1. Let $R$ be a ring and let $r \in R$. Given $n \in \mathbb{N}$, define $r^n = r \cdot \cdots \cdot r$. By convention, if $R$ has unity 1, set $r^0 = 1$. (However, the expression $r^n$, for $n < 0$, can only be defined if $r$ is a unit.) Show (informally) that $r^n \cdot r^m = r^{n+m}$ and that $(r^n)^m = r^{nm}$.

2. Let $R$ be a ring with unity and define $f: \mathbb{Z} \to R$ by $f(n) = n \cdot 1$. Show that $f$ is a (ring) homomorphism, that it is the unique homomorphism from $\mathbb{Z}$ to $R$ (with our conventions on homomorphisms from a ring with unity to another ring with unity) and that its image is the cyclic subgroup generated by 1. (You can use the fact that $(n \cdot r)s = r(n \cdot s) = n \cdot (rs)$ for all $r,s \in R$.)

3. Let $R$ be a commutative ring. Show:
   (a) For all $r,s \in R$, $(r + s)(r - s) = r^2 - s^2$. What is the correct statement if $R$ is not commutative?
   (b) For all $r,s \in R$, $(r + s)^2 = r^2 + 2 \cdot rs + s^2$. What is the correct statement if $R$ is not commutative?

4. Let $R$ and $S$ be two rings with unity and let $f: R \to S$ be a ring homomorphism (with the usual conventions). Show that, if $r \in R^*$, i.e. that $r$ is a unit in $R$, then $f(r)$ is a unit in $S$. Is the converse necessarily true, that if $f(r)$ is a unit in $S$, then $r$ is always a unit in $R$?

5. Let $\mathbb{Z}[i]$ be the ring $\{a+bi : a, b \in \mathbb{Z}\}$ defined in class. For $\alpha = a + bi \in \mathbb{Z}[i]$, define $\bar{\alpha} = a - bi$. Note that $\alpha \bar{\alpha} = |\alpha|^2 = a^2 + b^2$.
   (a) Show that $f: \mathbb{Z}[i] \to \mathbb{Z}[i]$ defined by $f(\alpha) = \bar{\alpha}$ is a (ring) isomorphism.
   (b) Show that, if $\alpha$ is a unit of $\mathbb{Z}[i]$, then so are $\bar{\alpha}$ and $|\alpha|^2$.
   (c) Conclude that, if $\alpha$ is a unit in $\mathbb{Z}[i]$, then $\alpha \in \{\pm 1, \pm i\}$.

6. Let $\mathbb{H}$ denote the quaternions: $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. Recall that $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.
   (a) Given $\alpha = x_0 + x_1i + x_2j + x_3k \in \mathbb{H}$, define the norm
      \[|\alpha| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}\]
and the conjugate \( \bar{\alpha} \) via:

\[
\bar{\alpha} = x_0 - x_1i - x_2j - x_3k.
\]

With some care due to the fact that multiplication of quaternions is not commutative, show that

\[
\bar{(\bar{\alpha})} = \alpha; \\
\bar{\alpha}\beta = \bar{\beta}\bar{\alpha}; \\
\alpha\bar{\alpha} = \bar{\alpha}\alpha = x_0^2 + x_1^2 + x_2^2 + x_3^2 = |\alpha|^2; \\
|\alpha\beta| = |\alpha||\beta|.
\]

Conclude that, if \( \alpha \neq 0 \), then \( \bar{\alpha}/|\alpha|^2 \) is a multiplicative inverse for \( \alpha \).

(b) Let \( \alpha = x_1i + x_2j + x_3k \in \mathbb{H} \). Compute \( \alpha^2 \). Conclude that there are an infinite number of \( \alpha \in \mathbb{H} \) such that \( \alpha^2 = -1 \).

7. Let \( R \) be a ring, not necessarily commutative or with unity. Define the center \( Z(R) \) to be the set

\[
\{ r \in R : rs = sr \text{ for all } s \in R. \}
\]

Show:

(a) If \( R \) has a unity 1, then 1 \( \in Z(R) \).

(b) \( Z(R) \) is a subring of \( R \).

(c) The center \( Z(\mathbb{H}) \) of the quaternions is just \( \mathbb{R} \subseteq \mathbb{H} \). (Be efficient: \( \alpha \in Z(\mathbb{H}) \iff \alpha \) commutes with \( i, j, k \).)

(d) In general, if \( D \) is a division ring, then the center \( Z(D) \) is a field.

(Note: using some linear algebra, it is possible to show that the center of \( M_n(\mathbb{R}) \) is the subring \( \{ t\text{Id} : t \in \mathbb{R} \} \) of all scalar multiples of the identity matrix; this subring is isomorphic to \( \mathbb{R} \).)