1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that $B \in O_2$ and $A \in U = \\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$. Compute $ABA^{-1}$ and $BAB^{-1}$. Is $O_2$ a normal subgroup of $GL_2(\mathbb{R})$? Is $U$ a normal subgroup of $GL_2(\mathbb{R})$?

2. Consider the following subgroup of $A_4$ (considered in Problem Set 10):

   $$H = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

   (i) Show that $H = \{\sigma \in S_4 : \sigma \text{ is even and } \sigma \text{ has order 1 or 2}\}$.

   (ii) Show that $H$ is a normal subgroup of $A_4$ and of $S_4$. (If $\sigma \in H$ and $\tau \in S_4$, what can you say about the shape of $\tau^{-1}\sigma \tau$?)

   (iii) Show that $A_4/H$ has order 3 and that $S_4/H$ has order 6. (In fact, one can show that $S_4/H \cong S_3$.)

   (iv) Show that $K = \{1, (1,2)(3,4)\}$ is a normal subgroup of $H$ but not of $A_4$. Thus $K \lhd H$ and $H \lhd A_4$ but $K \nmid A_4$.

3. (i) Let $G$ be a group and let $H$ be a normal subgroup of $G$. Let $a \in G$. Show that the order of $aH$ in the group $G/H$ is equal to the smallest positive integer $n$ such that $a^n \in H$ (and is infinite if no such integer exists).

   (ii) Give an example of a group $G$, a normal subgroup $H$ of $G$, and an element $a \in G$ such that $a$ has infinite order in $G$ but such that $aH$ has finite order.

   (iii) Show that, if $a$ has finite order $n$ in $G$, then the order of $aH$ is finite and divides $n$. Is the order of $aH$ always equal to $n$?

4. (i) Let $G$ be a group and let $H, K$ be two subgroups of $G$. Show that, if $H \lhd G$ and $K \lhd G$, then $H \cap K \lhd G$.

   (ii) With $G, H, K$ as in (i), show that, if $H \lhd G$, then $H \cap K \lhd K$.

5. Let $G_1$ and $G_2$ be groups and let $f : G_1 \to G_2$ be a homomorphism.

   (i) Show that, if $H_2$ is a normal subgroup of $G_2$, then $f^{-1}(H_2)$ is a normal subgroup of $G_1$. 


(ii) Show that, if $H_1$ is a normal subgroup of $G_1$, then $f(H_1)$ need not be a normal subgroup of $G_2$.

(iii) Show that, if $H_1$ is a normal subgroup of $G_1$, and $f$ is surjective, then $f(H_1)$ is a normal subgroup of $G_2$.

6. Let $G$ be a finite group and $f: G \to H$ a surjective homomorphism. Show that $\#(\text{Ker} \ f) = \#(G)/\#(H)$.

7. Let $G$ be a group $G \neq \{1\}$. Show that $G$ is simple if and only if, for every group $H$ and homomorphism $f: G \to H$, either $f$ is trivial (i.e. $\text{Im} \ f = \{1\}$) or $f$ is injective.

8. Let $G$ be a group, and let $H = \{1,g\}$ be a subgroup of $G$ with just two elements. Show that $H$ is a normal subgroup of $G$ if and only if $H \leq Z(G)$, where $Z(G)$ denotes the center of $G$.

9. For $n \geq 5$, and using the fact that $A_n$ is simple for $n \geq 5$, show that every normal subgroup $H$ of $S_n$ is either $S_n$, $A_n$, or $\{1\}$. (Hints: first, show that, by (ii) of Problem 4, either $H \cap A_n = A_n$ or $H \cap A_n = \{1\}$. If $H \cap A_n = A_n$, then $A_n \subseteq H$ so that $A_n \leq H \leq S_n$. What are the possibilities for the index $(S_n : H)$? If $H \cap A_n = \{1\}$, show that the induced homomorphism $\varepsilon : H \to \{\pm 1\}$ given by the sign homomorphism is injective. Conclude that $\#(H) \leq 2$. If $\#(H) = 2$, use the previous problem to conclude that $H \leq Z(S_n)$, the center of $S_n$, and then use Problem 7 from HW 9 to get a contradiction.)