# MODERN ALGEBRA I FALL 2020: SECOND PROBLEM SET

1. Let $f\colon X \to Y$ be a function. Show that, if $f$ has a left inverse $g$, then $f$ is injective. Conversely, suppose that $f$ is injective and that $X \neq \emptyset$. Show that $f$ has a left inverse. (However, if $X = \emptyset$ and $Y \neq \emptyset$, then the unique function $f\colon X \to Y$ is injective but does not have a left inverse.)

   Show that, if $f$ has a right inverse $g$, then $f$ is surjective. (While, conversely, if $f$ is surjective then it has a right inverse, this involves somewhat more serious set theory.)

2. (i) Let $f\colon \mathbb{N} \to \mathbb{N}$ be the function defined in Problem 3 of HW 1: $f(n) = n - 1$ for $n \geq 2$, and $f(1) = 30$. Show that $h(n) = n + 1$ is a right inverse for $f$. What are all possible right inverses for $f$?

   (ii) Since $h$ is a right inverse for $f$, we know that $f$ is a left inverse for $h$. What are all possible left inverses for $h$?

3. (i) How many elements are there in the set $S_2$, the set of all bijections from $\{1, 2\}$ to $\{1, 2\}$? Show that, for all $f, g \in S_2$, $f \circ g = g \circ f$. However, give an example of two functions (not assumed to be bijections) $f, g$ from $\{1, 2\}$ to $\{1, 2\}$ such that $f \circ g \neq g \circ f$.

   (ii) How many elements are there in $S_3$? Find two functions $f, g \in S_3$ such that $f \circ g \neq g \circ f$.

4. Write down, in the form $a + bi$, the elements in the set of all third roots of unity $\mu_3$. Do the same for: the elements of $\mu_4$, the set of all fourth roots of unity; the elements of $\mu_8$, the set of all eighth roots of unity.

5. (i) Compute $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ as well as $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Are they the same?

   (ii) Compute $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix}$. Can either factor have an inverse?

6. Let $A$ be an $m \times n$ matrix $A = (a_{ij})$. Define the *transpose matrix* ${}^tA$ to be the $n \times m$ matrix whose $(i, j)^{\text{th}}$ entry is $a_{ji}$. Clearly, ${}^t({}^tA) = A$.

   (a) Show that, for all standard basis vectors $\mathbf{e}_i \in \mathbb{R}^m$ and $\mathbf{e}_j \in \mathbb{R}^n$, $\langle \mathbf{e}_i, A\mathbf{e}_j \rangle = \langle {}^tA\mathbf{e}_i, \mathbf{e}_j \rangle$.

(b) Use this and the fact that $\langle \cdot, \cdot \rangle$ is linear in each variable, show that, for all $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$,

$$\langle \mathbf{v}, A\mathbf{w} \rangle = \langle {}^tA\mathbf{v}, \mathbf{w} \rangle.$$

Show that ${}^tA$ is in fact the unique such matrix with this property: if $B$ is an $n \times m$ matrix such that, for all $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$,

$$\langle \mathbf{v}, A\mathbf{w} \rangle = \langle B\mathbf{v}, \mathbf{w} \rangle,$$

then $B = {}^tA$. (Here, you can use the fact that, if $\mathbf{u}$ is a vector in $\mathbb{R}^n$ such that $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathbb{R}^n$, then $\mathbf{u} = 0$.)

(c) Using (b), show: If $A$ is an $m \times n$ matrix and $B$ is an $n \times k$ matrix, then

$${}^t(AB) = {}^tB\,{}^tA.$$

7. A matrix $A \in \mathbb{M}_n(\mathbb{R})$ is *symmetric* if ${}^tA = A$. Show by an example that the product of two $2 \times 2$ symmetric matrices need not be symmetric.

8. Let $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and let $B_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ be $2 \times 2$ orthogonal matrices (depending on a real number $\theta$), with $\det A_\theta = 1$ and $\det B_\theta = -1$. Finally, let $R = B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Compute $A_\theta^2$ (in terms of $\theta$), $A_\theta^{-1}$, $A_{\theta_1}A_{\theta_2}$ (in terms of $\theta_1$ and $\theta_2$), $B_\theta^2$ and $B_\theta^{-1}$. Show that $B_\theta = A_\theta R$ and that $R^2 = I$, so that $R = R^{-1}$. Show that $R^{-1}A_\theta R = RB_\theta = A_\theta^{-1} = A_{-\theta}$. Use this to give another computation of $B_\theta^2 = A_\theta R A_\theta R = A_\theta R B_\theta$.

Using the above identities, show: $A_{\theta_1}A_{\theta_2} = A_{\theta_1+\theta_2}$, $B_{\theta_1}B_{\theta_2} = A_{\theta_1-\theta_2}$, $A_{\theta_1}B_{\theta_2} = B_{\theta_1+\theta_2}$, $B_{\theta_1}A_{\theta_2} = B_{\theta_1-\theta_2}$. Also: $A_\theta R A_\theta^{-1} = A_\theta R A_{-\theta} = B_{2\theta}$.