MODERN ALGEBRA I SPRING 2016:
SECOND PROBLEM SET

1. Let $f: X \to Y$ be a function. Show that, if $f$ has a left inverse $g$, then $f$ is injective. Conversely, suppose that $f$ is injective and that $X \neq \emptyset$. Show that $f$ has a left inverse. (However, if $X = \emptyset$ and $Y \neq \emptyset$, then the unique function $f: X \to Y$ is injective but does not have a left inverse.)

Show that, if $f$ has a right inverse $g$, then $f$ is surjective. (While, conversely, if $f$ is surjective then it has a right inverse, this involves somewhat more serious set theory.)

2. Let $f: X \to Y$ be a function. Show that $f$ has an inverse function $f^{-1}$ if and only if $f$ is a bijection. (You can use the previous problem to show the $\implies$ direction. For the $\impliedby$ direction, show that, if $f$ is a bijection, then the transpose $^tG_f = \{(f(x), x) : x \in X\}$ of the graph of $f$ is the graph of a function $f'$: $Y \to X$, and then check that $f'$ is an inverse function to $f$.)

3. (i) How many elements are there in the set $S_1$, the set of all bijections from $\{1\}$ to $\{1\}$? Show that, for all $f, g \in S_1$, $f \circ g = g \circ f$.

(ii) How many elements are there in the set $S_2$, the set of all bijections from $\{1, 2\}$ to $\{1, 2\}$? Show that, for all $f, g \in S_2$, $f \circ g = g \circ f$. However, give an example of two functions $f, g$ from $\{1, 2\}$ to $\{1, 2\}$ such that $f \circ g \neq g \circ f$.

(iii) How many elements are there in $S_3$? Find two functions $f, g \in S_3$ such that $f \circ g \neq g \circ f$.

4. Write down, in the form $a + bi$, the elements in the set of all third roots of unity $\mu_3$. Do the same for: the elements of $\mu_4$, the set of all fourth roots of unity; the elements of $\mu_8$, the set of all eighth roots of unity.

5. (i) Compute $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ as well as $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Are they the same?

(ii) Compute $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix}$. Can either factor have an inverse?

(iii) Recall that a matrix $A \in M_n(\mathbb{R})$ is orthogonal if its columns $u_1, \ldots, u_n$ are an orthonormal basis, i.e. $\langle u_i, u_j \rangle = 0$ if $i \neq j$ and $= 1$
if \( i = j \). Which of the following matrices are orthogonal?

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix};
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix};
\begin{pmatrix}
2 & 3 \\
-3 & 2
\end{pmatrix};
\begin{pmatrix}
4 & 5 \\
1 & 1
\end{pmatrix};
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix};
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

(iv) Verify that

\[
A = \begin{pmatrix}
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}}
\end{pmatrix}
\]

is an orthogonal matrix.

(Note: it may be more efficient to write the columns of the matrix as 
\( \frac{1}{\sqrt{6}} (1, 1, 2) \), \( \frac{1}{\sqrt{2}} (-1, 1, 0) \), and \( \frac{1}{\sqrt{3}} (1, 1, -1) \).)