1 Review of complex numbers

1.1 Complex numbers: algebra

The set $\mathbb{C}$ of complex numbers is formed by adding a square root i of -1 to the set of real numbers: $i^2 = -1$. Every complex number can be written uniquely as $a + bi$, where a and b are real numbers. We usually use a single letter such as z to denote the complex number $a + bi$. In this case a is the *real part* of z, written $a = \operatorname{Re} z$, and b is the *imaginary part* of z, written $b = \operatorname{Im} z$. The complex number z is *real* if $z = \operatorname{Re} z$, or equivalently $\operatorname{Im} z = 0$, and it is *pure imaginary* if $z = (\operatorname{Im} z)i$, or equivalently $\operatorname{Re} z = 0$. In general a complex number is the sum of its real part and its imaginary part times i, and two complex numbers are equal if and only if they have the same real and imaginary parts.

We add and multiply complex numbers in the obvious way:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i;$$
$$(a_1 + b_1 i) \cdot (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i.$$

For example,

$$(2 + 3i)(-5 + 4i) = -22 - 7i.$$

In this way, addition and multiplication are associative and commutative, multiplication distributes over addition, there is an additive identity 0 and additive inverses $(-(a + bi) = (-a) + (-b)i)$, and there is a multiplicative identity 1. Note also that $\operatorname{Re}(z_1 + z_2) = \operatorname{Re} z_1 + \operatorname{Re} z_2$, and similarly for the imaginary parts, but a corresponding statement does not hold for multiplication. Before we discuss multiplicative inverses, let us recall *complex conjugation*: the *complex conjugate* $\bar{z}$ of a complex number $z = a + bi$ is by definition $\bar{z} = a - bi$. It is easy to see that:

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z});$$
$$\operatorname{Im} z = \frac{1}{2i}(z - \bar{z}).$$

Thus z is real if and only if $\bar{z} = z$ and pure imaginary if and only if $\bar{z} = -z$. More importantly, we have the following formulas which can be checked by direct calculation:

$$\begin{aligned}
\overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2; \\
\overline{z_1 \cdot z_2} &= \bar{z}_1 \cdot \bar{z}_2; \\
\overline{z^n} &= (\bar{z})^n; \\
\bar{\bar{z}} &= z; \\
z \cdot \bar{z} &= a^2 + b^2,
\end{aligned}$$

where in the last line $z = a + bi$. Thus, $z \cdot \bar{z} \geq 0$, and $z \cdot \bar{z} = 0$ if and only if $z = 0$. We set $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$, the *absolute value*, *length* or *modulus* of z.For example, $|2 + i| = \sqrt{5}$. Note that, for all $z_1, z_2 \in \mathbb{C}$,

$$|z_1|^2 |z_2|^2 = z_1 \bar{z}_1 z_2 \bar{z}_2 = z_1 z_2 \bar{z}_1 \bar{z}_2 = (z_1 z_2)(\overline{z_1 z_2}) = |z_1 z_2|^2,$$

and hence $|z_1||z_2| = |z_1 z_2|$. The link between the absolute value and addition is somewhat weaker; there is only the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

If $z \neq 0$, then z has a multiplicative inverse:

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

In terms of real and imaginary parts, this is the familiar procedure of dividing one complex number into another by "rationalizing the denominator:" if at least one of c, d is nonzero, then

$$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right)\left(\frac{c - di}{c - di}\right) = \frac{(a + bi)(c - di)}{c^2 + d^2}.$$

Thus it is possible to divide by any nonzero complex number. For example, to express $(2 + i)/(3 - 2i)$ in the form $a + bi$, we write

$$\frac{2 + i}{3 - 2i} = \left(\frac{2 + i}{3 - 2i}\right)\left(\frac{3 + 2i}{3 + 2i}\right) = \frac{(2 + i)(3 + 2i)}{3^2 + 2^2} = \frac{4 + 7i}{13} = \frac{4}{13} + \frac{7}{13}i.$$

If $z \neq 0$, then z^n is defined for every integer n, including the case $n < 0$, and the formula $\overline{z^n} = (\bar{z})^n$ still holds.

1.2 Complex numbers: geometry

Instead of thinking of a complex number z as $a + bi$, we can identify it with the point $(a, b) \in \mathbb{R}^2$. From this point of view, there is no difference between a complex number and a 2-vector, and we sometimes refer to $\mathbb{C}$ as the *complex plane*. The absolute value $|z|$ is then the same as $\|(a, b)\|$, the distance from the point (a, b) to the origin. Addition of complex numbers then corresponds to vector addition. However, multiplication of complex numbers is more complicated. One way to understand it is to use polar coordinates: if $z = a + bi$, where (a, b) corresponds to the polar coordinates (r, θ), then $r = |z|$ and $a = r\cos\theta$, $b = r\sin\theta$. Thus we may write $z = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$. This is sometimes called the *polar form* of z; $r = |z|$ is, as we have seen, called the *modulus* of z and θ is called the *argument*, sometimes written $\theta = \arg z$. Note that the argument is only well-defined up to an integer multiple of 2π. If $z = r(\cos\theta + i\sin\theta)$, then clearly r is real and nonnegative and $\cos\theta + i\sin\theta$ is a complex number of absolute value one; thus every complex number z is the product of a nonnegative real number times a complex number of absolute value 1. If $z \neq 0$, then this product expression is unique. (What happens if $z = 0$?) For example, the polar form of $1 + i$ is $\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$.

Given two complex numbers z_1 and z_2, with $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, we can ask for the polar form of z_1z_2:

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)) \\ &= r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)), \end{aligned}$$

where we have used the standard addition formulas for sine and cosine. (We will see in a minute where these addition formulas come from.) Thus the modulus of the product is the product of the moduli (this is just the formula $|z_1z_2| = |z_1||z_2|$ which we have already seen), but the really interesting formula is that the arguments add:

$$\arg(z_1z_2) = \arg z_1 + \arg z_2.$$

Of course, this has to be understood as up to possibly adding an integer multiple of 2π.

In this way, we can interpret geometrically the effect of multiplying by a complex number z. If z is real, multiplying by z is just ordinary scalar multiplication and has the usual geometric interpretation. If $z = \cos\theta + i\sin\theta$ has absolute value one, then multiplying a complex number $x + iy$ by

z is the same as rotating the point (x, y) by the angle θ. For a general z, multiplying the complex number $x + iy$ by z is a combination of these two operations: rotation by the angle θ followed by scalar multiplication by the nonnegative real number $|z|$.

Using the formula for multiplication, it is easy to see that if z has polar form $r(\cos\theta + i\sin\theta)$, then

$$\begin{aligned} z^n &= r^n(\cos n\theta + i\sin n\theta); \\ z^{-1} &= r^{-1}(\cos(-\theta) + i\sin(-\theta)) = r^{-1}(\cos\theta - i\sin\theta). \end{aligned}$$

Here the first formula, which is easily proved by mathematical induction, holds for all z and positive integers n, and the second holds for $z \neq 0$. From this it is easy to check that, for $z \neq 0$, the first formula holds for all integers n. This formula is called **De Moivre's Theorem**.

We can use De Moivre's Theorem to find powers and roots of complex numbers. For example, we have seen that $1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$. Thus

$$\begin{aligned} (1+i)^{20} &= (\sqrt{2})^{20}(\cos(20\pi/4) + i\sin(20\pi/4)) \\ &= 2^{10}(\cos(5\pi) + i\sin(5\pi)) = 2^{10}(-1) = -1024. \end{aligned}$$

De Moivre's Theorem can be used to generate identities for $\sin n\theta$ and $\cos n\theta$ via the binomial theorem. For example,

$$\begin{aligned} \cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta). \end{aligned}$$

Equating real and imaginary parts, we see that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, and likewise $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

It is more interesting to find roots. Let $z = r(\cos\theta + i\sin\theta)$ be a complex number, which we assume to be nonzero (since the only n^{th} root of 0 is zero–why?), and let $w = s(\cos\varphi + i\sin\varphi)$. Then $w^n = z$ if and only if $s^n = r$ and $n\varphi = \theta + 2k\pi$ for some integer k. Thus $s = r^{1/n}$, and $\varphi = \theta/n + 2k\pi/n$ for some integer k. But sometimes these numbers will be the same for different values of k: if $\varphi_1 = \theta/n + 2k_1\pi/n$ and $\varphi_2 = \theta/n + 2k_2\pi/n$, then

$$r^{1/n}(\cos\varphi_1 + i\sin\varphi_1) = r^{1/n}(\cos\varphi_2 + i\sin\varphi_2)$$

if and only if φ_1 and φ_2 differ by an integer multiple of 2π, if and only if $2k_1\pi/n$ and $2k_2\pi/n$ differ by an integer multiple of 2π, if and only if n

divides $k_1 - k_2$. Moreover, we can find a complete set of choices by taking the arguments

$$\theta/n, \theta/n + 2\pi/n, \ldots, \theta/n + 2(n-1)\pi/n.$$

Thus we see: *If n is a positive integer, then a nonzero complex number has exactly n distinct n^{th} roots given by the formula above.*

Examples: the two square roots of $i = \cos(\pi/2) + i\sin(\pi/2)$ are

$$\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i;$$
$$\cos\left(\frac{\pi}{4} + \pi\right) + i\sin\left(\frac{\pi}{4} + \pi\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$$

For another example, to find all of the fifth roots of $\sqrt{3} + i$, first write

$$\sqrt{3} + i = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

Thus the fifth roots are given by

$$2^{1/5}\left(\cos\left(\frac{\pi}{30} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{30} + \frac{2k\pi}{5}\right)\right), \quad k = 0, 1, 2, 3, 4.$$

We can apply the above to the complex number $1 = \cos 0 + i\sin 0$. Thus there are exactly n complex numbers z such that $z^n = 1$, called the n^{th} *roots of unity*: namely

$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \qquad k = 0, 1, \ldots, n-1.$$

It is easy to see that, once we have found one n^{th} root w of a nonzero complex number z, then all of the n^{th} roots of z are of the form

$$\left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)w,$$

for $k = 0, 1, \ldots, n-1$, i.e. any two n^{th} roots of a given nonzero complex number differ by multiplying by an n^{th} root of unity.

Warning: the usual rules for fractional exponents that hold for positive real numbers do **not** usually hold for complex roots; this is connected with the fact that there is not in general one preferred n^{th} root of a complex number. For example,

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} \neq \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

1.3 Complex numbers: the complex exponential function

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, we can try to substitute in complex values for x and see what we get. Here we shall just consider the usual power series for the exponential function $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We begin by substituting a purely imaginary complex number it, where t is real. This gives

$$e^{it} = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n} t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} t^{2n+1}}{(2n+1)!},$$

where we have simply broken the sum up into summing over even and odd positive integers. Using $i^{2n} = (-1)^n$, and hence $i^{2n+1} = (-1)^n i$, we see that the sum is equal to

$$e^{it} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \cos t + i \sin t.$$

This beautiful fact is known as **Euler's formula**. For example, $e^{i\pi} = -1$. We can thus write the polar form $r(\cos\theta + i\sin\theta)$ for a complex number as $re^{i\theta}$. Assuming the usual rules for exponents, we can see in another way that the arguments add under multiplication:

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i\theta_1 + i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

In particular, $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Equating real and imaginary parts, we see that this fact is equivalent to the usual addition formulas for the sine and cosine functions, and indeed is perhaps the best way of explaining why these somewhat mysterious looking addition formulas are true.

Euler's formula is a fundamental link between the basic constants of mathematics, e and π, and between the exponential and trigonometric functions. For example, since $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i \sin t$, we see that

$$\cos t = \operatorname{Re} e^{it} = \frac{1}{2}(e^{it} + e^{-it});$$
$$\sin t = \operatorname{Im} e^{it} = \frac{1}{2i}(e^{it} - e^{-it}).$$

Now suppose that we can substitute an arbitrary complex number $z = x + iy$ in the expression for e^x, and that the usual rules for exponentiation apply. Then

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

In particular, if r is a positive real number, then

$$e^{\ln r+i\theta} = r(\cos\theta + i\sin\theta).$$

Thus every nonzero complex number $z = r(\cos\theta + i\sin\theta)$ has a logarithm. In fact the possible solutions to $e^w = z$ are

$$w = \ln r + (\theta + 2n\pi)i, \qquad n \text{ an integer.}$$

We call any of these values a *logarithm* of z and write $w = \log z$. Of course, log is not a well-defined function. Note that, for **real** x, the exponential function e^x is one-to-one and its values are the positive real numbers; hence $\ln x$ is defined for positive x. For **complex** z, the exponential function is no longer one-to-one: $e^{z_1} = e^{z_2}$ exactly when $z_2 = z_2 + 2n\pi i$ for some integer n, and the values of the complex exponential are all nonzero complex numbers. Thus $\log z$ is "defined" for all $z \neq 0$, but only up to adding an arbitrary integer multiple of $2\pi i$.

For example, $\log(-1) = \log(e^{i\pi}) = i\pi + 2n\pi i = (2n+1)\pi i$, for any integer n.

It is then natural to try to define an expression of the form z^α, where z is a nonzero complex number and α is any complex number, by the formula

$$z^\alpha = e^{\alpha \log z}.$$

Since $\log z$ is only well-defined up to adding an integer multiple of $2\pi i$, this says that, given any particular choice of value, say w, for z^α, then $we^{2n\alpha\pi i}$ is also a value for z^α, for every integer n. If $\alpha = 1/k$ for some integer k, or more generally if α is a rational number, then there are only finitely many possible values for the expression z^α; for example, if $\alpha = 1/k$, we just see the k different k^{th} roots of z described earlier. But if α is not rational, then the expression z^α will define infinitely many different complex numbers! For example,

$$\begin{aligned}
2^{\sqrt{2}} &= e^{\sqrt{2}\log 2} = e^{\sqrt{2}(\ln 2 + 2n\pi i)} = e^{\sqrt{2}\ln 2 + 2n\sqrt{2}\pi i)} \\
&= 2^{\sqrt{2}} e^{2n\sqrt{2}\pi i)} = 2^{\sqrt{2}}(\cos(2n\sqrt{2}\pi) + i\sin(2n\sqrt{2}\pi),
\end{aligned}$$

where, in the second line, the expression $2^{\sqrt{2}}$ means the usual real valued expression $e^{\sqrt{2}\ln 2}$, and n is an arbitrary integer. For another example,

$$2^i = e^{i\log 2} = e^{i(\ln 2 + 2n\pi i)} = e^{i\ln 2 - 2n\pi} = e^{-2n\pi}(\cos(\ln 2) + i\sin(\ln 2)).$$

1.4 Homework

1. Write in the form $a + bi$:

   (a) $(2 + i) - (3 + i)$; (b) $(1 + 4i)(2 + 4i)$; (c) $(2 - 3i)(2 + 3i)$.

2. Write in the form $a + bi$:

   (a) $\dfrac{2 + i}{3 + i}$; (b) $\dfrac{1 + 4i}{2 + 8i}$; (c) $\dfrac{2 - 3i}{3 + 2i}$.

3. Write in polar form:

   (a) $1 - \sqrt{3}i$; (b) $-5 + 5i$; (c) πi.

4. Write the complex number $3 - 2i$ in polar form (using inverse trigonometric functions if necessary).

5. (i) Write in the form $a + bi$:

   (a) $e^{-\pi i/4}$; (b) $e^{1+\pi i}$; (c) e^{3+i}.

6. Evaluate (in the form $a + bi$): $(\sqrt{3} - i)^7$; $(1 + i)^9$.

7. Find all complex numbers z such that $z^5 = -2 - 2i$. (You can leave z in polar form.) How many different ones are there?

8. Find all solutions in complex numbers z of the equation $(z + 1)^5 = z^5$. (Note: you should find exactly four different solutions.)

9. What are all possible values (in the form $a + bi$) of the following expressions?

   (a) $\log(1 + i)$ (b) i^i (c) i^e (d) $(1 + i)^\pi$.

   How many are real? Pure imaginary?

   What are all possible values of e^i, interpreting e^i as (a) the value of the complex exponential function on i? (b) as the complex number e raised to the power i?

10. Beginning with the formula $\cos t = \frac{1}{2}(e^{it} + e^{-it})$, find a formula for $\cos^{-1} x$ in terms of log and square roots. (Hint: let $x = \cos t$ and $z = e^{it}$, so that $e^{-it} = 1/z$. Multiply both sides of the above formula by $2z$ and apply the quadratic formula to solve first for z and then for t.)