Examples of changing the order in triple integrals

Example 1: A tetrahedron $T$ is defined by the inequalities $x, y, z \geq 0$ and $2x + 3y + z \leq 6$. The tetrahedron has three faces which are triangles in the coordinate planes. For example, the face of $T$ in the $xy$-plane is given by $x, y \geq 0$ and $2x + 3y \leq 6$. The remaining face of $T$ is the triangle with vertices $(3, 0, 0), (0, 2, 0)$ and $(0, 0, 6)$. It’s straightforward to draw a picture of $T$, as we did in class. If we want to describe $T$ as a standard region, corresponding to the order of integration given by $dx \, dy \, dz$, then $T$ is defined by the inequalities

$$0 \leq x \leq 3;$$
$$0 \leq y \leq \frac{1}{3}(6 - 2x);$$
$$0 \leq z \leq 6 - 2x - 3y.$$

If we wanted to use a triple integral to compute the volume of $T$, we would get

$$\text{volume}(T) = \iiint_T 1 \, dV = \int_0^3 \int_0^{\frac{1}{3}(6-2x)} \int_0^{6-2x-3y} 1 \, dz \, dy \, dx$$
$$= \int_0^3 \int_0^{\frac{1}{3}(6-2x)} (6 - 2x - 3y) \, dy \, dx = \int_0^3 \left[(6 - 2x)y - \frac{3}{2}y^2\right]_0^{\frac{1}{3}(6-2x)} \, dx$$
$$= \int_0^3 \left[\frac{(6 - 2x)^2}{3} - \frac{3}{2} \frac{(6 - 2x)^2}{9}\right] \, dx$$
$$= \frac{1}{6} \int_0^3 (6 - 2x)^2 \, dx = \left(\frac{1}{6}\right) \cdot \left(-\frac{1}{6}\right) \left[(6 - 2x)^3\right]_0^6$$
$$= \frac{6^3}{6^2} = 6.$$

(Here, we have skipped a few steps in the computation. Also, note that we have saved ourselves a lot of calculation by not expanding out the squared terms but rather by grouping them carefully.) The area of the base of $T$ is one half the base times the height $= \frac{1}{2}(2)(3) = 3$, and the volume of $T$ is one third the area of the base times the height $= \frac{1}{3}(3)(6) = 6$, which agrees with the computation above.

If we wanted to change the order of integration above to, say, $dx \, dy \, dz$, then $T$ would lie to the front of the triangle in the $yz$-plane given by $y, z \geq 0$.
and $3y + z \leq 6$. The inequalities defining $T$ would then take the form

\begin{align*}
0 &\leq z \leq 6; \\
0 &\leq y \leq \frac{1}{3}(6 - z); \\
0 &\leq x \leq \frac{1}{2}(6 - 3y - z).
\end{align*}

The corresponding integral expressing the volume of $T$ would then be

\[
\iiint_T 1 \, dV = \int_0^6 \int_0^{\frac{1}{3}(6-z)} \int_0^{\frac{1}{2}(6-3y-z)} 1 \, dx \, dy \, dz.
\]

A similar calculation shows that this triple integral is equal to 6 (as it must).

**Example 2:** Consider the triple integral

\[
\int_0^1 \int_0^{\sqrt{x}} \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx,
\]

where $f(x, y, z)$ is a generic function of three variables. This integral is an integral over the region in $\mathbb{R}^3$ defined by the inequalities

\begin{align*}
0 &\leq x \leq 1; \\
\sqrt{x} &\leq y \leq 1; \\
0 &\leq z \leq 1 - y. 
\end{align*} \tag{*}

It’s not that difficult to draw a picture of the region defined by these inequalities, as we did in class, but it’s not too helpful for what follows. If we just want to change the order to $dz \, dx \, dy$, we just want to change the Type I region in the plane defined by

\begin{align*}
0 &\leq x \leq 1; \\
\sqrt{x} &\leq y \leq 1 
\end{align*}

to a Type II region, and we have seen how to do this: the Type II region is

\begin{align*}
0 &\leq y \leq 1; \\
0 &\leq x \leq y^2.
\end{align*}

(You can and should draw a picture of this!) So the integral becomes

\[
\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z) \, dz \, dx \, dy.
\]
It’s more complicated to change the order to \(dy \, dx \, dz\). To do so, we look for the strongest inequalities we can get on \(z\) in terms of constants alone, on \(x\) in terms of \(z\), and on \(y\) in terms of \(x\) and \(z\). For example, we see that \(0 \leq z\). Also, \(z \leq 1 - y\) is the only inequality we have on \(z\) in the other direction. But we need an inequality only involving constants. However, since \(y \geq 0\), \(1 - y \leq 1\), so we get a combined inequality \(0 \leq z \leq 1 - y \leq 1\). Similarly, for \(x\) we see that \(0 \leq x\) and \(\sqrt{x} \leq y\), so \(x \leq y^2\). However, we can’t use an inequality for \(x\) which involves \(y\), so we also use \(z \leq 1 - y\), hence \(y \leq 1 - z\), and thus \(0 \leq x \leq y^2 \leq (1 - z)^2\). Finally, for \(y\), we have \(\sqrt{x} \leq y \leq 1\) and also \(y \leq 1 - z\), which is a better inequality since \(z \geq 0\) and hence \(1 - z \leq 1\). So in all we get three inequalities

\[
\begin{align*}
0 & \leq z \leq 1; \\
0 & \leq x \leq (1 - z)^2; \\
\sqrt{x} & \leq y \leq 1 - z.
\end{align*}
\]

How do we know that we have done this correctly? What this means is, how do we know that the system of inequalities (\#) and (**) are equivalent, i.e. that they define the same set of points \((x, y, z)\) in \(\mathbb{R}^3\)? We have showed that, if \((x, y, z)\) satisfy the system (\#) of inequalities, then they satisfy the system (**). If we show conversely that every \((x, y, z)\) satisfying the system (**) also satisfy the system (\#), then we see that the regions defined by (\#) and (**) are the same. While we won’t normally ask for this on HW or exams, let us just check it in this case. Starting with \((x, y, z)\) satisfying (**), we see that \(0 \leq x \leq (1 - z)^2 \leq 1\), since \(0 \leq z \leq 1\). Hence \(0 \leq x \leq 1\). Next, \(\sqrt{x} \leq y \leq 1 - z \leq 1\), since \(z \geq 0\). Hence \(\sqrt{x} \leq y \leq 1\). Finally, \(0 \leq z\) and, from \(y \leq 1 - z\), we see that \(z \leq 1 - y\). (Of course, we also have \(z \leq 1\), but \(z \leq 1 - y\) is stronger, and \(\sqrt{x} \leq 1 - z\), hence \(z \leq 1 - \sqrt{x}\), but since \(\sqrt{x} \leq y\), \(z \leq 1 - y\) is a stronger inequality because \(1 - y \leq 1 - \sqrt{x}\).) This gives \(0 \leq z \leq 1 - y\). So the system (**) of inequalities implies the system (\#), and hence they are equivalent in the sense of defining the same points in \(\mathbb{R}^3\).

In terms of triple integrals,

\[
\int_0^1 \int_0^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_0^{(1-z)^2} \int_0^{1-z} f(x, y, z) \, dy \, dx \, dz.
\]
Example 3: Consider the triple integral

\[ \int_0^1 \int_0^{1-x} \int_0^{2-2z} f(x, y, z) \, dy \, dz \, dx. \]

Thus corresponds to the system of inequalities

\[
\begin{align*}
0 & \leq x \leq 1; \\
0 & \leq z \leq 1 - x; \\
0 & \leq y \leq 2 - 2z
\end{align*}
\]

The first two inequalities define a triangle in the \(xz\)-plane bounded by the lines \(x = 0\), \(z = 0\), and \(x + z = 1\). The region \(R\) defined by these inequalities is all of the \(y\) lying to the right of this region in the \(xz\)-plane and to the left of the plane \(y + 2z = 2\). If you draw the picture, you can see that \(R\) is also described as a pyramid, whose base is the rectangle in the \(xy\)-plane defined by \(0 \leq x \leq 1\) and \(0 \leq y \leq 2\), and whose vertex is the point \((0, 0, 1)\).

Now let’s try the order \(dz \, dy \, dx\). First, we see that \(0 \leq x \leq 1\), and the inequality \(z \leq 1 - x\), which is the same as \(x \leq 1 - z\), doesn’t give anything better since \(z\) has to disappear from the final answer. Also, \(0 \leq y \leq 2 - 2z\) and the inequality \(x \leq 1 - z\), i.e. \(2x \leq 2 - 2z\), is not comparable with this. As we have to make \(z\) disappear from the final inequality, we use \(z \geq 0\) to conclude that \(0 \leq y \leq 2 - 2z \leq 2\).

What about inequalities for \(z\)? We see that \(0 \leq z \leq 1 - x\) and also, from \(y \leq 2 - 2z\), that \(2z \leq 2 - y\), or equivalently that \(z \leq 1 - \frac{1}{2}y\). Neither of the inequalities \(z \leq 1 - x\) or \(z \leq 1 - \frac{1}{2}y\) is necessarily better that the other, so for the moment we just record the inequalities we have learned:

\[
\begin{align*}
0 & \leq x \leq 1; \\
0 & \leq y \leq 2; \\
0 & \leq z \leq 1 - x; \\
0 & \leq z \leq 1 - \frac{1}{2}y.
\end{align*}
\]

The way to understand this system of inequalities is as follows. The first two define a rectangle in the \(xy\)-plane. When \(1 - x \geq 1 - \frac{1}{2}y\), the second inequality for \(z\) is stronger than the first (i.e. imposes a stronger condition on \(z\)). If \(1 - x \leq 1 - \frac{1}{2}y\), the first inequality for \(z\) is stronger than the second (i.e. imposes a stronger condition on \(z\)). This says that we should divide up the rectangle \(0 \leq x \leq 1, 0 \leq y \leq 2\), according to whether \(1 - x \geq 1 - \frac{1}{2}y\) or \(1 - x \leq 1 - \frac{1}{2}y\). The first condition is that \(y \geq 2x\), the second that
$y \leq 2x$. So we define two regions $R_1$ and $R_2$ as follows: $R_1$ is the region where $0 \leq x \leq 1$, $0 \leq y \leq 2$, and $y \geq 2x$, and then we use the inequality $0 \leq z \leq 1 - \frac{1}{2}y$. The second region $R_2$ is the region where $0 \leq x \leq 1$, $0 \leq y \leq 2$, and $y \leq 2x$, and then we use the inequality $0 \leq z \leq 1 - x$. Thus, the region $R_1$ is defined by

\begin{align*}
0 &\leq x \leq 1; \\
2x &\leq y \leq 2; \\
0 &\leq z \leq 1 - \frac{1}{2}y.
\end{align*}

(††_1)

The region $R_2$ is defined by

\begin{align*}
0 &\leq x \leq 1; \\
0 &\leq y \leq 2x; \\
0 &\leq z \leq 1 - x.
\end{align*}

(††_2)

You can also see this from a good picture of the pyramid: over the triangle in the $xy$-plane below the diagonal line $y = 2x$ in the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$, the upper boundary of the pyramid is the plane $z = 1 - x$. Over the triangle in the rectangle lying above the diagonal line $y = 2x$, the upper boundary of the pyramid is the plane $z = 1 - \frac{1}{2}y$.

The upshot is that, while the original region $R$ is a standard region for the order $dy\,dz\,dx$, it is not a standard region for the order $dz\,dy\,dx$. In fact, it is a union of two standard regions $R_1$ and $R_2$ for the order $dz\,dy\,dx$. Thus we can write (for a generic function $f(x,y,z)$)

\[
\iiint f(x,y,z)\,dV = \int_0^1 \int_0^{1-x} \int_0^{2-2x} f(x,y,z)\,dy\,dz\,dx \\
= \iiint f(x,y,z)\,dV + \iiint f(x,y,z)\,dV \\
= \int_0^1 \int_0^{2x} f(x,y,z)\,dz\,dy\,dx + \int_0^1 \int_0^{1-x} f(x,y,z)\,dz\,dy\,dx.
\]

Other examples of this type are in the HW for Monday (Stewart 15.6, exercises 34, 36).