?) Show that φ is surjective. (In fact, it is not hard to show that $\text{Ker } \varphi = I$ and hence that φ is an isomorphism from $F[x]/I$ to $F \times F$.)