An example

An example of implicit differentiation: Let \( f(x, y, x) = e^z - xyz^2 \).
Note that \( f(e, 1, 1) = e - e = 0 \). Assume that \( f(x, y, z) = 0 \) defines \( z \) implicitly as a function of \( x \) and \( y \): there is a function \( z(x, y) \) defined at least for some \( (x, y) \) close to \( (e, 1) \), such that \( z(e, 1) = 1 \) and \( f(x, y, z(x, y)) \) is identically zero. We want to compute \( \partial z/\partial x \) and \( \partial z/\partial y \). The method of implicit differentiation tells us that
\[
\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z};
\]
\[
\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.
\]
Computing, we find that
\[
\frac{\partial f}{\partial x} = -yz^2;
\]
\[
\frac{\partial f}{\partial y} = -xz^2;
\]
\[
\frac{\partial f}{\partial z} = e^z - 2xyz.
\]
Thus, \( \frac{\partial f}{\partial x}(e, 1, 1) = -1 \), \( \frac{\partial f}{\partial y}(e, 1, 1) = -e \), and \( \frac{\partial f}{\partial z}(e, 1, 1) = e - 2e = -e \).
Then \( \frac{\partial z}{\partial x}(e, 1) = -(-1)/(-e) = 1/e \) and \( \frac{\partial z}{\partial y}(e, 1) = -(e)/(-e) = -1 \).
The equation for the tangent plane to the graph of \( z = z(x, y) \) at the point \( (e, 1, 1) \) is then
\[
z = z(e, 1) + \frac{\partial z}{\partial x}(e, 1)(x - e) + \frac{\partial z}{\partial y}(e, 1)(y - 1)
\]
\[
= 1 + (-1/e)(x - e) + (-1)(y - 1) = -x/e - y + 3.
\]
We can rewrite this as
\[
x + ey + ez = 3e.
\]
The example as an example of using the gradient to compute a tangent plane to a level surface: Another way to compute the tangent plane to the level surface defined by \( f(x, y, z) = e^z - xyz^2 = 0 \) at the point \( (e, 1, 1) \) is as follows: we compute the gradient
\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right) = (-yz^2, -xz^2, e^z - 2xyz).
\]
Then $\nabla f(e, 1, 1) = (-1, -e, -e)$ is the normal vector to the tangent plane at $(e, 1, 1)$. The equation for the tangent plane is then

$$\nabla f(e, 1, 1) \cdot (x - e, y - 1, z - 1) = 0.$$  

Writing this out gives

$$(-1, -e, -e) \cdot (x - e, y - 1, z - 1) = 0 = -x + e - ey + e - ez + e,$$

and hence we see again that the equation for the tangent plane is

$$x + ey + ez = 3e.$$

Note that the method of implicit differentiation gives the normal vector to the graph of $z = z(x, y)$ at $(e, 1, 1)$ as

$$\left( -\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, 1 \right) = \left( -\frac{-1}{e}, -(-1), 1 \right) = \left( \frac{1}{e}, 1, 1 \right) = -\frac{1}{e}(-1, -e, -e).$$

Thus we see that, up to a scalar multiple, the two normal vectors to the tangent plane at the point $(e, 1, 1)$ agree, as they must.

An example of directional derivatives: Let’s also compute some directional derivatives for the function $f(x, y, z) = e^z - xyz^2$ at the point $(e, 1, 1)$. For example, if $u = \frac{1}{\sqrt{11}}(-1, 3, 1)$, then the directional derivative $D_u f(e, 1, 1)$ of the function $f(x, y, z)$ at the point $(e, 1, 1)$ in the unit direction $u = \frac{1}{\sqrt{11}}(-1, 3, 1)$ is:

$$\nabla f(e, 1, 1) \cdot u = (-1, -e, -e) \cdot \frac{1}{\sqrt{11}}(-1, 3, 1) = \frac{1}{\sqrt{11}}(-1, -e, -e) \cdot (1 - 3e - e) = \frac{-4e + 1}{\sqrt{11}}.$$

The unit direction where the directional derivative is the largest (“the direction of steepest increase”) is:

$$\frac{\nabla f(e, 1, 1)}{\|\nabla f(e, 1, 1)\|} = \frac{1}{\sqrt{1 + 2e^2}}(-1, -e, -e) = \left( -\frac{1}{\sqrt{1 + 2e^2}}, -\frac{e}{\sqrt{1 + 2e^2}}, -\frac{e}{\sqrt{1 + 2e^2}} \right).$$

The value of the directional derivative in the unit direction $\frac{\nabla f(e, 1, 1)}{\|\nabla f(e, 1, 1)\|}$ is then

$$\|\nabla f(e, 1, 1)\| = \sqrt{1 + 2e^2}.$$  

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