Characters I

Throughout, \( G \) denotes a finite group.

1 The character of a representation

**Definition 1.1.** Let \( V \) (or \( \rho_V \)) be a \( G \)-representation. Then the character \( \chi_V \) (or \( \chi_{\rho_V} \)) of \( V \) is the function \( \chi_V : G \to \mathbb{C} \) defined by:

\[
\chi_V(g) = \text{Tr} \rho_V(g).
\]

Note that, for all \( g \in G \), \( \chi_V(g) \) is a sum of roots of unity.

**Example 1.2.**

1. If \( V \) is the trivial representation (i.e. \( \dim V = 1 \) and \( \rho_V(g) = \text{Id} \) for all \( g \in G \)), then \( \chi_V(g) = 1 \) for all \( g \in G \). We sometimes write \( \chi_1 \) or just 1 for this character.

2. More generally, if \( \dim V = 1 \) and \( \rho_V(g)(v) = \lambda(g) \), where \( \lambda : G \to \mathbb{C}^* \) is a homomorphism, then \( \chi_V = \lambda \). For example, for the one-dimensional representation \( V \) of \( \mathbb{Z}/n\mathbb{Z} \) on \( \mathbb{C} \) for which \( \lambda(k) = e^{2\pi i k/n} \), we have \( \chi_V(k) = \lambda(k) = e^{2\pi i k/n} \).

3. The group \( D_n \) is generated by elements \( \sigma \) and \( \tau \), where \( \sigma \) is a counterclockwise rotation by the angle \( 2\pi k/n \) and \( \tau \) is reflection in the \( x \)-axis. For the representation of \( D_n \) on \( V = \mathbb{C}^2 \) for which

\[
\rho_V(\sigma^k) = A_{2\pi k/n} = \begin{pmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{pmatrix};
\]

\[
\rho_V(\sigma^k \tau) = B_{2\pi k/n} = \begin{pmatrix} \cos 2\pi k/n & \sin 2\pi k/n \\ -\sin 2\pi k/n & -\cos 2\pi k/n \end{pmatrix},
\]

we clearly have:

\[
\chi_V(\sigma^k) = 2 \cos 2\pi k/n; \quad \chi_V(\sigma^k \tau) = 0.
\]
4. For the 2-dimensional representation $V$ of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ described previously, we have
\[
\chi_V(1) = 2; \quad \chi_V(-1) = -2; \quad \chi_V(\pm i) = \chi_V(\pm j) = \chi_V(\pm k) = 0.
\]

5. For the standard representation of $S^n$ on $\mathbb{C}^n$, the corresponding character $\chi$ satisfies:
\[
\chi(\sigma) = \text{the number of } i \text{ such that } \sigma(i) = i.
\]
Hence, if $\sigma = \gamma_1 \cdots \gamma_k$ is a product of disjoint cycles $\gamma_i$ of lengths $\ell_i > 1$, then
\[
\chi(\sigma) = n - \sum_{i=1}^{k} \ell_i.
\]

6. More generally, if $X$ is a $G$-set and $\rho_{\mathbb{C}[X]}$ is the corresponding permutation representation on $\mathbb{C}[X]$, with character $\chi_{\mathbb{C}[X]}$, then
\[
\chi_{\mathbb{C}[X]}(g) = \#(X^g),
\]
where $X^g$ is the fixed set of $g$: $X^g = \{x \in X : g \cdot x = x\}$. In particular, if $X = G$, where $G$ acts on itself by left multiplication, then $\mathbb{C}[G]$ is the regular representation. We write $\chi_{\text{reg}}$ for the character $\chi_{\mathbb{C}[G]}$. For the left multiplication action, given $g \in G$ and $x \in G$, $g$ fixes $x$, i.e. $gx = x \iff g = 1$, and the element 1 fixes every $x \in G$. In other words, $G^g = \emptyset$ if $g \neq 1$ and $G^1 = G$. Thus:
\[
\chi_{\text{reg}}(g) = \begin{cases} 
\#(G), & \text{if } g = 1; \\
0, & \text{if } g \neq 1.
\end{cases}
\]

We list some basic properties of characters.

1. If $V$ is the trivial representation, then $\chi_V(g) = 1$ for all $g \in G$, i.e. $\chi_V$ is the constant function 1.

2. For every representation $V$,
\[
\chi_V(1) = \dim V = \deg \rho_V.
\]
This follows since $\rho_V(1) = \text{Id}$ corresponds to the $d \times d$ identity matrix $I$, where $d = \dim V = \deg \rho_V$, and $\text{Tr} I = d$.

3. For all $g, h \in G$,
\[
\chi_V(gh^{-1}) = \chi_V(g).
\]
This follows since by definition
\[
\chi_V(gh^{-1}) = \text{Tr}(\rho_V(h) \circ \rho_V(g) \circ \rho_V(h)^{-1}) = \text{Tr} \rho_V(g) = \chi_V(g).
\]
4. For every $g \in G$, 

$$\chi_V(g^{-1}) = \overline{\chi_V(g)}.$$ 

To see this, note that, for every $g \in G$, $\rho_V(g) \in \text{Aut} V$ has finite order. Hence $\rho_V(g)$ is diagonalizable and its eigenvalues are roots of unity, in particular complex numbers of absolute value 1. By a homework problem, 

$$\chi_V(g^{-1}) = \text{Tr} \rho_V(g^{-1}) = \text{Tr} \rho_V(g)^{-1} = \overline{\text{Tr} \rho_V(g)} = \overline{\chi_V(g)}.$$ 

Next, we see how the character behaves with respect to the standard constructions of linear algebra: Suppose that $V_1$, $V_2$, and $V$ are $G$-representations. Then:

1. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$

This is an immediate consequence of the formula $\text{Tr}(F_1 \oplus F_2) = \text{Tr} F_1 + \text{Tr} F_2$. Applying this inductively, we see that $\chi_{V_1 \oplus \cdots \oplus V_k} = \chi_{V_1} + \cdots + \chi_{V_k}$. Also, if we let $V^n = V \oplus \cdots \oplus V$, then $\chi_{V^n} = n \chi_V$.

2. $\chi_{V^*} = \overline{\chi_V}$

To see this, first recall that $\text{Tr} F = \text{Tr} F^*$. Now $\rho_{V^*} = (\rho_V)^{-1}$, and hence

$$\chi_{V^*}(g) = \text{Tr}(\rho_{V^*}(g)) = \text{Tr}((\rho_V(g)^{-1})^*) = \text{Tr}(\rho_V(g)^{-1}) = \overline{\text{Tr}(\rho_V(g))} = \overline{\chi_V(g)}.$$ 

3. $\chi_{\text{Hom}(V_1, V_2)} = \overline{\chi_{V_1}} \chi_{V_2}$

The argument for this is similar to the argument for (2): Suppose that $F_1 \in \text{Hom}(V_1, V_1)$ and that $F_2 \in \text{Hom}(V_2, V_2)$. Then $(F_2)_* \circ (F_1)^* \in \text{Hom}(V_1, V_2)$. We have, by a homework problem, 

$$\text{Tr}((F_2)_* \circ (F_1)^*) = \text{Tr}(F_1)/(\text{Tr} F_2).$$

By definition, $\rho_{\text{Hom}(V_1, V_2)} = (\rho_{V_2})_* \circ (\rho_{V_1}^{-1})^*$. Thus, 

$$\chi_{\text{Hom}(V_1, V_2)}(g) = \text{Tr} \rho_{\text{Hom}(V_1, V_2)}(g)) = \text{Tr}((\rho_{V_2}(g))_* \circ (\rho_{V_1}(g)^{-1})^*)
= \text{Tr}(\rho_{V_2}(g)) \text{Tr}(\rho_{V_1}(g)^{-1})) = \overline{\chi_{V_1}(g)} \chi_{V_2}(g).$$

4. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}$

This follows from the fact that $\text{Tr}(F_1 \otimes F_2) = (\text{Tr} F_1)/(\text{Tr} F_2)$.

In particular, we see that the sum, product, and complex conjugates of characters are characters.
2 Orthogonality relations

There are many identities involving characters which are called orthogonality relations. To begin, recall that, given a $G$-representation $V$, we have defined a projection map $p: V \rightarrow V^G$ by

$$p(v) = \frac{1}{\#(G)} \sum_{g \in G} \rho_V(g)(v).$$

We also know that $\text{Tr} p = \dim V^G$ by general linear algebra results about traces. Computing the trace of $p$ in two different ways then gives

$$\dim V^G = \frac{1}{\#(G)} \sum_{g \in G} \chi_V(g).$$

Applying this formula to $\text{Hom}(V_1, V_2)$ gives:

$$\dim \text{Hom}^G(V_1, V_2) = \frac{1}{\#(G)} \sum_{g \in G} \overline{\chi_{V_1}(g)} \chi_{V_2}(g) = \frac{1}{\#(G)} \sum_{g \in G} \chi_{V_1}(g) \overline{\chi_{V_2}(g)}.$$

Finally, if $V_1$ and $V_2$ are irreducible, and using Schur’s lemma, this becomes:

**Proposition 2.1.** If $V_1$ and $V_2$ are irreducible, then

$$\frac{1}{\#(G)} \sum_{g \in G} \chi_{V_1}(g) \overline{\chi_{V_2}(g)} = \dim \text{Hom}^G(V_1, V_2) = \begin{cases} 1, & \text{if } V_1 \cong V_2; \\ 0, & \text{if } V_1 \text{ is not isomorphic to } V_2. \end{cases}$$

It’s convenient to introduce the $G$-invariant positive definite Hermitian inner product on the vector space $\mathbb{C}(G)$, viewed as the space $L^2(G)$ of functions $f: G \rightarrow \mathbb{C}$:

$$\langle f_1, f_2 \rangle = \frac{1}{\#(G)} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Thus we can restate the above proposition as: If $V_1$ and $V_2$ are irreducible, then

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \text{Hom}^G(V_1, V_2) = \begin{cases} 1, & \text{if } V_1 \cong V_2; \\ 0, & \text{if } V_1 \text{ is not isomorphic to } V_2. \end{cases}$$
Corollary 2.2. Write \( V \cong V_{1}^{m_{1}} \oplus \cdots \oplus V_{k}^{m_{k}} \), where \( V_{i} \) is irreducible, \( V_{i} \) is not isomorphic to \( V_{j} \) if \( i \neq j \), and \( V_{i}^{m_{i}} \) is shorthand for the direct sum \( V_{i} \oplus \cdots \oplus V_{i} \), \( m_{i} \) times.

Then

\[
\langle \chi_{V}, \chi_{V} \rangle = \sum_{i=1}^{k} m_{i}^2
\]

In particular, \( V \) is irreducible \( \iff \langle \chi_{V}, \chi_{V} \rangle = 1 \).

Proof. By our formulas, \( \chi_{V} = \sum_{i=1}^{k} m_{i} \chi_{V_{i}} \). Then, expanding out the inner product gives

\[
\langle \chi_{V}, \chi_{V} \rangle = \sum_{i,j} m_{i} m_{j} \langle \chi_{V_{i}}, \chi_{V_{j}} \rangle.
\]

As \( \langle \chi_{V_{i}}, \chi_{V_{j}} \rangle = 1 \) if \( i = j \) and 0 otherwise, the sum becomes \( \sum_{i} m_{i}^2 \) as claimed. The final statement follows since, if the \( m_{i} \) are positive integers, then \( \sum_{i=1}^{k} m_{i}^2 = 1 \iff k = 1 \) and \( m_{1} = 1 \), which clearly happens \( \iff V \) is irreducible.

Corollary 2.3. Write \( V \cong V_{1}^{m_{1}} \oplus \cdots \oplus V_{k}^{m_{k}} \) as in the previous corollary. Let \( W \) be an irreducible representation. Then

\[
\langle \chi_{W}, \chi_{V} \rangle = \begin{cases} m_{i}, & \text{if } W \cong V_{i}; \\ 0, & \text{if } W \text{ is not isomorphic to } V_{i} \text{ for any } i. \end{cases}
\]

Hence two representations \( V \) and \( V' \) are isomorphic \( \iff \chi_{V} = \chi_{V'} \). In other words:

The character \( \chi_{V} \) determines the representation \( V \) up to isomorphism.

Proof. We have seen that \( \chi_{V} = \sum_{i=1}^{k} m_{i} \chi_{V_{i}} \), and hence

\[
\langle \chi_{W}, \chi_{V} \rangle = \sum_{i=1}^{k} m_{i} \langle \chi_{W}, \chi_{V_{i}} \rangle.
\]

But \( \langle \chi_{W}, \chi_{V_{i}} \rangle = 1 \iff W \cong V_{i} \), which can happen for at most one \( i \) by the assumption that \( V_{i} \) is not isomorphic to \( V_{j} \) if \( i \neq j \). Hence \( \langle \chi_{W}, \chi_{V} \rangle = m_{i} \) if \( W \cong V_{i} \) and \( \langle \chi_{W}, \chi_{V} \rangle = 0 \) if \( W \) is not isomorphic to any \( V_{i} \).
To see the final statement, clearly, if $V \cong V'$, then $\chi_V = \chi_{V'}$. Conversely, suppose that $\chi_V = \chi_{V'}$. Write $V \cong V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$ as above. Then

$$\langle \chi_V, \chi_{V'} \rangle = \langle \chi_V, \chi_V \rangle = m_i,$$

and $\langle \chi_W, \chi_{V'} \rangle = \langle \chi_W, \chi_V \rangle = 0$ if $W$ is an irreducible representation not isomorphic to $V_i$ for some $i$. Hence $V' \cong V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$, and thus $V' \cong V$.

**Definition 2.4.** If $V$ is a representation and $W$ is an irreducible representation, we define the *multiplicity of $W$ in $V$* to be the nonnegative integer $\langle \chi_W, \chi_V \rangle$.

### 3 The regular representation

Our goal now will be to apply the results of the previous section to the regular representation $\mathbb{C}[G]$, whose character $\chi_{\mathbb{C}[G]} = \chi_{\text{reg}}$ we have computed. In fact, $\chi_{\text{reg}}(1) = #(G)$ and $\chi_{\text{reg}}(g) = 0$ if $g \neq 1$.

**Proposition 3.1.** Let $W$ be an irreducible representation. Then

$$\langle \chi_W, \chi_{\text{reg}} \rangle = \dim W$$

**Proof.** By definition and the above remarks,

$$\langle \chi_W, \chi_{\text{reg}} \rangle = \frac{1}{#(G)} \sum_{g \in G} \chi_W(g) \overline{\chi_{\text{reg}}(g)} = \frac{\chi_W(1) \cdot #(G)}{#(G)} = \dim W.$$  

\[\square\]

**Corollary 3.2.** Write

$$\mathbb{C}[G] \cong W_1^{d_1} \oplus \cdots \oplus W_h^{d_h},$$

where the $W_i$ are irreducible and, for $i \neq j$, $W_i$ is not isomorphic to $W_j$. Then:

(i) $d_i = \dim W_i$.

(ii) Every irreducible representation of $V$ is isomorphic to $W_i$ for a unique $i$. In particular, there are only finitely many irreducible $G$-representations up to isomorphism.
Proof. The first statement follows from the previous proposition and Corollary 2.3. The second follows similarly, since if $W$ is an irreducible representation, then $\left< \chi_W, \chi_{\text{reg}} \right> = \dim W > 0$ and hence $W \cong W_i$ for some $i$. \hfill \Box

**Corollary 3.3.** If $W_1, \ldots, W_h$ are the finitely many distinct irreducible $G$-representations up to isomorphism and $d_i = \dim W_i$, then

$$
\sum_{i=1}^{h} d_i^2 = \#(G)
$$

$$
\sum_{i=1}^{h} d_i \chi_{W_i}(g) = \begin{cases} 
\#(G), & \text{if } g = 1; \\
0, & \text{if } g \neq 1.
\end{cases}
$$

Proof. We prove the second identity first. Since $\mathbb{C}[G] \cong W_1^{d_1} \oplus \cdots \oplus W_h^{d_h}$,

$$
\chi_{\text{reg}} = \sum_{i=1}^{h} d_i \chi_{W_i}.
$$

The result then follows from our calculation of $\chi_{\text{reg}}$. The first identity is then a consequence, since, for every $i$, $\chi_{W_i}(1) = \dim W_i = d_i$. \hfill \Box