Commutative Algebra II Spring 2008
Third Problem Set due February 18

1. Let R be a reduced Noetherian ring. Let S be the multiplicative subset of R consisting of all elements which are not zero divisors. Show that $S^{-1}R$ is a product of fields as follows:

(i) Show that $S^{-1}R$ is a reduced Noetherian ring in which every element is either a divisor of zero or a unit.

(ii) Let R be a reduced Noetherian ring in which every element is either a divisor of zero or a unit. Show that R is Artinian. (If $(0) = P_1 \cap \cdots \cap P_n$ is a minimal primary decomposition of (0), argue that every prime ideal P is equal to P_i for some i.)

(iii) If an Artinian local ring is reduced, show that it is a field.

2. Let k be a field and let R be a finitely generated k-algebra. Show that R is Artinian $\iff$ R is a finite-dimensional k-vector space. (In one direction, use the fact that, if $\mathfrak{m}$ is a maximal ideal of R, then, by the Nullstellensatz, the field $R/\mathfrak{m}$ is a finite extension of k.)

3. Let R be a local Noetherian integral domain with maximal ideal $\mathfrak{m}$. Show that, if $\mathfrak{m} = (t)$ is a principal ideal, then every nonzero ideal is of the form (t^k) for some $k \geq 0$. (By the previous problem set, every nonzero element of R can be written as ht^k where t does not divide h. In fact, the statement of this problem holds without the assumption that R is an integral domain.)

4. Let R be a UFD and let I be an ideal of R containing two relatively prime elements r, s of R. Show that if $\lambda \in K^*$ and $\lambda I \subseteq R$, then $\lambda \in R$. Conclude that, if $I \neq R$, then I is not invertible. In particular, for $R = k[x_1, \dots, x_n]$, $n \geq 1$ and $I = (x_1, \dots, x_k)$, $k > 1$, I is not an invertible fractional ideal of R. Finally show that, with R and I as above, $\mathrm{Hom}_R(I, R) \cong R$, and in fact that the natural homomorphism $R \to \mathrm{Hom}_R(I, R)$ given by multiplication is an isomorphism.