5.1 Definition of cohomology

Let $X$ be a topological space. We list some basic definitions concerning sheaves on $X$. If $F$ is a sheaf of $R$-modules on $X$, we denote by $\Gamma(X; F)$ the $R$-module of global sections of $F$. For a complex $(C^\bullet, d)$ of sheaves (of $R$-modules) on $X$, the convention will always be that $d$ has degree $+1$. We will usually just write $C^\bullet$ when the differential $d$ is clear from the context. By convention, all complexes of sheaves will be bounded below.

Definition 1. (i) Let $(C^\bullet, d)$ and $(D^\bullet, d)$ be two complexes of sheaves on $X$. A quasi-isomorphism $f: C^\bullet \rightarrow D^\bullet$ is a morphism of complexes of sheaves which induces an isomorphism on the cohomology sheaves. The complexes $(C^\bullet, d)$ and $(D^\bullet, d)$ are quasi-isomorphic if there exists some quasi-isomorphism $f: C^\bullet \rightarrow D^\bullet$.

(ii) Two morphisms $f: C^\bullet \rightarrow D^\bullet$, $g: C^\bullet \rightarrow D^\bullet$ are chain homotopic if there exists an $R$-module homomorphism $T: C^\bullet \rightarrow D^\bullet$ of degree $-1$ such that $f - g = dT +Td$.

(iii) A sheaf $I$ on $X$ is injective if it is an injective object in the abelian category of sheaves of $R$-modules on $X$: given sheaves $F$ and $G$ on $X$ and a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & F & \longrightarrow & G \\
& & f \downarrow & & \\
& & I & & \\
\end{array}
$$

there exists a map $\tilde{f}: G \rightarrow I$ making the diagram commute.

(iv) A resolution $I^\bullet$ of a sheaf $F$ is a complex $I^\bullet$ with $I^k = 0$ for $k < 0$, together with an injection $F \rightarrow I^0$, such that $0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ is exact. In particular, the map $F[0] \rightarrow I^\bullet$ is a quasi-isomorphism, where $F[0]$ denotes the complex with a single term in degree 0 consisting of $F$. An injective resolution $I^\bullet$ of $F$ is a resolution such that $I^k$ is injective for all $k$. 

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(v) More generally, if $C$ is a complex of sheaves, then an injective resolution of $C$ is an injective quasi-isomorphism $f : C \to I$, where $I^k$ is injective for all $k$.

**Lemma 2.**  
(i) Let $F$ be a sheaf on $X$. Then an injective resolution $(I, d)$ of $F$ exists.

(ii) If $I$ is a resolution of $F$ and $J$ is an injective resolution of $F$, then there exists a quasi-isomorphism $f : I \to J$ which is compatible with the two inclusions $F \to I^0$, $F \to J^0$.

(iii) If $I$ and $J$ are two injective resolutions of $F$ and $f : I \to J$ and $g : I \to J$ are two quasi-isomorphisms compatible with the inclusions of $F$ in degree 0, then $f$ and $g$ are chain homotopic.

**Definition 3.** If $F$ is a sheaf of $R$-modules on $X$, let $I$ be an injective resolution of $F$ and define

$$H^i(X; F) = H^i(\Gamma(X; I), d).$$

It is independent of the choice of injective resolution $I$ by the above lemma, in the sense that if $J$ is another such, then there is a canonical isomorphism $H^i(\Gamma(X; I), d) \cong H^i(\Gamma(X; J), d)$. Finally the sheaf cohomology groups $H^i(X; F)$ satisfy the usual properties. In particular, $H^0(X; F) = \Gamma(X; F)$ and, given a short exact sequence $0 \to F' \to F \to F'' \to 0$ of sheaves on $X$, there is a long exact cohomology sequence

$$\cdots \to H^{i-1}(X; F'') \xrightarrow{\delta} H^i(X; F') \to H^i(X; F) \to H^i(X; F'') \xrightarrow{\delta} H^{i+1}(X; F') \to \cdots,$$

where the connecting homomorphisms $\delta$ are also functorial in a natural sense with respect to commutative diagrams of short exact sequences of complexes of sheaves on $X$.

**Remark 4.** The cohomology groups $H^k(X; F)$ are the derived functors of the global section functor $\Gamma$.

In practice, it is hard to work with injective resolutions, and we need other methods for computing cohomology.

**Definition 5.** Let $K$ be a sheaf on $X$. Then $K$ is acyclic if $H^i(X; K) = 0$ for all $i > 0$. An acyclic resolution $(K, d)$ of a sheaf $F$ is a resolution $F \to K$ such that $K^i$ is acyclic for every $i$. 

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Lemma 6. Let $(K', d)$ be an acyclic resolution of a sheaf $F$. Then, for all $i \geq 0$,
\[ H^i(X; F) \cong H^i(\Gamma(X; K'), d). \]

Proof. Since there is an exact sequence
\[ 0 \to F \to K^0 \to K^1, \]
and the global section functor is left exact, we see that
\[ H^0(X; F) = \text{Ker}\{d: H^0(X; K^0) \to H^0(X; K^1)\}. \]
Also, using the exact sequence
\[ 0 \to F \to K^0 \to K^0/F \to 0 \]
and the vanishing of $H^1(X; K^0)$, we see that
\[ H^1(X; F) = \text{Coker}\{d: H^0(X; K^0) \to H^0(X; K^0/F)\}. \]
Moreover, using the exact sequence
\[ 0 \to K^0/F \to K^1 \to K^2, \]
we see that
\[ H^0(X; K^0/F) = \text{Ker}\{d: H^0(X; K^1) \to H^0(X; K^2)\}. \]
Thus $H^1(X; F)$ is isomorphic to
\[ \frac{\text{Ker}\{d: H^0(X; K^1) \to H^0(X; K^2)\}}{\text{Im}\{d: H^0(X; K^0) \to H^0(X; K^1)\}} = H^1(\Gamma(X; K'), d). \]
Finally, for $k \geq 2$, we argue by induction on $k$, assuming inductively that the result has been proved for all $k - 1 \geq 1$ and all sheaves $G$. Then, given the sheaf $F$, there is the induced acyclic resolution of $K^0/F$:
\[ 0 \to K^0/F \to K^1 \to K^2 \to \cdots. \]
Moreover, the long exact cohomology sequence associated to
\[ 0 \to F \to K^0 \to K^0/F \to 0 \]
and the vanishing of $H^i(X; K^0)$ for $i \geq 1$ shows that, for $k \geq 2$,
\[ H^k(X; F) \cong H^{k-1}(X; K^0/F) \cong H^{k-1}(\Gamma(X; K^+1), d) = H^k(\Gamma(X; K'), d). \]
This completes the inductive step. \qed
5.2 Some examples of acyclic sheaves

To make use of Lemma 6, we need to produce some examples of acyclic sheaves. Of course, an injective sheaf $I$ is acyclic, because the complex $I[0]$ which is $I$ in degree zero and zero elsewhere is an injective resolution of $I$. Hence, $H^i(X; I) = 0$ for all $i > 0$. The following gives another very broad class of acyclic sheaves.

**Definition 7.** A sheaf $F$ is flasque if, for every pair of open subsets $U \subseteq V$ of $X$, the restriction morphism $\Gamma(V; F) \to \Gamma(U; F)$ is surjective.

The method of constructing injective resolutions given in Hartshorne shows that, given an arbitrary sheaf $F$, there exists an injective resolution $(I \cdot, d)$ of $F$ such that all of the $I^k$ are flasque. However, this is a somewhat superfluous argument since the following is easy to show directly:

**Lemma 8.** An injective sheaf is flasque. \hfill \Box

A Zorn’s lemma argument shows:

**Lemma 9.** Let $F$ be flasque and suppose that there is an exact sequence of sheaves

$$0 \to F \to A \to B \to 0.$$  

Then the induced map $\Gamma(X; A) \to \Gamma(X; B)$ is surjective, i.e.

$$0 \to H^0(X; F) \to H^0(X; A) \to H^0(X; B) \to 0$$

is exact. \hfill \Box

**Lemma 10.** (i) Given an exact sequence

$$0 \to F' \to F \to F'' \to 0,$$

if $F'$ and $F$ are flasque, then so $F''$.

(ii) Suppose that

$$0 \to F^0 \to F^1 \to \cdots$$

is an exact sequence of flasque sheaves. Then the associated sequence of global sections

$$0 \to \Gamma(X; F^0) \to \Gamma(X; F^1) \to \cdots$$

is also exact. \hfill \Box

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Finally, we have:

**Lemma 11.** An flasque sheaf is acyclic.

*Proof.* There exists an injective sheaf $I$ and an injective homomorphism $F \to I$. Thus we have an exact sequence

$$0 \to F \to I \to I/F \to 0.$$  

By Lemma 9, the map $H^0(X; I) \to H^0(X; I/F)$ is surjective and thus $H^1(X; F) = 0$ since $H^1(X; I) = 0$. Now assume inductively that we have shown, for all $k - 1 \geq 1$, and for all flasque sheaves $G$, that $H^{k-1}(X; G) = 0$. For $k \geq 2$, we have an isomorphism

$$H^k(X; F) \cong H^{k-1}(X; I/F).$$

By (i) of Lemma 10, $I/F$ is flasque. Hence, by the inductive assumption, $H^{k-1}(X; I/F) = 0$ for all $k \geq 2$. Hence $H^k(X; F) = 0$ as well, completing the inductive step. \qed

The usefulness of flasque sheaves is that it enables us to construct a canonical flasque resolution (due to Godement).

**Definition 12.** Let $F$ be a sheaf on $X$, and define $C^0_{Go}(F)$, the sheaf of germs of discontinuous sections of $F$, by: for every open subset $U$ of $X$,

$$C^0_{Go}(F)(U) = \prod_{x \in U} F_x.$$  

If $V \subseteq U$ are two open subsets, there is the obvious restriction map

$$C^0_{Go}(F)(U) \to C^0_{Go}(F)(V),$$

making $C^0_{Go}(F)$ a sheaf (not just a presheaf) of $R$-modules on $X$. Clearly, $C^0_{Go}(F)$ is flasque. There is a natural injection of sheaves $F \to C^0_{Go}(F)$, which associates to a section $s \in F(U)$ the element $(s_x) \in \prod_{x \in U} F_x$. Then we define

$$C^1_{Go}(F) = C^0_{Go}(C^0_{Go}(F)/F),$$

with a natural map

$$d: C^0_{Go}(F) \to C^0_{Go}(F)/F \to C^0_{Go}(C^0_{Go}(F)/F) = C^1_{Go}(F).$$

Inductively, suppose that we have defined $C^r_{Go}(F)$ and the map $C^{r-1}_{Go}(F) \to C^r_{Go}(F)$. Then set

$$C^{r+1}_{Go}(F) = C^0_{Go}(C^r_{Go}(F)/\text{Im} C^{r-1}_{Go}(F)).$$
We then have the composition

\[ d_{\text{Go}} : C_{\text{Go}}^r(F) \to C_{\text{Go}}^r(F)/\text{Im } C_{\text{Go}}^{r-1}(F) \to C_{\text{Go}}^{r+1}(F), \]

with the first map surjective with kernel \( \text{Im } C_{\text{Go}}^{r-1}(F) \) and the second map injective. It follows that \((C_{\text{Go}}(F), d_{\text{Go}})\) is exact except in dimension zero, and is a resolution of \( F \) by flasque sheaves. We will refer to it as the canonical flasque resolution or Godement resolution of \( F \).

We can view \((C_{\text{Go}}(\cdot), d_{\text{Go}})\) as a functor from the abelian category of sheaves on \( X \) to the category of complexes on \( X \), because a homomorphism \( f : F \to G \) defines a morphism of complexes \( \tilde{f} : C_{\text{Go}}(F) \to C_{\text{Go}}(G) \) in a functorial way. Then:

**Lemma 13.** (i) The functor \((C_{\text{Go}}(\cdot), d_{\text{Go}})\) is exact. In other words, given an exact sequence

\[ F' \to F \to F'' \]

of sheaves on \( X \), the sequence of complexes

\[ C_{\text{Go}}(F') \to C_{\text{Go}}(F) \to C_{\text{Go}}(F'') \]

is exact.

(ii) If \( \alpha : F' \to F \) is a homomorphism of sheaves and \( \tilde{\alpha} : C_{\text{Go}}(F') \to C_{\text{Go}}(F) \) is the induced homomorphism, then

\[ \text{Ker } \tilde{\alpha} = C_{\text{Go}}(\text{Ker } \alpha) \quad \text{and} \quad \text{Im } \tilde{\alpha} = C_{\text{Go}}(\text{Im } \alpha). \]

(iii) The functor \((C_{\text{Go}}(\cdot), d_{\text{Go}})\) commutes with taking cohomology. In other words, given a complex of sheaves on \( X \)

\[ \ldots \to F^{i-1} \xrightarrow{\delta} F^i \xrightarrow{\delta} F^{i+1} \xrightarrow{\delta} \ldots, \]

and if \( \tilde{\delta} \) is the induced map \( C_{\text{Go}}(F^i) \to C_{\text{Go}}(F^{i+1}) \), then the corresponding cohomology sheaves satisfy

\[ H^i(C_{\text{Go}}(\cdot), \tilde{\delta}) = C_{\text{Go}}(H^i F). \]

**Proof.** (i) Clearly the sequence

\[ C_{\text{Go}}^0(F') \to C_{\text{Go}}^0(F) \to C_{\text{Go}}^0(F'') \]
is exact. To prove (i), given the inductive construction of \( C_{\text{Go}}(F) \), it suffices to show that the sequence

\[
C_{\text{Go}}^0(F')/F' \to C_{\text{Go}}^0(F)/F \to C_{\text{Go}}^0(F'')/F''
\]

is exact. This is a straightforward exercise.

(ii) We have an exact sequence

\[
0 \to \text{Ker } \alpha \to F' \overset{\alpha}{\to} F.
\]

Using (i), the sequence

\[
0 \to C_{\text{Go}}(\text{Ker } \alpha) \to C_{\text{Go}}(F') \overset{\tilde{\alpha}}{\to} C_{\text{Go}}(F)
\]

is also exact. Thus \( \text{Ker } \tilde{\alpha} = C_{\text{Go}}(\text{Ker } \alpha) \). As for \( \text{Im } \alpha \), we have a commutative diagram

\[
\begin{array}{ccc}
F' & \longrightarrow & \text{Im } \alpha \\
\downarrow \alpha & & \downarrow \\
F & \longrightarrow & F
\end{array}
\]

Thus, using (i) again, there is a corresponding diagram

\[
\begin{array}{ccc}
C_{\text{Go}}(F') & \longrightarrow & C_{\text{Go}}(\text{Im } \alpha) \\
\downarrow \tilde{\alpha} & & \downarrow \\
C_{\text{Go}}(F) & \longrightarrow & C_{\text{Go}}(F)
\end{array}
\]

It then follows easily that the image of \( C_{\text{Go}}(\text{Im } \alpha) \) is contained in \( \text{Im } \tilde{\alpha} \) and that the corresponding homomorphism \( C_{\text{Go}}(\text{Im } \alpha) \to \text{Im } \tilde{\alpha} \) is both injective and surjective, hence an isomorphism.

(iii) This follows easily from (ii).

\( \square \)

Here (ii) and (iii) are general properties of exact functors. The other class of acyclic sheaves we will consider are the fine sheaves.

**Definition 14.** A sheaf \( K \) on \( X \) is fine if, for every open cover \( \{U_\alpha\} \) of \( X \), there exists a family \( \varphi_\alpha : K \to K \) of endomorphisms of \( K \), satisfying

1. The support of \( \varphi_\alpha \) is contained in \( U_\alpha \). In other words, there is a closed subset \( S_\alpha \) of \( U_\alpha \) such that \( \varphi_\alpha|_{X - S_\alpha} = 0 \).

2. (Local finiteness). For all \( x \in X \), there exists an open neighborhood \( V \) of \( x \) such that there are only finitely many \( \alpha \) with \( \varphi_\alpha|_V \neq 0 \).
3. $\sum \varphi_a = \text{Id.}$ (Here the sum is meaningful by (2).)

The primary example of fine sheaves is given by:

**Example 15.** If $M$ is a $C^\infty$ manifold and $C^\infty_M = A^0_M$ denotes the sheaf of rings of $C^\infty$ functions on $M$, then, for every open cover $\{U_\alpha\}$ of $M$, there exists a $C^\infty$ partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$. Then, if $K$ is any sheaf of $C^\infty_M$-modules, multiplication by $f_\alpha$ defines an endomorphism of $K$ satisfying the conditions of the above definition. Thus, such a $K$ is a fine sheaf.

In particular, the sheaves $A^k_M$ of $C^\infty$ differential forms on $M$, as well as the sheaves $A^{p,q}_M$ in case $M$ is a complex manifold, are fine sheaves.

**Proposition 16.** A fine sheaf $K$ is acyclic.

**Proof.** Consider the Godement resolution $C^*_G(K)$. We need to show that the complex $\Gamma(X; C^*_G(K))$ has no cohomology in positive degrees. Given $k > 0$, let $\xi \in \text{Ker}(\Gamma(X; C^k_G(K)) \to \Gamma(X; C^{k+1}_G(K)))$. By local exactness, there exists an open cover $\{U_\alpha\}$ of $X$ such that, for all $\alpha$, there exists an $\eta_\alpha \in C^{k-1}_G(K)(U_\alpha)$ such that $d_G\eta_\alpha = \xi|U_\alpha$. Let $\varphi_\alpha : K \to K$ be a family of endomorphisms of $K$ satisfying (1)–(3) of Definition 14 for the open cover $\{U_\alpha\}$. Since $C^*_G$ is functorial, the $\varphi_\alpha$ lift to endomorphisms $\tilde{\varphi}_\alpha$ of the complex $C^*_G(K)$, and it is easy to check that the $\tilde{\varphi}_\alpha$ satisfy the same conditions as $\varphi_\alpha$. Define

$$\eta = \sum_{\alpha} \tilde{\varphi}_\alpha(\eta_\alpha).$$

Here, $\tilde{\varphi}_\alpha(\eta_\alpha)$ is a priori only defined on $U_\alpha$, but by the support condition it extends over $X$. Also, locally around every point $x$ of $X$, the sum $\sum_{\alpha} \tilde{\varphi}_\alpha(\eta_\alpha)$ is actually a finite sum, and so the section $\eta$ is well defined.

Then we compute (again with the understanding that this computation is meaningful on all sufficiently small open sets):

$$d_G\eta = \sum_{\alpha} d_G\tilde{\varphi}_\alpha(\eta_\alpha) = \sum_{\alpha} \tilde{\varphi}_\alpha(d_G\eta_\alpha) = \sum_{\alpha} \tilde{\varphi}_\alpha(\xi) = \xi.$$ 

Thus $H^k(\Gamma(X; C^*_G(K)), d_G) = 0$ for $k > 0$, so that $H^k(X; K) = 0$ for $k > 0$ and $K$ is acyclic.

**Corollary 17** (de Rham isomorphism). If $M$ is a $C^\infty$ manifold, then $H^k(M; \mathbb{C}) = H^k(A(M), d)$. More generally, if $V$ is a flat vector bundle on $M$, then $H^k(M; V) = H^k(A(M; V), d)$. 

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Proof. By the Poincaré lemma, $(\mathcal{A}_M, d)$ is a resolution of $\mathbb{C}$. Example 15 shows that the $\mathcal{A}_M^k$ are fine sheaves, hence acyclic. By Lemma 6, $H^k(M; \mathbb{C})$ is computed by taking the cohomology of the complex of global sections $\Gamma(M; \mathcal{A}_M) = A(M)$. The case of a flat vector bundle is similar. □

Corollary 18 (Dolbeault isomorphism). If $M$ is a complex manifold, then $H^q(M; \Omega^p_M) = H^p_q(M)$. More generally, if $V$ is a holomorphic vector bundle on $M$, then $\check{H}^q(M; \Omega^p_M \otimes V) = H^p_q(M; V)$.

Proof. The proof is the same as the proof of the de Rham isomorphism, using the $\bar{\partial}$-Poincaré lemma instead of the usual Poincaré lemma. □

5.3 Čech cohomology and Leray’s theorem

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $X$, where we assume the index set $I$ to be totally ordered, and let $F$ be a sheaf on $X$. We define the Čech cohomology of $F$ with respect to the cover $\{U_\alpha\}$ in the usual way: for each finite sequence $\alpha_0 < \alpha_1 < \cdots < \alpha_k$, define

$$U_{\alpha_0, \alpha_1, \ldots, \alpha_k} = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k},$$

and define the Čech complex $\check{C}^\ast(\{U_\alpha\}; F)$ via

$$\check{C}^k(\{U_\alpha\}; F) = \prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_k} \Gamma(U_{\alpha_0, \alpha_1, \ldots, \alpha_k}; F).$$

We define the Čech differential $\delta: \check{C}^k(\{U_\alpha\}; F) \to \check{C}^{k+1}(\{U_\alpha\}; F)$ by: for a section $t = (t_{\alpha_0, \alpha_1, \ldots, \alpha_k})$ of $\check{C}^k(\{U_\alpha\}; F)$,

$$\delta t_{\alpha_0, \alpha_1, \ldots, \alpha_k} = \sum_{i=0}^{k+1} (-1)^i t_{\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}}|U_{\alpha_0, \alpha_1, \ldots, \alpha_{k+1}},$$

where the notation $\hat{\alpha}_i$ means that this term is omitted. It is straightforward to check that $\delta \circ \delta = 0$. Then we define $\check{H}^\ast(\{U_\alpha\}; F)$ to be the cohomology of the complex $(\check{C}^\ast(\{U_\alpha\}; F), \delta)$. By the sheaf axioms, it is easy to check that $\check{H}^0(\{U_\alpha\}; F) = \Gamma(X; F)$.

There is also a sheaf version: define

$$\check{C}^k(\{U_\alpha\}; F) = \prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_k} (j_{\alpha_0, \alpha_1, \ldots, \alpha_k})_* (F|U_{\alpha_0, \alpha_1, \ldots, \alpha_k}),$$

where $j_{\alpha_0, \alpha_1, \ldots, \alpha_k}: U_{\alpha_0, \alpha_1, \ldots, \alpha_k} \to X$ is the inclusion. There is also a differential $\delta: \check{C}^k(\{U_\alpha\}; F) \to \check{C}^{k+1}(\{U_\alpha\}; F)$ which is defined by analogy with $\delta: \check{C}^k(\{U_\alpha\}; F) \to \check{C}^{k+1}(\{U_\alpha\}; F)$. A combinatorial argument using the sheaf axioms shows:
Proposition 19. The complex \((\tilde{\mathcal{C}}^k(\{U_\alpha\}; F), \delta)\) is a resolution of \(F\). \(\square\)

Note that \(\Gamma(X; \tilde{\mathcal{C}}(\{U_\alpha\}; F)) = \tilde{\mathcal{C}}(\{U_\alpha\}; F)\) (compatibly with \(\delta\)).

Lemma 20. If \(F\) is flasque, then \(\check{H}^k(\{U_\alpha\}; F) = 0\) for \(k > 0\).

Proof. Let \(U\) be an open subset of \(X\), and let \(i: U \to X\) be the inclusion. An easy argument shows that, if \(F\) is a flasque sheaf on \(X\), then \(F|U\) is a flasque sheaf on \(U\), and that, if \(G\) is a flasque sheaf on \(U\), then \(i_* G\) is a flasque sheaf on \(X\). Thus, the sheaves \(\tilde{\mathcal{C}}^k(\{U_\alpha\}; F)\) are flasque, so that \((\tilde{\mathcal{C}} \cdot (\{U_\alpha\}; F), \delta)\) is a flasque resolution of \(F\). It follows that

\[ H^k(X; F) = H^k(\Gamma(\tilde{\mathcal{C}}^k(\{U_\alpha\}; F), \delta)) = \check{H}^k(\{U_\alpha\}; F). \]

Thus \(\check{H}^k(\{U_\alpha\}; F) = 0\) for \(k > 0\). \(\square\)

By (ii) of Lemma 2, if \(I\) is an injective resolution of \(F\), there is an induced morphism of complexes \(\tilde{\mathcal{C}}^k(\{U_\alpha\}; F) \to I\). Taking global sections, there is a morphism of complexes \(\mathcal{C}(\{U_\alpha\}; F) \to \Gamma(X; I)\) and hence an induced homomorphism

\[ \check{H}(\{U_\alpha\}; F) \to H(X; F). \]

With an appropriate notion of refinements, we can define

\[ \check{H}(X; F) = \lim_{\to} \check{H}(\{U_\alpha\}; F). \]

The homomorphisms \(\check{H}(\{U_\alpha\}; F) \to H(X; F)\) fit together to give a homomorphism \(\check{H}(X; F) \to H(X; F)\).

Theorem 21. For every open cover, \(\check{H}^0(\{U_\alpha\}; F) \cong H^0(X; F)\). Moreover, \(\check{H}^1(X; F) \cong H^1(X; F)\) via the natural homomorphism. If \(X\) is paracompact, then, for every \(k\), \(\check{H}^k(X; F) \cong H^k(X; F)\). \(\square\)

The really computationally useful version of \(\check{C}\)ech cohomology is the following:

Theorem 22 (Leray’s theorem). Let \(F\) be a sheaf on \(X\) and let \(\{U_\alpha\}\) be an open cover such that, for every sequence \(\alpha_0 < \alpha_1 < \cdots < \alpha_k\) and for every \(k > 0\), \(H^k(U_{\alpha_0, \alpha_1, \ldots, \alpha_k}; F) = 0\). Then \(\check{H}(\{U_\alpha\}; F) \to H(X; F)\) is an isomorphism.
Proof. Choose a flasque resolution $K \cdot$ of $F$ or more generally any resolution which has the property that $K \cdot$ and $K \cdot|U_{\alpha_0,\alpha_1,\ldots,\alpha_k}$ are acyclic for all $\alpha_0, \alpha_1, \ldots, \alpha_k$, and consider the double complex $(\mathcal{C}(\{U_{\alpha}\}; K^{\cdot}), \delta, d)$, where $\delta$ is the Čech differential and $d$ is induced by the differential on $K^{\cdot}$. For the associated single complex $s(\mathcal{C}(\{U_{\alpha}\}; K^{\cdot}), D)$, with $D = \delta + (-1)^p d$, we can consider the two spectral sequences associated to the double complex $\mathcal{C}(\{U_{\alpha}\}; K^{\cdot})$. For the one with $E^{p,q}_1$ given by taking the cohomology first with respect to $\delta$, we get

$$E^{p,q}_1 = \tilde{H}^q(\{U_{\alpha}\}; K^p) = \begin{cases} 0, & \text{if } q > 0; \\ \Gamma(X; K^p), & \text{if } q = 0, \end{cases}$$

where we have used Lemma 20 to conclude that $\tilde{H}^q(\{U_{\alpha}\}; K^p) = 0$ for $q > 0$. Thus the $E_2 = E_\infty$ term is $H^p(\Gamma(X; K^{\cdot}), (-1)^p d) = H^p(X; F)$.

Looking at the other spectral sequence, the $E_1$ term is the $d$-cohomology of the complex

$$\mathcal{C}^p(\{U_{\alpha}\}; K^0) \to \mathcal{C}^p(\{U_{\alpha}\}; K^1) \to \cdots.$$

Since $K^{\cdot}|U_{\alpha_0,\alpha_1,\ldots,\alpha_k}$ is flasque, it is an acyclic resolution of $F|U_{\alpha_0,\alpha_1,\ldots,\alpha_k}$. Then $H^p(U_{\alpha_0,\alpha_1,\ldots,\alpha_k}; F)$ is the cohomology of the complex $\Gamma(U_{\alpha_0,\alpha_1,\ldots,\alpha_k}; K^{\cdot})$. But then the $d$-cohomology of $\mathcal{C}^p(\{U_{\alpha}\}; K^0)$ in degree $q$ is

$$\prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_p} H^q(U_{\alpha_0,\alpha_1,\ldots,\alpha_p}; F).$$

Since $H^q(U_{\alpha_0,\alpha_1,\ldots,\alpha_p}; F) = 0$ for $q > 0$, the $d$-cohomology of the complex $\mathcal{C}(\{U_{\alpha}\}; K^{\cdot})$ is just $\mathcal{C}(\{U_{\alpha}\}; F)$ and the $E_2$ term is $\tilde{H}^p(\{U_{\alpha}\}; F)$. Comparing, we see that $\tilde{H}^p(\{U_{\alpha}\}; F) \cong H^p(X; F)$. \qed

Remark 23. (i) Note that, with our sign conventions, the proof identifies $\tilde{H}^p(\{U_{\alpha}\}; F)$ with the cohomology of the complex $(\Gamma(X; K^{\cdot}), (-1)^p d)$, and in fact there is an inclusion $(\Gamma(X; K^{\cdot}), (-1)^p d) \to (s(\mathcal{C}^p(\{U_{\alpha}\}; K^{\cdot}), D)$ which is a quasi-isomorphism by the above proof. Of course, the cohomology of $(\Gamma(X; K^{\cdot}), (-1)^p d)$ is the same as the cohomology of $(\Gamma(X; K^{\cdot}), d)$ and indeed the complexes $(\Gamma(X; K^{\cdot}), (-1)^p d)$ and $(\Gamma(X; K^{\cdot}), d)$ are isomorphic by the map $\Gamma(X; K^{\cdot}) \to \Gamma(X; K^{\cdot})$ which is $(-1)^{p(p-1)/2} \text{Id}$ in degree $p$.

(ii) Even without the hypothesis that $H^k(U_{\alpha_0,\alpha_1,\ldots,\alpha_k}; F) = 0$ for $k > 0$, the proof exhibits a homomorphism

$$E^{p,0}_2 = \tilde{H}^p(\{U_{\alpha}\}; F) \to E^{p,0}_\infty \to H^p(X; F)$$

which is the homomorphism described in the remarks before the statement of Theorem 21.
**Definition 24.** Given a sheaf $F$ on $X$, an open cover which satisfies the hypotheses of Leray’s theorem will be called a **Leray cover** for $F$.

To use Leray’s theorem, we need in general a class of sheaves and Leray covers for all of the sheaves in the given class. There are three broad classes where this applies:

1. The class of constant sheaves and contractible open subsets (for example open subsets which are homeomorphic to convex subsets of $\mathbb{R}^n$).

2. The class of quasicoherent sheaves and affine open subsets.

3. The class of coherent analytic sheaves and Stein spaces (spaces biholomorphic to closed analytic subspaces of $\mathbb{C}^N$ for some $N$, for example polydisks or more generally strictly pseudoconvex domains).

As another application, recall that we previously wanted to compare a class in $\check{H}^1(C; \mathcal{O}_C)$ which came from a class in $H^1(C; \mathcal{O}_C)$ with a corresponding class in $\check{H}^0(\Gamma(\mathcal{X}; A^1))$. Let us discuss a very general procedure of which this is a special case: $X$ is a topological space, $\{U_\alpha\}$ is an open cover of $X$, $F$ is a sheaf on $X$, and $(A, d)$ is an acyclic resolution of $F$. More precisely, we assume that $A|_{U_{\alpha_0, \alpha_1, \ldots, \alpha_k}}$ is acyclic for every open subset $U_{\alpha_0, \alpha_1, \ldots, \alpha_k}$. Note that Leray’s theorem then implies that

$$\check{H}^q(\{U_\alpha\}; A^k) = 0$$

for all $q > 0$ and all $k$. We want to pass from an element of $\check{H}^1(\{U_\alpha\}; F)$ to an element of $\check{H}^1(\Gamma(\mathcal{X}; A^1), d)$.

**Claim 25.** The recipe for doing so is as follows: given an element $h_{\alpha\beta} \in \check{C}^1(\{U_\alpha\}; F)$ inducing a class $\xi \in \check{H}^1(\{U_\alpha\}; F)$, let $t_{\alpha\beta} \in \check{C}^1(\{U_\alpha\}; A^0)$ be its image, and write $t_{\alpha\beta} = s_\alpha - s_\beta$ for some $s = (s_\alpha) \in \check{C}^0(\{U_\alpha\}; A^0)$, which is possible since $\check{H}^1(\{U_\alpha\}; A^0) = 0$. Then $\xi$ corresponds to the image of the element $ds_\alpha = ds_\beta \in \check{H}^0(X; A^1) = H^0(X; A^1)$ in $\check{H}^1(\Gamma(\mathcal{X}; A^1), d)$, at least up to a sign.

**Proof.** Consider the double complex $(\check{C}(\{U_\alpha\}; A^1), \delta, d)$, where $\delta$ is the Čech differential and $d$ is induced by the differential on $A$. We have the associated single complex $(s(\check{C}(\{U_\alpha\}; A^1), D))$, with $D = \delta + (-1)^p d$. We will examine this complex directly, as a concrete way of understanding the associated spectral sequences. Note that, for $(c_\alpha) \in \check{C}^0(\{U_\alpha\}; A^0)$,

$$D(c_\alpha) = (c_\alpha - c_\beta, dc_\alpha) \in \check{C}^1(\{U_\alpha\}; A^0) \oplus \check{C}^0(\{U_\alpha\}; A^1),$$
and, for \((b_{\alpha\beta}, \varphi_\alpha) \in \check{C}^1(\{U_\alpha\}; A^0) \oplus \check{C}^0(\{U_\alpha\}; A^1),\)
\[D(b_{\alpha\beta}, \varphi_\alpha) = (\delta(b_{\alpha\beta}), -db_{\alpha\beta} + (\varphi_\alpha - \varphi_\beta), -d\varphi_\alpha) \in \check{C}^2(\{U_\alpha\}; A^0) \oplus \check{C}^1(\{U_\alpha\}; A^1) \oplus \check{C}^0(\{U_\alpha\}; A^2).\]

We can then see directly that
\[H^1(s(\check{C}^\cdot(\{U_\alpha\}; A^\cdot)), D) = H^1(\Gamma(X; A^\cdot), d) = H^1(X; F)\]

as follows: if \(D(b_{\alpha\beta}, \varphi_\alpha) = 0,\) then \(\delta(b_{\alpha\beta}) = 0.\) Since \(\check{H}^1(\{U_\alpha\}; A^0) = 0,\) we can write \(b_{\alpha\beta} = \delta(c_\alpha)\) for some \((c_\alpha) \in \check{C}^0(\{U_\alpha\}; A^0),\) unique up to adding a \(\sigma \in H^0(X; A^0).\) Thus, after modifying \((b_{\alpha\beta}, \varphi_\alpha)\) by \(D(c_\alpha),\) we can assume that \(b_{\alpha\beta} = 0\) for all \(\alpha, \beta,\) and that \(\varphi_\alpha\) satisfies: \(\varphi_\alpha = \varphi_\beta\) for all \(\alpha, \beta,\) hence \(\varphi_\alpha\) is the restriction of \(\varphi \in \Gamma(X; A^1)\) such that \(d\varphi = 0.\) Hence there is a representative for the class \((b_{\alpha\beta}, \varphi_\alpha)\) of the form \((0, \varphi),\) and it is unique up to adding \(D\sigma = (0, d\sigma)\) for an arbitrary \(\sigma \in \Gamma(X; A^0).\) This identifies \(H^1(s(\check{C}^\cdot(\{U_\alpha\}; A^\cdot)), D)\) with \(H^1(\Gamma(X; A^\cdot), d).\)

Going back to the original construction, given a class \(\xi \in \check{H}^1(\{U_\alpha\}; F)\) represented by \(b_{\alpha\beta} \in \check{C}^1(\{U_\alpha\}; F),\) with image \(t_{\alpha\beta} \in \check{C}^1(\{U_\alpha\}; A^0)\) and such that \(t_{\alpha\beta} = s_\alpha - s_\beta\) for some \(s = (s_\alpha) \in \check{C}^0(\{U_\alpha\}; A^0),\) we have
\[D(s_\alpha) = t_{\alpha\beta} + ds_\alpha \in \check{C}^1(\{U_\alpha\}; A^0) \oplus \check{C}^0(\{U_\alpha\}; A^1).\]

Also \(Dt_{\alpha\beta} = ds_\alpha - ds_\beta = 0,\) and likewise \(D(ds_\alpha) = 0.\) Thus, \(t_{\alpha\beta}\) and \(-ds_\alpha\) both define classes in \(H^1(s(\check{C}^\cdot(\{U_\alpha\}; A^\cdot)), D),\) and these classes are equal. Tracing through the various identifications shows that \(t_{\alpha\beta}\) is the image of \(\xi \in \check{H}^1(\{U_\alpha\}; F)\) under the map \(\check{H}^1(\{U_\alpha\}; F) \to H^1(X; F),\) and that it is equal to the class corresponding to \(-ds_\alpha.\)

5.4 Hypercohomology

We begin by collecting some basic results about injective resolutions of complexes.

**Proposition 26.** Let \(K^\cdot\) be a complex of sheaves on \(X,\) let \(f: K^\cdot \to L^\cdot\) be an injective quasi-isomorphism, and let \(g: K^\cdot \to I^\cdot\) be a morphism of complexes, where \(I^\cdot\) is injective. Then there exists an extension of \(g,\ i.e.\ a\ morphism \(\tilde{g}: L^\cdot \to I^\cdot\) with \(g = \tilde{g} \circ f,\) and any two such extensions of \(g\) are chain homotopic.

Recall that an injective resolution of a complex \(K^\cdot\) is an injective quasi-isomorphism of complexes \(f: K^\cdot \to I^\cdot,\) where \(I^\cdot\) is injective.
Proposition 27. Let $K^\cdot$ be a complex of sheaves on $X$.

(i) An injective resolution $f: K^\cdot \rightarrow I^\cdot$ exists.

(ii) If $f: K^\cdot \rightarrow I^\cdot$ and $g: K^\cdot \rightarrow J^\cdot$ are two injective resolutions, then $I^\cdot$ and $J^\cdot$ are quasi-isomorphic, via an quasi-isomorphism $\phi: I^\cdot \rightarrow J^\cdot$ such that $\phi \circ f = g$.

(iii) With $I^\cdot, J^\cdot, f, g$ as in (ii), any two quasi-isomorphisms $\phi_1: I^\cdot \rightarrow J^\cdot$ and $\phi_2: I^\cdot \rightarrow J^\cdot$ such that $\phi_i \circ f = g$, $i = 1, 2$, are chain homotopic.

Definition 28. For $K^\cdot$ a complex of sheaves on $X$, we define the hypercohomology $H^\cdot(X; K^\cdot)$ to be the cohomology of the complex $(\Gamma(X; I^\cdot), d)$, where $I^\cdot$ is an injective resolution of $K^\cdot$. By the previous proposition, $H^\cdot(X; K^\cdot)$ does not depend on the choice of an injective resolution, up to a canonical isomorphism. Its properties are similar to those of ordinary cohomology: it is functorial, and given a short exact sequence of complexes of sheaves, there is a long exact sequence of hypercohomology groups, with functorial connecting homomorphisms.

For example, if $K^\cdot = K[0]$ is a single sheaf $K$ in dimension 0, then $H^0(X; K[0]) = H^0(X; K)$.

As before, we would like to compute hypercohomology by using acyclic sheaves.

Proposition 29. Let $K^\cdot$ be a complex of sheaves on $X$, and let $f: K^\cdot \rightarrow L^\cdot$ be a quasi-isomorphism, where the complex $L^\cdot$ is acyclic. We call such a quasi-isomorphism an acyclic resolution of $K^\cdot$. Then

$$H^\cdot(X; K^\cdot) \cong H^\cdot(\Gamma(X; L^\cdot), d).$$

Note: in the above we do not need to assume that $f$ is injective. It is then easy to show:

Corollary 30. If $f: K^\cdot \rightarrow L^\cdot$ is a quasi-isomorphism of complexes of sheaves, then $f$ induces an isomorphism

$$H^\cdot(X; K^\cdot) \cong H^\cdot(X; L^\cdot).$$

We can also use the Godement resolutions of each $K^i$ to construct an acyclic resolution of the complex $K^\cdot$. In fact, we get a double complex $(C^\cdot_{Go}(K^\cdot), d_{Go}, d)$, where $d_{Go}$ is the differential $C^i_{Go}(K^\cdot) \rightarrow C^{i+1}_{Go}(K^\cdot)$ coming from the Godement resolution for $K^\cdot$ and $d: C^i_{Go}(K^j) \rightarrow C^i_{Go}(K^{j+1})$ is the differential induced by the functoriality of the Godement resolution. Note that, in this case, $d_{Go}$ and $d$ commute, so if we want to form the associated single complex $(s(C^\cdot_{Go}(K^\cdot)), d)$, we have to use $D = d_{Go} + (-1)^p d$. 

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Lemma 31. The associated single complex \((s(C_{Go}(K^-)), D)\) is an acyclic resolution of \(K^\cdot\).

Proof. Quite generally, we have the following lemma (whose statement and proof could have come much earlier):

Lemma 32. Suppose that \(K^\cdot\) is a complex, that \(A^\cdot\) is a double complex, and that \(K^\cdot \to s(A^\cdot)\) is a morphism of complexes such that, for each \(p\), \((A^p\cdot, d^p)\) is a resolution of \(K^p\). Then \(K^\cdot \to s(A^\cdot)\) is a quasi-isomorphism.

Proof. Working in the abelian category of sheaves on \(X\), we look at the spectral sequence of the double complex \(s(A^\cdot)\) whose \(E_1\) term is \(E_1^{p,q} = H^q(X; K^p)\). By assumption, this is 0 for \(q > 0\) and \(H^0_0(A^p\cdot) = K^p\). Viewing \(K^\cdot\) as a filtered complex of sheaves with the trivial filtration in degree zero, the morphism \(K^\cdot \to s(A^\cdot)\) is then a morphism of filtered complexes inducing an isomorphism on the \(E_1\) terms of the corresponding spectral sequences. Hence \(K^\cdot \to s(A^\cdot)\) is a quasi-isomorphism. \(\square\)

Returning to the proof of Lemma 31, we see that \(K^\cdot \to s(C_{Go}(K^-))\) is a resolution of \(K^\cdot\), and it is acyclic since the terms of \(C_{Go}(K^-)\) are flasque. \(\square\)

Corollary 33. \(\mathbb{H}(X; K^-) \cong H^\cdot(\Gamma(X; s(C_{Go}(K^-))), D)\).

Looking now at the double complex of global sections \(\Gamma(X; s(C_{Go}(K^-)))\), with its two differentials \(d_{Go}\) and \(d\), we have the following:

Theorem 34. There are two spectral sequences abutting to \(\mathbb{H}(X; K^-)\). The first has \(E_1\) term given by

\[ E_1^{p,q} = H^q(X; K^p), \]

and the differential \(d_1\) is induced by \(d:\ K^p \to K^{p+1}\). The second spectral sequence has \(E_2\) term given by

\[ E_2^{p,q} = H^p(X; \mathcal{H}^q(K^-)). \]

Proof. The first statement follows by looking at the spectral sequence which has \(E_0^{p,q} = \Gamma(X; C_{Go}^p(K^q))\) and \(d_0 = d_{Go}\). Then \(E_1^{p,q} = H^q(X; K^p)\).

For the second spectral sequence, consider the spectral sequence which has \(E_0^{p,q} = \Gamma(X; C_{Go}^p(K^q))\) and \(d_0 = d\) (up to a sign). For the \(E_1\) term, we have to compute the cohomology of the complex \((\Gamma(X; C_{Go}^p(K^-)), d)\). An easy argument using Lemma 13 shows that

\[ H^q(\Gamma(X; C_{Go}^p(K^-), d)) = \Gamma(X; \mathcal{H}^q(C_{Go}^p(K^-), d)) = \Gamma(X; C_{Go}^p(\mathcal{H}^qK^-)), \]

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with the differential equal to $d_{Go}$. Then the $E_2$ term satisfies
\[ E_2^{p,q} = H^p(X; \mathcal{H}^q(K^-)) \]
as claimed. \hfill \Box

**Remark 35.** We can recover the above theorem in the general context of injective resolutions by using Cartan-Eilenberg resolutions.

There is also a Čech version of hypercohomology: given a complex $K^-$ of sheaves and an open cover $\{ U_\alpha \}$ of $X$, we can define \( \check{\mathbb{H}} \left( \{ U_\alpha \}; K^- \right) \) by taking the single complex associated to the double complex $\check{C}^- \left( \{ U_\alpha \}; K^- \right)$ with the two commuting differentials $\delta$ and $d$ ($\delta$ is the Čech differential and $d$ is the differential induced by the differential on $K^-$) and total differential $D = \delta + (-1)^p d$. Note that, as in Remark 23, the inclusion $\Gamma(X; K^-) \subseteq \check{C}^0 \left( \{ U_\alpha \}; K^- \right)$ defines an inclusion of complexes

\[
(\Gamma(X; K^-), (-1)^p d) \subseteq (s(\check{C}^- \left( \{ U_\alpha \}; K^- \right)), D).
\]

A straightforward spectral sequence argument shows:

**Proposition 36.** If $\{ U_\alpha \}$ is a Leray cover for $K^p$ for every $p$, then
\[
\check{\mathbb{H}} \left( \{ U_\alpha \}; K^- \right) \cong H^\bullet(X; K^-).
\]

Finally, we discuss the hypercohomology of filtered complexes of sheaves. Suppose that $K^-$ is a complex of sheaves on $X$, with a decreasing filtration $F^\cdot K^-$ by subcomplexes of of sheaves. Then there is an induced filtration of the Godement resolution, $F^\cdot C_{Go}(K^-)$, by defining

\[
F^p C_{Go}(K^-) = \text{Im} \{ C_{Go}(F^p K^-) \to C_{Go}(K^-) \}.
\]

Note that $C_{Go}$ preserves injections. Also, by the exactness of the functor $C_{Go}$,
\[
\text{gr}^F_p C_{Go}(K^-) = C_{Go}(\text{gr}^F_p K^-).
\]

There is then an induced filtration on $\Gamma(X; s(C_{Go}(K^-)))$ and hence a spectral sequence with
\[
E_1^{p,q} = H^{p+q}(X; \text{gr}^F_p K^-) \implies H^{p+q}(X; K^-).
\]

Of course, we can also work this out via injective resolutions of the appropriate kind (filtered injective resolutions, cf. Stacks 13.26).
Example 37. There are two general filtrations on a complex $K^\cdot$ of sheaves:

1. The **trivial or naive or stupid filtration** (French: *filtration bête*):

   \[
   \sigma^{\geq p}K^q = \begin{cases} 
   0; & \text{if } q < p; \\
   K^q; & \text{if } q \geq p.
   \end{cases}
   \]

   Thus the associated graded complex $\text{gr}^\sigma K^\cdot$ is the complex $K^p[-p]$ with a single nonzero term $K^p$ in degree $p$ (and hence all differentials are 0).

2. The **canonical filtration**, which is an increasing filtration defined by:

   \[
   \tau_{\leq p}K^q = \begin{cases} 
   0, & \text{if } q > p; \\
   \text{Ker}\{d: K^p \to K^{p+1}\}, & \text{if } q = p; \\
   K^q; & \text{if } q < p.
   \end{cases}
   \]

   Thus the associated graded complex $\text{gr}^\tau K^\cdot$ is the complex $K^{p-1}/\text{Ker } d \to \text{Ker } d$,

   where the first $\text{Ker } d$ is the kernel of $d: K^{p-1} \to K^p$ and the second is the kernel of $d: K^p \to K^{p+1}$, and the two terms are located in degrees $p-1$ and $p$ respectively. Note that the map $\text{Ker}\{d: K^p \to K^{p+1}\} \to (\mathcal{H}^pK^\cdot)[-p]$ defines a quasi-isomorphism from $\text{gr}^\tau K^\cdot$ to the complex $(\mathcal{H}^pK^\cdot)[-p]$ (viewed as a complex with the single nonzero term $\mathcal{H}^pK^\cdot$ in degree $p$).

Each of the above filtrations defines a corresponding spectral sequence. The spectral sequence arising from the filtration $\sigma^{\geq p}K^\cdot$ has

\[
E_1^{p,q} = H^{p+q}(X; K^p[-p]) = H^q(X; K^p).
\]

In fact, it is easy using the definitions to identify this spectral sequence with the first hypercohomology spectral sequence of Theorem 34.

For the canonical filtration, we first have to reindex by $-p$ because $\tau_{\leq p}K^\cdot$ is an increasing filtration. Once we do so, we get a spectral sequence with $E_1$ term

\[
E_1^{p,q} = H^{p+q}(X; \text{gr}_\tau K^\cdot) = H^{p+q}(X; (\mathcal{H}^{-p})K^\cdot[p]) = H^{2p+q}(X; \mathcal{H}^{-p}K^\cdot).
\]

Up to a shift in indices, this is the same as the $E_2$ term of the second hypercohomology spectral sequence. Note $d_1: E_1^{p,q} \to E_1^{p+1,q}$ is a map

\[
H^{2p+q}(X; \mathcal{H}^{-p}K^\cdot) \to H^{2p+2+q}(X; \mathcal{H}^{-p-1}K^\cdot),
\]
which can be identified with \( d_2 \) for the second hypercohomology spectral sequence. And in fact one can show that the two spectral sequences agree after an appropriate shift.

### 5.5 Applications to complex manifolds and schemes

Let \( M \) be a complex manifold. By the holomorphic Poincaré lemma, the complex of sheaves \( (\Omega_M, d) \) is a resolution of \( \mathbb{C} \). Thus:

**Theorem 38.** For all \( k \), \( H^k(M; \mathbb{C}) \cong \mathbb{H}^k(M; \Omega_M) \). \( \square \)

By Theorem 34, there is a spectral sequence with \( E_1^{p,q} = H^q(M; \Omega^p_M) \) abutting to \( \mathbb{H}(M; \Omega_M) = H^k(M; \mathbb{C}) \). Of course, we already know of such a spectral sequence, the Hodge spectral sequence. The following proposition states, somewhat informally, that these are the same spectral sequence:

**Proposition 39.** There is a natural identification of the hypercohomology spectral sequence for \( \mathbb{H}^k(M; \Omega_M) \) with the Hodge spectral sequence.

**Proof.** We have seen that we can identify the hypercohomology spectral sequence with the spectral sequence arising from the filtration \( \sigma \geq p \Omega_M \). On the other hand, by Lemma 32, we have a quasi-isomorphism of complexes \( \Omega_M \to s(A_M) \), and the \( \bar{\partial} \)-Poincaré lemma implies that this induces a quasi-isomorphism of filtered complexes \( (\Omega_M, \sigma \geq p) \to (s(A_M), \mathcal{F}) \), where \( \mathcal{F} = \bigoplus_{r \geq p} A^r \). Hence we get a morphism of spectral sequences which is an isomorphism on the \( E_1 \) terms, hence an isomorphism on \( E_r \) for all \( r \).

In particular, for \( M \) a smooth projective variety, or more generally the compact complex manifold associated to a smooth scheme proper over \( \text{Spec} \mathbb{C} \), the above spectral sequence degenerates at \( E_1 \).

Let \( X \) be a smooth scheme over \( \text{Spec} \mathbb{C} \), not necessarily proper, and let \( X^{\text{an}} \) be the corresponding complex manifold. We can define the coherent algebraic sheaves \( \Omega^i_X/\mathbb{C} \) in the Zariski topology, as well as the de Rham differential \( d \). Note however that \( d \) is not \( O_X \)-linear, for the sheaves \( \Omega^i_X/\mathbb{C} \) or the analytic sheaves \( \Omega^i_X^{\text{an}} \). We can then consider the “algebraic” hypercohomology, i.e. the hypercohomology \( \mathbb{H}^i_{\text{Zar}}(X; \Omega^j_X/\mathbb{C}) \) computed with respect to the Zariski topology.
For formal reasons, we have a homomorphism of hypercohomology groups

\[ \mathbb{H}_{\text{Zar}}(X; \Omega_{X/\mathbb{C}}) \to \mathbb{H}(X^\text{an}; \Omega_{X^\text{an}}). \]

In fact, there is a morphism of filtered complexes \( \Omega_{X/\mathbb{C}} \to \pi_* \Omega_{X^\text{an}} \), where \( \pi: X^\text{an} \to X \) is the identity map, but where \( X^\text{an} \) has the usual (Hausdorff) topology but \( X \) has the Zariski topology. Then we have a morphism \( \mathbb{H}_{\text{Zar}}(X; \pi_* \Omega_{X^\text{an}}) \to \mathbb{H}(X^\text{an}; \Omega_{X^\text{an}}) \), which we shall describe in more detail in remarks on the Leray spectral sequence. The filtrations on all complexes are the trivial filtration \( \sigma \geq p \). Thus there is a morphism of the corresponding spectral sequences. For the \( E_1 \) term, this morphism is the corresponding morphism

\[ H^q_{\text{Zar}}(X; \Omega^p_{X/\mathbb{C}}) \to H^q(X^\text{an}; \Omega^p_{X^\text{an}}). \]

As a consequence, we obtain:

**Theorem 40.** Suppose that \( X \) is a smooth projective variety, or more generally is proper over \( \text{Spec} \mathbb{C} \). Then the natural map

\[ \mathbb{H}_{\text{Zar}}(X; \Omega_{X/\mathbb{C}}) \to \mathbb{H}(X^\text{an}; \Omega_{X^\text{an}}) \]

is an isomorphism.

**Proof.** In this case, by GAGA for \( X \) projective, and by Grothendieck’s generalization to the proper case, \( H^q_{\text{Zar}}(X; \Omega^p_{X/\mathbb{C}}) \to H^q(X^\text{an}; \Omega^p_{X^\text{an}}) \) is an isomorphism for every \( p, q \). Hence the map on the \( E_1 \) terms of the hypercohomology spectral sequence is an isomorphism, so the same is true for the abutments. \( \square \)

In fact, Theorem 40 holds without any assumption of properness. The main point is the following:

**Theorem 41** (Grothendieck’s algebraic de Rham theorem). Let \( X \) be a smooth affine variety over \( \mathbb{C} \). Then

\[ H^k(X^\text{an}; \mathbb{C}) \cong H^k(\Gamma(X; \Omega_{X/\mathbb{C}}), d). \]  \( \square \)

Note that Theorem 40 implies in particular that, in case \( X \) is proper over \( \text{Spec} \mathbb{C} \), the hypercohomology spectral sequence for \( \mathbb{H}_{\text{Zar}}(X; \Omega_{X/\mathbb{C}}) \) degenerates at \( E_1 \).

One way to interpret Theorem 40 is that de Rham cohomology can be defined purely algebraically. In particular, suppose that \( k \) is a subfield of \( \mathbb{C} \), for example a number field, and that \( X \) is a scheme defined over \( k \), and
for simplicity proper over Spec $k$. Let $X_C = X \times_{\text{Spec} \, k} \text{Spec} \, \mathbb{C}$. Then we can consider the finite dimensional $k$-vector space $\mathbb{H}^2_{\text{Zar}}(X; \Omega_{X/k}^1)$, and general results imply that

$$\mathbb{H}^2_{\text{Zar}}(X; \Omega_{X/k}) \otimes_k \mathbb{C} \cong \mathbb{H}^2_{\text{Zar}}(X_C; \Omega_{X_C}^1) \cong \mathbb{H}^2(X^\text{an}_C; \Omega_{X^\text{an}_C}^1) \cong H^2(X_C; \mathbb{C}).$$

In other words, $H^k(X_C; \mathbb{C})$ is the complexification of a $k$-vector space. But even if $k = \mathbb{Q}$, the rational structure induced on $H^k(X_C; \mathbb{C})$ via this isomorphism is very different from the rational structure given by $H^k(X_C; \mathbb{C}) \cong H^k(X_C; \mathbb{Q}) \otimes \mathbb{Q}$. This difference accounts for the mysterious powers of $2\pi\sqrt{-1}$ in our discussion of Tate twists and polarizations. For example, suppose that $L = \mathcal{O}_X(D)$ is the line bundle on $X$ associated to the divisor $D$, defined over $k$. Then the integral cohomology class of $D$, or equivalently $c_1(L)$, is given as follows: take the transition functions $h_{\alpha\beta}$ of $L$ with respect to an open cover $\{U_\alpha\}$ of $X^\text{an}_C$, where we can assume that the $U_\alpha \cap U_\beta$ are simply connected. Then

$$c_1(L) = \frac{1}{2\pi\sqrt{-1}} \delta \log h_{\alpha\beta},$$

where $\delta$ is the Čech coboundary, a class in $H^2(\{U_\alpha\}; \mathbb{Z})$ which maps to an element of $H^2(X^\text{an}_C; \mathbb{Z})$. However, using instead a cover by Zariski open subsets $V_{ij}$ such that $L|_{V_{ij}}$ is (algebraically) trivial, with transition functions $g_{ij} \in \mathcal{O}_X^*$, then clearly $dg_{ij}/g_{ij} \in H^1(X; \Omega_{X/k}^1)$. In fact, there is an analogue $c(L)$ of the first Chern class $c_1(L)$ with values in $\mathbb{H}^2_{\text{Zar}}(X; \Omega_{X/k}^1)$, more precisely in $F^1\mathbb{H}^2_{\text{Zar}}(X; \Omega_{X/k}^1)$, which we can see as follows: the map $\mathcal{O}_X^* \to \Omega_{X/k}^1$ defined by $h \mapsto dh/h$ gives rise to a morphism of complexes

$$\mathcal{O}_X^*[-1] \to \sigma^{\geq 1}\Omega_{X/k}^1.$$

Here as usual $\mathcal{O}_X^*[-1]$ means the complex with a single term in degree one, and hence trivial differential, and we get a morphism of complexes because $dh/h$ is a closed form. There is thus an induced map on hypercohomology

$$c: \mathbb{H}^2_{\text{Zar}}(X; \mathcal{O}_X^*[-1]) = \mathbb{H}^1_{\text{Zar}}(X; \mathcal{O}_X^*) \to \mathbb{H}^2_{\text{Zar}}(X; \sigma^{\geq 1}\Omega_{X/k}^1) \to F^1\mathbb{H}^2_{\text{Zar}}(X; \Omega_{X/k}^1),$$

which is compatible with the natural map $F^1\mathbb{H}^2_{\text{Zar}}(X; \Omega_{X/k}^1) \to H^1(X; \Omega_{X/k}^1)$. However, the image of $c(L)$ in $F^1\mathbb{H}^2_{\text{Zar}}(X_C; \Omega_{X_C/k}^1) \cong H^2(X; \mathbb{C})$ only agrees with the integral class $c_1(L)$ after multiplying by the factor of $1/2\pi\sqrt{-1}$, which measures the difference between the $k$-structure and the integral (or rational) structure on $H^2(X^\text{an}_C; \mathbb{C})$.  

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5.6 The Leray spectral sequence

In this section, we describe the Leray spectral sequence, one of the most important examples of the Grothendieck spectral sequence for the derived functors of a composition of two functors.

Let $f: X \to Y$ be a continuous map of topological spaces. If $F$ is a sheaf on $X$, we have the direct image sheaf $f_*F$. The functor $f_*$ from sheaves on $X$ to sheaves on $Y$ is left exact, and we define the higher direct image sheaves $R^if_*$ by: choose an injective resolution $F \to I$. Then $R^if_*$ is the $i$th cohomology sheaf of the complex $f_*I$. As before, this definition is independent of the choice of an injective resolution, and the $R^if_*$ are the derived functors of $f_*$. A straightforward argument (cf. Hartshorne) shows that $R^if_*F$ is the sheaf associated to the presheaf on $Y$ defined by:

$$U \mapsto H^i(f^{-1}(U); F).$$

As in the case of cohomology, we would like to compute higher direct images using a broader class of sheaves than just injective ones.

Definition 42. A sheaf $K$ is $f_*$-acyclic if $R^if_*K = 0$ for all $i > 0$. For example, an injective sheaf is $f_*$-acyclic. An $f_*$-acyclic resolution $F \to K$ of $F$ is a resolution such that $K^i$ is $f_*$-acyclic for all $i$.

The argument of Lemma 6 then shows:

Lemma 43. Let $K \to$ be an $f_*$-acyclic resolution of $F$. Then, for all $i \geq 0$, $R^if_*F = \mathcal{H}^i f_*K$.

Similarly, the arguments of Lemma 11 show:

Lemma 44. If $K$ is a flasque sheaf on $X$, then $K$ is $f_*$-acyclic. Thus a flasque resolution is $f_*$-acyclic.

Also, it is an easy exercise to show:

Lemma 45. If $K$ is a flasque sheaf on $X$, then $f_*K$ is a flasque sheaf on $Y$.

Theorem 46 (Leray spectral sequence). Let $F$ be a sheaf on $X$. There is a spectral sequence abutting to $H^*(X; F)$ with $E_2$ term given by

$$E_2^{p,q} = H^p(Y; R^qf_*F).$$
Proof. Choose a flasque resolution \( C^\cdot \) of \( F \), for example the Godement resolution, and consider the hypercohomology \( \mathbb{H}(Y; f_* C^\cdot) \) of the direct image complex \( f_* C^\cdot \). The first hypercohomology spectral sequence has \( E_1 \) term

\[
E_1^{p,q} = H^q(Y; f_* C^p).
\]

By Lemma 45, \( f_* C^p \) is flasque for all \( p \). Thus \( H^q(Y; f_* C^p) = 0 \) for \( q > 0 \). So \( E_1^{p,q} = 0 \) if \( q > 0 \), and \( E_1^{p,0} = H^0(Y; f_* C^p) = H^0(X; C^p) \). The differential \( d_1: E_1^{0,0} \to E_1^{0,1} \) is the differential induced by \( d: C^p \to C^{p+1} \). Thus, \( E_2^{0,0} = H^0(X; F) = E_\infty^{0,0} \), the spectral sequence degenerates at \( E_2 \), and the filtration on \( \mathbb{H}^p(Y; f_* C^\cdot) \) is trivial. Hence \( \mathbb{H}^p(Y; f_* C^\cdot) \cong H^p(X; F) \).

The second spectral sequence for \( \mathbb{H}(Y; f_* C^\cdot) \) has \( E_2 \) term

\[
E_2^{p,q} = H^p(Y; \mathcal{H}^q f_* C^p).
\]

By Lemmas 43 and 44, \( \mathcal{H}^q f_* C^p \cong R^q f_* F \). Putting the results about the two spectral sequences together gives the statement of the theorem. \( \square \)

**Remark 47.** The edge homomorphisms induce homomorphisms \( E_2^{0,0} = H^0(Y; f_* F) \to H^0(X; F) \) and \( H^p(X; F) \to E_2^{0,p} = H^0(Y; R^p f_* F) \). In fact, we can see the homomorphism \( H^p(Y; f_* F) \to H^p(X; F) \) quite directly. More generally, let \( G \) be a sheaf on \( Y \). Then there are homomorphisms of cohomology groups \( H^p(Y; G) \to H^p(X; f^{-1}G) \) defined as follows. First, it is easy to check that \( f^{-1}C^p_{Go}(G) \cong C^p_{Go}(f^{-1}G) \). Moreover, \( f^{-1} \) is an exact functor, so we get: \( f^{-1}C^p_{Go}(G) \cong C^p_{Go}(f^{-1}G) \) is a flasque resolution of \( f^{-1}G \). The homomorphism \( \Gamma(Y; G) \to \Gamma(X; f^{-1}G) \) extends to a morphism

\[
\Gamma(Y; C^p_{Go}(G)) \to \Gamma(X; f^{-1}C^p_{Go}(G)) \cong \Gamma(X; C^p_{Go}(f^{-1}G)).
\]

Taking cohomology yields \( H^p(Y; G) \to H^p(X; f^{-1}G) \). Finally, setting \( G = f_* F \) and using the canonical homomorphism \( f^{-1}f_* F \to F \) gives the desired homomorphism \( H^p(Y; f_* F) \to H^p(X; F) \).

We can also generalize the above to hyperdirect images: Let \( K^\cdot \) be a complex of sheaves on \( X \). Choosing an injective resolution \( K^\cdot \to \Gamma \), we define

\[
\mathbb{R}^i f_* K^\cdot = \mathcal{H}^i f_* \Gamma.
\]

Arguments similar to those given for the proof of Theorem 34 show:

**Theorem 48.** There are two spectral sequences abutting to \( \mathbb{R}^k f_* K^\cdot \). The first has \( E_1 \) term given by

\[
E_1^{p,q} = R^q f_* K^p,
\]
and the differential \(d_1\) is induced by \(d: K^p \to K^{p+1}\). The second spectral sequence has \(E_2\) term given by

\[
E_2^{p,q} = R^q f_* \mathcal{H}^q K. \quad \square
\]

Similar but slightly more involved arguments prove the existence of the Leray spectral sequence for hypercohomology:

**Theorem 49.** Let \(K\) be a complex of sheaves on \(X\). There is a spectral sequence abutting to \(\mathbb{H}(X; K)\) with \(E_2\) term given by

\[
E_2^{p,q} = H^p(Y; R^q f_* F). \quad \square
\]

### 5.7 Products

We give a very brief treatment of products in Čech cohomology and Čech hypercohomology.

**Definition 50.** Suppose that \(F_1\) and \(F_2\) are two sheaves on \(F\) and that we have an \(R\)-bilinear map from \(F_1 \times F_2\) to a third sheaf \(G\), i.e. a homomorphism \(F_1 \otimes F_2 \to G\). We denote the value on sections \(s_1\) of \(F_1\) and \(s_2\) of \(F_2\) by \(s_1 \cdot s_2\). Let \(\{U_\alpha\}\) be an open cover of \(X\). Define the cup product

\[
\cup: \check{C}^n(\{U_\alpha\}; F_1) \times \check{C}^m(\{U_\alpha\}; F_2) \to \check{C}^{n+m}(\{U_\alpha\}; G)
\]

by the formula

\[
(s \cup t)_{\alpha_0, \alpha_1, \ldots, \alpha_k} = s_{\alpha_0, \alpha_1, \ldots, \alpha_n} \cdot t_{\alpha_n, \alpha_1, \ldots, \alpha_{n+m}}.
\]

A straightforward calculation shows that

\[
\delta(s \cup t) = \delta(s) \cup (-1)^n s \cup \delta(t).
\]

Hence \(\cup\) induces an \(R\)-bilinear map, also denoted \(\cup\):

\[
\cup: \check{H}^n(\{U_\alpha\}; F_1) \times \check{H}^m(\{U_\alpha\}; F_2) \to \check{H}^{n+m}(\{U_\alpha\}; G).
\]

Now suppose we have a bilinear morphism of **complexes** of sheaves, \(\varphi: A \otimes K \to L\). This means that we have a graded \(R\)-bilinear map \(\varphi\), i.e. \(\varphi(A^p \otimes K^q) \subseteq L^{p+q}\), which we again denote by \(\varphi(s, t) = s \cdot t\), satisfying

\[
d(s \cdot t) = ds \cdot t + (-1)^p s \cdot dt.
\]

Equivalently, the induced map \(A \otimes K \to L\) is a morphism of complexes, where \(A \otimes K\) has the tensor product differential \(d(s \otimes t) = ds \otimes t + (-1)^p s \otimes dt\).
A basic example of this is as follows: Given a complex $K'$, define the complex $\text{Hom}^k(K', K')$ as follows:

$$\text{Hom}^k(K', K') = \{ \varphi \in \text{Hom}(K', K') : \varphi(K^p) \subseteq K^{p+k} \text{ for all } p \}.$$ 

Thus by definition $d \in \text{Hom}^1(K', K')$. Define the differential $d$ by: if $\varphi \in \text{Hom}^k(K', K')$, then

$$d \varphi = d \circ \varphi + (-1)^{k+1} \varphi \circ d \in \text{Hom}^{k+1}(K', K').$$

Equivalently, $d \varphi = [d, \varphi]$, where $[\cdot, \cdot]$ is the graded Lie bracket on $\text{Hom}(K', K')$.

We have the evaluation map $\text{Hom}^p(K', K') \otimes K^q \rightarrow K^r$, which by definition satisfies

$$\text{Hom}^p(K', K') \otimes K^q \subseteq K^{p+q}.$$ 

Also, for sections $\varphi \in \text{Hom}^p(K', K')$ and $s \in K^q$,

$$(d \varphi)(t) + (-1)^p \varphi(dt) = d(\varphi(t)) + (-1)^{p+1} \varphi(dt) + (-1)^p \varphi(dt) = d(\varphi(t)).$$

Hence the evaluation morphism is a morphism of complexes in the above sense.

Returning to the general picture of a morphism $\varphi: A \otimes K' \rightarrow L'$ of complexes in the above sense, we would like to define analogous cup products

$$\hat{C}^r({\{U_\alpha\}}; A') \otimes \hat{C}^s({\{U_\alpha\}}; K') \rightarrow \hat{C}^r({\{U_\alpha\}}; L') :$$

For $s \in \hat{C}^p({\{U_\alpha\}}; A^q)$ and $t \in \hat{C}^r({\{U_\alpha\}}; K^s)$, set

$$s \bowtie t = (-1)^{rq}s \bowtie t.$$ 

Then $\bowtie: \hat{C}^p({\{U_\alpha\}}; A^q) \otimes \hat{C}^r({\{U_\alpha\}}; K^s) \rightarrow \hat{C}^{p+q}({\{U_\alpha\}}; L^{r+s})$ and one checks that

$$D(s \bowtie t) = D(s) \bowtie t + (-1)^n s \bowtie D(t),$$

in case $s$ is homogeneous of degree $n$. Thus there is an induced cup product

$$\hat{H}^r({\{U_\alpha\}}; A') \otimes \hat{H}^s({\{U_\alpha\}}; K') \rightarrow \hat{H}^{r+s}({\{U_\alpha\}}; L').$$

Returning to the original setting of products, suppose that $F_1$ and $F_2$ are sheaves on $X$ and that there is a bilinear map $F_1 \otimes F_2 \rightarrow G$. It is a
general fact that we can also define pairings of complexes for the Godement resolution, or more generally for injective resolutions, giving a cup product

\[ H^n(X; F_1) \otimes H^m(X; F_2) \to H^{n+m}(X; G). \]

Moreover, if \( \{ U_\alpha \} \) is a Leray cover for \( F_1, F_2, \) and \( G \), then these cup products agree under the isomorphisms. Finally, suppose that the cohomology of \( F_1, F_2, \) and \( G \) can be computed by complexes of sheaves \( A, K, \) which are \textit{acyclic} with respect to the \( p \)-fold intersections of the open sets \( \{ U_\alpha \} \) and such that there is a pairing of complexes \( \varphi : A \otimes K \to L \) as above which is compatible with the inclusions \( F_1 \to A, F_2 \to K, G \to L \). Then the cup product

\[ \mathbb{H}((U_\alpha); A') \otimes \mathbb{H}((U_\alpha); K') \to \mathbb{H}((U_\alpha); L') \]

reduces to a pairing

\[ H(\Gamma(X; A'), d) \otimes H(\Gamma(X; K'), d) \to H(\Gamma(X; L'), d) \]

which is the induced pairing on global sections, and it is straightforward to check that it also agrees with the cup product

\[ \mathcal{H}^p((U_\alpha); F_1) \times \mathcal{H}^q((U_\alpha); F_2) \to \mathcal{H}^{p+q}((U_\alpha); G) \]

as defined above.

For example, if \( M \) is a manifold, taking \( F_1 = F_2 = G = \mathbb{C} \) and \( A = K = L = \mathbb{A}_M \) with wedge product, we can identify wedge product on de Rham cohomology with usual cup product on Čech cohomology (and on singular cohomology with a little more work). A similar argument, for a complex manifold \( M \) and a Stein cover \( \{ U_\alpha \} \), identifies cup product

\[ \mathcal{H}^q((U_\alpha); \Omega^p_M) \times \mathcal{H}^s((U_\alpha); \Omega^r_M) \to \mathcal{H}^{q+s}((U_\alpha); \Omega_M^{p+r}) \]

with wedge product of forms

\[ H^p,q_\partial(M) \otimes H^r,s_\partial(M) \to H^{p+r,q+s}_\partial(M). \]

We can similarly add a holomorphic vector bundles to the mix, but shall not write down the general statement here.