MODERN ALGEBRA II SPRING 2019:
TENTH PROBLEM SET

1. Test the following polynomials in $\mathbb{Z}[x]$ for irreducibility in $\mathbb{Q}[x]$ and in $\mathbb{Z}[x]$. In each case, give a reason why the polynomial is irreducible or find its complete factorization in $\mathbb{Q}[x]$ and in $\mathbb{Z}[x]$. You may use any of the criteria we have developed in class or on the homework (including Problem 3 of HW 9).

(a) $2x^4 - 50x^3 + 100x^2 - 750x + 60$; (b) $x^3 - 2x^2 + x + 1$
(c) $2x^3 + 3x^2 + 3x + 1$; (d) $x^4 + 3x^2 + 2$; (e) $2x^{16} - 88$;
(f) $x^4 - 6x^2 + 9$; (g) $x^4 + 1$; (h) $x^{28} - 28$
(i) $x^4 - 7x^2 + 12$; (j) $5x^6 - 540x^5 + 90x^3 - 360x + 60$.

2. Show that the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}[\sqrt{-3}][x]$ but not in $\mathbb{Q}(\sqrt{-3})[x]$ as follows:

(i) Show that $x^2 + x + 1$ has a root in $\mathbb{Q}(\sqrt{-3})$ but not in $\mathbb{Z}[\sqrt{-3}]$.
(ii) Show that, if there is a factorization $x^2 + x + 1 = (ax + b)(cx + d)$ with $a, b, c, d \in \mathbb{Z}[\sqrt{-3}]$, then $a$ and $c$ are units, contradicting the fact that there is no root of $x^2 + x + 1$ in $\mathbb{Z}[\sqrt{-3}]$.
(iii) Finally, rule out a factorization of the form $x^2 + x + 1 = rg$ where $r \in \mathbb{Z}[\sqrt{-3}]$ is not a unit.

3. Let $R$ be an integral domain which is not a UFD and let $F$ be its field of quotients. Suppose that $\alpha, \beta \in R$ are nonzero elements and $\gamma$ is an irreducible element of $R$ which satisfy: (i) $\gamma$ divides $\alpha\beta$; (ii) $\gamma$ does not divide either $\alpha$ or $\beta$.

Show that the polynomial
\[ f = \gamma \cdot \left( x - \frac{\alpha}{\gamma} \right) \left( x - \frac{\beta}{\gamma} \right) = \gamma x^2 - (\alpha + \beta)x + \frac{\alpha\beta}{\gamma} \]
has coefficients in $R[x]$, is reducible in $F[x]$, but does not factor into polynomials of degree one in $R[x]$. If in addition $\gamma^2$ does not divide $\alpha\beta$, show that $f$ is irreducible in $R[x]$. Applying the above to $R = \mathbb{Z}\sqrt{-5}[x]$, $\alpha = 1 + \sqrt{-5}$, $\beta = 1 - \sqrt{-5}$, and $\gamma = 2$, conclude that $2x^2 - 2x + 3$ is irreducible in $\mathbb{Z}\sqrt{-5}[x]$ but not in $\mathbb{Q}(\sqrt{-5})[x]$.

4. Let $R$ be a UFD and let $f, g \in R[x]$ be two nonzero polynomials. Show that $c(fg) = c(f)c(g)$. (Hint: Gauss lemma.)