1. Let $k = \bar{k}$ be an algebraically closed field, let $F \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d > 0$. Let $X = V_+(F) \subseteq \mathbb{P}_k^n$ scheme-theoretically, i.e. $X$ is the closed subscheme of $\mathbb{P}_k^n$ associated to the graded ideal $(F)$. Show that the singular locus $X_{\text{sing}} = X - X_{\text{reg}} = \{ x \in X : \mathcal{O}_{X, x} \text{ is not regular} \}$ is the closed subset $V_+(F, \partial F/\partial x_0, \ldots, \partial F/\partial x_n)$. Using Euler’s lemma, that, for $F \in k[x_0, \ldots, x_n]$, 
\[
    \sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = dF,
\]
conclude that, if the characteristic of $k$ is 0, or more generally does not divide $d$, then $X_{\text{sing}} = V_+(\partial F/\partial x_0, \ldots, \partial F/\partial x_n)$. Is this necessarily true if $\text{char } k > 0$?

2. (i) Let $X$ be a scheme and let $Y$ be a closed subscheme defined by the ideal sheaf $\mathcal{I}_Y$. Let $\text{Bl}_Y X$ denote the blowup of $X$ along $Y$. Suppose that $\mathcal{I}_Y$ is a sheaf of nilpotent ideals, for example $Y = X$. Show that $\text{Bl}_Y X = \emptyset$.

(ii) Let $X$ be an integral Noetherian scheme and let $Y$ be a proper closed subscheme of $X$. Let $\pi : \widetilde{X} = \text{Bl}_Y X \to X$ be the blowup of $Y$ and let $E$ be the exceptional divisor. Let $Z \neq \emptyset$ be an integral subscheme of $X$ not contained in $Y$ and let $\rho : Z' \to Z$ be the blowup of $Z$ at the subscheme
defined by \( \mathcal{I}_Y \mathcal{O}_Z \). Show that there is a morphism from \( Z' \) to \( \tilde{X} \) which embeds \( Z' \) as a closed subscheme of \( \tilde{X} \). (In fact \( Z' \) is the closure of \( Z' - E \approx Z - Y \) in \( \tilde{X} \) and is the unique component \( Z' \) of the inverse image \( \pi^{-1}(Z) \), the subscheme of \( \tilde{X} \) defined by the ideal sheaf \( \mathcal{I}_Z \mathcal{O}_{\tilde{X}} \), not contained in \( E \).)

(iii) In the situation of (ii), suppose that \( X = \mathbb{A}^n_k \), where \( k = \overline{k} \) is an algebraically closed field, and that \( Y = \{0\} \), i.e. that \( \mathcal{I}_Y \) corresponds to the maximal ideal \( \mathfrak{m} = (x_1, \ldots, x_n) \) of \( k[x_1, \ldots, x_n] \). Let \( I_Z \subseteq k[x_1, \ldots, x_n] \) be the ideal corresponding to \( Z \); note that \( Z \) is not contained in \( Y \) \( \iff \) \( I_Z \neq \mathfrak{m} \). Thus \( E \approx \mathbb{P}^{n-1}_k \). Show that, as schemes, \( Z' \cap E = V_+(\text{in}(I_Z)) \), where \( \text{in}(I_Z) \) is the homogeneous ideal in \( k[x_1, \ldots, x_n] \) defined by

\[
\text{in}(I_Z) = \{ \text{in}(f) : f \in I_Z \},
\]

and, for \( f \in k[x_1, \ldots, x_n] \), \( \text{in}(f) \) is the initial form of \( f \): if \( f = \sum_{\nu=0}^{\infty} f_\nu \) is the expression of \( f \) as a sum of homogeneous polynomials \( f_\nu \) of degree \( \nu \), then \( \text{in}(f) = f_\nu \), where \( \nu \) is the smallest nonnegative integer such that \( f_\nu \neq 0 \) (here \( \text{in}(0) = 0 \) by convention).

(iv) In the situation of (iii), suppose further that \( Z = V(F) \), where \( F \in k[x_1, \ldots, x_n] \) is homogeneous of degree \( d \geq 2 \). Suppose moreover that \( V_+(F) \subseteq \mathbb{P}^{n-1}_k \) is smooth in the sense of the previous problem. Show that \( Z_{\text{sing}} = \{0\} \), that \( Z' \) is smooth, and that \( Z' \cap E = V_+(F) \).

3. Let \( X = \mathbb{A}^2_k = \text{Spec} k[x, y] \), where \( k = \overline{k} \) is an algebraically closed field, and let \( Y \) be the closed subscheme defined by the ideal \( (x, y^d) \). Let \( \pi : \text{Bl}_Y X \to X \) be the blowup of \( X \) with respect to the subscheme \( Y \) and let \( E \) be the exceptional divisor. Show that \( E \approx \mathbb{P}^1 \), but that \( \text{Bl}_Y X \) is not smooth if \( d > 1 \), and in fact it has a unique singular point with a Zariski open neighborhood isomorphic to

\[
V(z^d - xy) \subseteq \mathbb{A}^3_k = \text{Spec} k[x, y, z].
\]

4. Let \( X \) be an integral smooth projective curve of genus \( g \) over an algebraically closed field \( k = \overline{k} \), and let \( K_X \) be the canonical line bundle.

(i) Show that, for all closed points \( p \in X \), the line bundle \( K_X \otimes \mathcal{O}_X(p) \) has a base point. For \( g = 0 \), show that every point of \( X \) is a base point, but that there is a unique base point if \( g > 0 \).
(ii) Show that, for all closed points \( p, q \in X \), the line bundle \( K_X \otimes \mathcal{O}_X(p+q) \) is base point free. What is the dimension \( h^0(X; K_X \otimes \mathcal{O}_X(p + q)) \)?

(iii) Suppose that \( p \neq q \) in (ii). Let \( \varphi: X \to \mathbb{P}^N \) be the morphism corresponding to the line bundle \( K_X \otimes \mathcal{O}_X(p + q) \). Show that \( \varphi(p) = \varphi(q) \).

(iv) With notation and assumptions as in (iii), suppose also that \( g \geq 2 \). Show that either \( \varphi \) is birational, and in fact that \( \varphi(x) = \varphi(y) \) for two distinct points \( x, y \in X \iff \{x, y\} = \{p, q\} \), or \( X \) is hyperelliptic, \( \dim |p + q| = 2 \), and \( \varphi(x) = \varphi(y) \) for all \( x, y \) such that \( x + y \in |p + q| \), thus in this last case \( \varphi \) factors through the unique degree two morphism \( X \to \mathbb{P}^1 \) and the image of \( \varphi \) is a rational normal curve.