1. Let \( A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) and let \( B_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \) be 2 \( \times \) 2 orthogonal matrices (depending on a real number \( \theta \)), with \( \det A_\theta = 1 \) and \( \det B_\theta = -1 \). Finally, let \( R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

(i) Show that every element of \( O_2 \), the set of 2 \( \times \) 2 orthogonal matrices, is either equal to \( A_\theta \) for some \( \theta \) or equal to \( B_\theta \) for some \( \theta \). Thus \( SO_2 \) is just the subset of \( O_2 \) consisting of matrices of the form \( A_\theta \) for some \( \theta \).

(ii) Compute \( A_\theta^2 \) (in terms of \( \theta \)), \( A_\theta^{-1} \), \( A_\theta A \theta_2 \) (in terms of \( \theta_1 \) and \( \theta_2 \)), \( B_\theta^2 \) and \( B_\theta^{-1} \). Show that \( B_\theta = A_\theta R \) and that \( R^2 = I \), so that \( R = R^{-1} \). Show that \( R^{-1} A_\theta R = B_\theta = A_\theta^{-1} = A_{-\theta} \). Use this to give another computation of \( B_\theta^2 = A_\theta RA_\theta R = A_\theta RB_\theta \).

(iii) Using the above identities, show: \( A_{\theta_1} A_{\theta_2} = A_{\theta_1 + \theta_2} \), \( B_\theta B_{\theta_2} = A_{\theta_1 - \theta_2} \), \( A_\theta B_\theta = B_{\theta_1 + \theta_2} \), \( B_\theta A_\theta = B_{\theta_1 - \theta_2} \). Also: \( A_\theta RA_\theta^{-1} = A_\theta RA_{-\theta} = B_{2\theta} \).

(iv) Show that there exist orthogonal unit vectors \( u_1 \) and \( u_2 \) such that \( B_\theta u_1 = u_1 \) and \( B_\theta u_2 = -u_2 \), in other words \( B_\theta \) is reflection about the line through the origin and \( u_1 \), and hence we see again that \( B_\theta^2 = I \). To see this, use the last identity in (iii), \( B_\theta = A_{\theta/2} R A_{\theta/2}^{-1} \).

Then set \( u_1 = A_{\theta/2} e_1 \) and \( u_2 = A_{\theta/2} e_2 \), where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) are the standard unit vectors. Show that \( u_1 \) and \( u_2 \) are orthogonal unit vectors and that \( B_\theta u_1 = u_1 \), \( B_\theta u_2 = -u_2 \).

When does \( A_\theta \) have a nonzero (real) eigenvector, and how can you explain this geometrically?

2. For \( v, w \in \mathbb{R}^n \), define \( v \sim w \) if there exists a matrix \( A \in O_n \) such that \( v = Aw \).

(a) Show that \( \sim \) is an equivalence relation. Can you describe the equivalence classes geometrically? (Do the case \( n = 2 \) if you have trouble visualizing the general case.)

(b) For \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \), recall that the norm of \( v \) is given by the formula

\[
\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}.
\]
Show that, if \([v]\) is the equivalence class of \(v\) for the equivalence relation \(\sim\), then the function \(f: (\mathbb{R}^n/ \sim) \to \mathbb{R}\) defined by \(f([v]) = \|v\|\) is well-defined. What is its image?

(c) Let \(g: \mathbb{R}^n \to \mathbb{R}\) be defined by: \(g(v) = g(v_1, \ldots, v_n) = v_1\), i.e. \(g\) is the projection onto the first coordinate. Show that \(g\) does not induce a well-defined function on \(\mathbb{R}^n/ \sim\).

(d) Suppose instead that we define the relation \(\sim\) via: \(v \sim w\) if there exists a matrix \(A \in GL_n(\mathbb{R})\) such that \(v = Aw\). What are the equivalence classes for \(\sim\) in this case?

3. In \(\mathbb{Z}/17\mathbb{Z}\), compute (by writing in the form \([a]\), with 0 \(\leq a \leq 16\)) \([3] + [14]; [3] \cdot [14]; [12] + [12]; [12] \cdot [12] = [12]^2\). Can you find an integer \(k\) such that \([3] \cdot [k] = [1]\) in \(\mathbb{Z}/17\mathbb{Z}\)? In \(\mathbb{Z}/12\mathbb{Z}\), compute \([3] \cdot [4]\) and \([2] \cdot [6]\) (again, by writing the answer in the form \([a]\), with 0 \(\leq a \leq 11\)). Can you find an integer \(k\) such that \([3] \cdot [k] = [1]\) in \(\mathbb{Z}/12\mathbb{Z}\)?

4. For a natural number \(n \in \mathbb{N}\), we sometimes denote the congruence class of an integer mod \(n\) as \([a]_n\), for example if we want to discuss congruences to different moduli.

(a) Is the “function” \(f: \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}\) defined by \(f([a]_n) = a\) well-defined?

(b) For \(a, k \in \mathbb{Z}\), show that \([a + k]_n = [a]_n \iff n | k\).

(c) Using (b), show that the “function” \(f: \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}\) defined by \(f([a]_n) = [a]_m\) is well-defined \(\iff m | n\).

5. (i) Let \((X_1, \ast_1)\) and \((X_2, \ast_2)\) be two binary structures and let \(f: X_1 \to X_2\) be an isomorphism of the binary structures. Show that \(f^{-1}: X_2 \to X_1\) is also an isomorphism.

(ii) Let \((X_1, \ast_1)\), \((X_2, \ast_2)\), and \((X_3, \ast_3)\) be three binary structures and let \(f: X_1 \to X_2\) and \(g: X_2 \to X_3\) be isomorphisms of the binary structures. Show that \(g \circ f: X_1 \to X_3\) is also an isomorphism.

6. Fix an \(n \in \mathbb{N}\). Show that the function \(f: \mathbb{Z}/n\mathbb{Z} \to \mu_n\) defined by \(f([a]_n) = e^{2\pi i / n}\) is well-defined. Then show that \(f\) is an isomorphism from \(\mathbb{Z}/n\mathbb{Z}\) to \(\mu_n\).