1. Let $k$ be a field, let $k[\varepsilon] = k[x]/(x^2)$ be the ring of dual numbers, where $\varepsilon$ is the image of $x$. Thus $\varepsilon \neq 0, \varepsilon^2 = 0$ and

$$k[\varepsilon] = \{a + b\varepsilon : a, b \in k\}.$$ 

What is Spec $k[\varepsilon]$? Show that $\text{Hom}_k(k[\varepsilon], k[\varepsilon]) \cong k$, where $\text{Hom}_k$ denotes the set of $k$-algebra homomorphisms, via

$$\varphi \in \text{Hom}_k(k[\varepsilon], k[\varepsilon]) \mapsto \varphi(\varepsilon).$$

Thus $\varphi$ is far from being determined by the induced map $\varphi^* : \text{Spec } k[\varepsilon] \to \text{Spec } k[\varepsilon]$.

2. (i) Let $R$ be a discrete valuation ring, for example $R = \mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$, the ring of $p$-adic integers, and let $K$ be the field of quotients of $R$. Show that the inclusion $R \to K$ is not a local homomorphism.

(ii) What are the points of Spec $R$? Describe the collection of open subsets of Spec $R$.

(iii) Show that, if $\psi : \text{Spec } K \to \text{Spec } R$ is the morphism corresponding to the inclusion $R \subseteq K$, then the image of the unique point of Spec $K$ is the generic point of Spec $R$. Show that there is a morphism of ringed spaces which sends the point of Spec $K$ to the closed point of Spec $R$, but which is not a morphism of locally ringed spaces and therefore does not correspond to a homomorphism $R \to K$.

3. Let $X$ be a scheme and let $R$ be a ring. The goal of this exercise is to show that there is a natural bijection from the set $\text{Mor}_{\text{schemes}}(X, \text{Spec } R)$ of morphisms of schemes from $X$ to Spec $R$ to the set $\text{Hom}_{\text{rings}}(R, \Gamma(X, \mathcal{O}_X))$ of ring homomorphisms from $R$ to the ring of global sections of $\mathcal{O}_X$. Equivalently, to give a morphism of schemes from $X$ to Spec $R$ is the same thing as giving $\mathcal{O}_X(U)$ the structure of an $R$-algebra, so that the restriction homomorphisms are $R$-algebra homomorphisms. To prove this, we define a function

$$F : \text{Mor}_{\text{schemes}}(X, \text{Spec } R) \to \text{Hom}_{\text{rings}}(R, \Gamma(X, \mathcal{O}_X)).$$
as follows: If $\psi: X \to \text{Spec } R$ is a morphism, we have $\psi^*: \mathcal{O}_{\text{Spec } R} \to \psi_*\mathcal{O}_X$. We let $F(\psi)$, also denoted $\psi^*$, be the induced ring homomorphism on global sections:

$$\psi^*: \mathcal{O}_{\text{Spec } R}(\text{Spec } R) \to \psi_*\mathcal{O}_X(\text{Spec } R) = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X).$$

To find a map $G$ in the other direction, write $X = \bigcup U_\alpha$, where $U_\alpha = \text{Spec } S_\alpha$ is an affine scheme. Given $\varphi: R \to \Gamma(X, \mathcal{O}_X)$, via restriction there is a homomorphism $\varphi_\alpha: R \to \Gamma(U_\alpha, \mathcal{O}_{U_\alpha}) = S_\alpha$, and we have seen in class that $\varphi = \psi^*$ for a unique morphism $\psi_\alpha: \text{Spec } S_\alpha \to \text{Spec } R$. Show that $\psi_\alpha|_{U_\alpha \cap U_\beta} = \psi_\beta|_{U_\alpha \cap U_\beta}$, hence the $\psi_\alpha$ glue to give a morphism $G(\varphi) = \psi: X \to \text{Spec } R$, and finally show that $F$ and $G$ are inverse functions.

4. Let $X$ be a scheme and let $x \in X$. Show that there is a canonical morphism $\text{Spec } k(x) \to X$. More generally, if $K$ is a field, then there is a bijection from the set of morphisms $\psi: \text{Spec } K \to X$ with $\psi(\text{Spec } K) = \{x\}$ and field extensions $k(x) \subseteq K$.

5. Let $X = \text{Spec } S$ and $Y = \text{Spec } R$ be affine schemes, and let $\psi: X \to Y$ be a morphism corresponding to the ring homomorphism $\varphi: R \to S$. Let $\tilde{M}$ be the sheaf on $X$ corresponding to the $S$-module $M$. Show that the sheaf $\psi_*\tilde{M}$ is equal to $\tilde{M}_\varphi$, the sheaf associated to the $R$-module $M_\varphi$, where $M_\varphi = M$ as abelian groups and the action of an element $f$ on $m \in M_\varphi$ is: $f \cdot m = \varphi(f)m$.

6. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be two sheaves of $\mathcal{O}_X$-modules. (i) Show that the presheaf $\underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ defined by

$$U \mapsto \text{Hom}_{\mathcal{O}_X}(\mathcal{F}|U, \mathcal{G}|U)$$

is a sheaf of $\mathcal{O}_X$-modules, and that there is a morphism (of sheaves of $\mathcal{O}_X$-modules) from $\underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to $\underline{\text{Hom}}'_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, where $\underline{\text{Hom}}'_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheafification of the presheaf

$$U \mapsto \text{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U)).$$

(ii) Show that the functors $\text{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{G})$ and $\underline{\text{Hom}}_{\mathcal{O}_X}(\cdot, \mathcal{G})$ are left exact, i.e. if

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is an exact sequence of sheaves, then

$$0 \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G})$$

is exact, and similarly for $\text{Hom}_{\mathcal{O}_X}$.

(iii) Suppose that $\mathcal{F} = \mathcal{O}_X$. Show that $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \cong \mathcal{G}(X)$, and conclude that

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \cong \text{Hom}_{\mathcal{O}_X}'(\mathcal{O}_X, \mathcal{G}) \cong \mathcal{G}$$

via the natural maps.