An example

An example of implicit differentiation: Let $f(x, y, x) = e^z - xyz^2$. Note that $f(e, 1, 1) = e - e = 0$. Assume that $f(x, y, z) = 0$ defines z implicitly as a function of x and y: there is a function $z(x, y)$ defined at least for some (x, y) close to $(e, 1)$, such that $z(e, 1) = 1$ and $f(x, y, z(x, y))$ is identically zero. We want to compute $\partial z/\partial x$ and $\partial z/\partial y$. The method of implicit differentiation tells us that

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z};$$
$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

Computing, we find that

$$\frac{\partial f}{\partial x} = -yz^2;$$
$$\frac{\partial f}{\partial y} = -xz^2;$$
$$\frac{\partial f}{\partial z} = e^z - 2xyz.$$

Thus, $\frac{\partial f}{\partial x}(e, 1, 1) = -1$, $\frac{\partial f}{\partial y}(e, 1, 1) = -e$, and $\frac{\partial f}{\partial z}(e, 1, 1) = e - 2e = -e$. Then $\frac{\partial z}{\partial x}(e, 1) = -(-1)/(-e) = -1/e$ and $\frac{\partial z}{\partial y}(e, 1) = -(-e)/(-e) = -1$. The equation for the tangent plane to the graph of $z = z(x, y)$ at the point $(e, 1, 1)$ is then

$$\begin{aligned} z &= z(e, 1) + \frac{\partial z}{\partial x}(e, 1)(x - e) + \frac{\partial z}{\partial y}(e, 1)(y - 1) \\ &= 1 + (-1/e)(x - e) + (-1)(y - 1) = -x/e - y + 3. \end{aligned}$$

We can rewrite this as

$$x + ey + ez = 3e.$$

The example as an example of using the gradient to compute a tangent plane to a level surface: Another way to compute the tangent plane to the level surface defined by $f(x, y, z) = e^z - xyz^2 = 0$ at the point $(e, 1, 1)$ is as follows: we compute the gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}\right) = (-yz^2, -xz^2, e^z - 2xyz).$$

Then $\nabla f(e,1,1) = (-1,-e,-e)$ is the normal vector to the tangent plane at $(e,1,1)$. The equation for the tangent plane is then

$$\nabla f(e,1,1) \cdot (x-e, y-1, z-1) = 0.$$

Writing this out gives

$$(-1,-e,-e) \cdot (x-e, y-1, z-1) = 0 = -x + e - ey + e - ez + e,$$

and hence we see again that the equation for the tangent plane is

$$x + ey + ez = 3e.$$

Note that the method of implicit differentiation gives the normal vector to the graph of $z = z(x,y)$ at $(e,1,1)$ as

$$\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = \left(-\frac{(-1)}{e}, -(-1), 1\right) = \left(\frac{1}{e}, 1, 1\right) = -\frac{1}{e}(-1,-e,-e).$$

Thus we see that, up to a scalar multiple, the two normal vectors to the tangent plane at the point $(e,1,1)$ agree, as they must.

An example of directional derivatives: Let's also compute some directional derivatives for the function $f(x,y,z) = e^z - xyz^2$ at the point $(e,1,1)$. For example, if $\mathbf{u} = \frac{1}{\sqrt{11}}(-1,3,1)$, then the directional derivative $D_{\mathbf{u}} f(e,1,1)$ of the function $f(x,y,z)$ at the point $(e,1,1)$ in the unit direction $\mathbf{u} = \frac{1}{\sqrt{11}}(-1,3,1)$ is:

$$\nabla f(e,1,1) \cdot \mathbf{u} = (-1,-e,-e) \cdot \frac{1}{\sqrt{11}}(-1,3,1) = \frac{1}{\sqrt{11}}(1 - 3e - e) = \frac{-4e+1}{\sqrt{11}}.$$

The unit direction where the directional derivative is the largest ("the direction of steepest increase") is:

$$\frac{\nabla f(e,1,1)}{\|\nabla f(e,1,1)\|} = \frac{1}{\sqrt{1+2e^2}}(-1,-e,-e) = \left(-\frac{1}{\sqrt{1+2e^2}}, -\frac{e}{\sqrt{1+2e^2}}, -\frac{e}{\sqrt{1+2e^2}}\right).$$

The value of the directional derivative in the unit direction $\dfrac{\nabla f(e,1,1)}{\|\nabla f(e,1,1)\|}$ is then

$$\|\nabla f(e,1,1)\| = \sqrt{1+2e^2}.$$