MODERN ALGEBRA I FALL 2020: FIRST PROBLEM SET

1. (i) Let X_1 and X_2 be two sets. Show that $X_1 \subseteq X_1 \cup X_2$ and that $X_2 \subseteq X_1 \cup X_2$. Moreover, if Y is any set such that $X_1 \subseteq Y$ and $X_2 \subseteq Y$, then $(X_1 \cup X_2) \subseteq Y$. (This says that $X_1 \cup X_2$ is the smallest set containing both X_1 and X_2.)

   (ii) Let X_1 and X_2 be two sets. Show that $X_1 \cap X_2 \subseteq X_1$ and that $X_1 \cap X_2 \subseteq X_2$. Moreover, if Y is any set such that $Y \subseteq X_1$ and $Y \subseteq X_2$, then $Y \subseteq (X_1 \cap X_2)$. (This says that $X_1 \cap X_2$ is the largest set which is contained both in X_1 and in X_2.)

2. Let X be a set and let Δ_X be the diagonal in $X \times X$:

$$\Delta_X = \{(x, x) : x \in X\}.$$

   Show that, if X has at least two elements, then Δ_X is not of the form $A \times B$ for subsets A, B of X.

3. Let $\mathbb{R}$ be the set of real numbers. For each of the following functions $g\colon \mathbb{R} \to \mathbb{R}$, determine if g is (a) injective; (b) surjective; (c) a bijection, and describe the image of g.

   (i) $g(x) = e^x$; (ii) $g(x) = 5x - 12$; (iii) $g(x) = x^3$ (iv) $g(x) = x^3 - 3x$

   (For (iv), you can use some calculus to sketch the graph of g.)

4. Let $f\colon \mathbb{N} \to \mathbb{N}$ be the function defined by $f(n) = n - 1$ if $n \geq 2$, and $f(1) = 30$ (here $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers). Is f injective? Surjective? Find the following subsets of $\mathbb{N}$:

   (a) $f(\{1, 2, 3, 5\})$; (b) $f(\{1, 31\})$; (c) $f^{-1}(1)$;
   (d) $f^{-1}(\{1, 2, 3\})$ (e) $f^{-1}(30)$; (f) $f^{-1}(\{1, 30\})$.

5. Let Y be a nonempty set and let $f\colon X \to Y$ be a constant function: there exists a $c \in Y$ such that, for all $x \in X$, $f(x) = c$. Equivalently, the graph of f is equal to $X \times \{c\} \subseteq X \times Y$. When is f injective? When is f surjective? When is f a bijection?

6. Using the precise definition of a function and a little logic, show that, for every set Y, there is exactly one function f from $\emptyset$ to Y. When is f injective? Surjective?

   Let X be a set. Show that there are either **no** functions from X to $\emptyset$ or exactly one such function, depending on whether $X \neq \emptyset$ or $X = \emptyset$.

7. Let X and Y be two nonempty sets. Define functions $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ by:

$$\pi_1(x, y) = x; \qquad \pi_2(x, y) = y.$$

The function π_1 is called the *first projection* or the *projection of $X \times Y$ to the first component*, and similarly for π_2. When is π_1 injective? Surjective? What happens when either $X = \emptyset$ or $Y = \emptyset$?

8. Let X and Y be two sets and define a function $t \colon X \times Y \to Y \times X$ by:

$$t(x, y) = (y, x).$$

Argue from the definitions that t is injective and surjective, hence a bijection.

9. Let X be a set and let $A \subseteq X$.

   (i) Let A be a subset of X. Define a function $\chi_A \colon X \to \{0, 1\}$ by the rule

   $$\chi_A(x) = \begin{cases} 0, & \text{if } x \notin A; \\ 1, & \text{if } x \in A. \end{cases}$$

   The function χ_A is called the *characteristic function of A*. Show that $\chi_A^{-1}(0) = X - A$ and that $\chi_A^{-1}(1) = A$. When is χ_A a constant function? For which sets X and subsets A is χ_A injective? Surjective? A bijection?

   (ii) Suppose that $f \colon X \to \{0, 1\}$ is any function. Define a subset S_f of X by taking $S_f = f^{-1}(1)$. Show that f is the characteristic function of S_f.

   (iii) Combining (i) and (ii), show that $S_{\chi_A} = A$ and that $\chi_{S_f} = f$.