1. Let $G$ be a finite group and $V$, $W$ be two irreducible representations of $G$, with characters $\chi_V, \chi_W$.

(i) Show that

$$F_{V,\chi_W} = \sum_{g \in G} \chi_W(g) \rho_V(g) = \frac{\#(G) \langle \chi_W, \overline{\chi_V} \rangle}{\dim V} \text{Id}.$$ 

Define $e_W = \frac{\dim W}{\#(G)} \chi_W \in L^2(G)$, and conclude that

$$F_{V,e_W} = \begin{cases} 
\text{Id}, & \text{if } W \cong V^*; \\
0, & \text{otherwise}.
\end{cases}$$

(ii) (Convolution of characters) Prove the convolution identity

$$\chi_V \ast \chi_W = \frac{\#(G)}{\dim V} \langle \chi_V, \chi_W \rangle \chi_V = \begin{cases} 
\frac{\#(G)}{\dim V} \chi_V, & \text{if } V \cong W; \\
0, & \text{otherwise}.
\end{cases}$$

(Hint: begin with the expression for $F_{V,\chi_W}$ above. Replacing $\chi_W$ by $\overline{\chi_W}$ and $g \in G$ by $x^{-1}$, $x \in G$, using $\overline{\chi_W}(x^{-1}) = \chi_W(x)$, gives

$$\sum_{g \in G} \overline{\chi_W}(g) \rho_V(g) = \sum_{x \in G} \chi_W(x) \rho_V(x^{-1}) = \frac{\#(G) \langle \chi_W, \overline{\chi_V} \rangle}{\dim V} \text{Id}.$$ 

Note that $\langle \chi_W, \overline{\chi_V} \rangle = \langle \chi_W, \chi_V \rangle$. Finally, multiply both sides by $\rho_V(g)$ on the right and take traces.)

(iii) Let $V_1, \ldots, V_h$ be the distinct irreducible representations of the finite group $G$ up to isomorphism, with $\dim V_i = d_i$. For each $i$, define the element $e_i \in L^2(G)$ by

$$e_i = e_{V_i^*} = \frac{d_i}{\#(G)} \chi_{V_i^*} = \frac{d_i}{\#(G)} \overline{\chi_i},$$

where $d_i = \dim V_i = \dim V_i^*$. (We use $V_i^*$ instead of $V_i$ to make things work out in the explanation given below.) Show that

$$e_i \ast e_j = \begin{cases} 
e_i, & \text{if } i = j; \\
0, & \text{otherwise}.
\end{cases}$$
Also (using standard results) show that \( \delta_1 = e_1 + \cdots + e_h \). (Here \( \delta_1 \) is the \( \delta \)-function associated to \( 1 \in G \), and it is the multiplicative identity in \( L^2(G) \) under convolution, corresponding to \( 1 \in \mathbb{C}[G] \).)

**Explanation** (you do not have to hand this part in): Let \( V_i \) be an irreducible representation and define \( e_i = e_{V_i^*} \) as above. For an arbitrary representation \( V \), not necessarily irreducible, we have defined \( \rho_V(e_i) = F_{V,e_i} \). Part (i) shows that it is the projection onto all of the summands of \( V \) isomorphic to \( V_i \).

If \( \alpha \in \mathbb{C}[G] \) and \( m_\alpha: \mathbb{C}[G] \rightarrow \mathbb{C}[G] \) is multiplication by \( \alpha \), then \( \rho_{\text{reg}}(\alpha) = m_\alpha \). It is easy to see that \( m_\alpha \) is a \( G \)-morphism of the representation \( \mathbb{C}[G] \iff \alpha \in \mathbb{Z}[\mathbb{C}[G]] \). Applying this to the elements \( e_i \), viewed as elements of \( \mathbb{Z}[\mathbb{C}[G]] \) instead of \( L^2(G) \), we see that \( m_{e_i} \) is a \( G \)-morphism for every \( i \). Moreover, since \( e_i * e_i = e_i \), it follows that \( m_{e_i} \) is projection onto a subspace of \( \mathbb{C}[G] \). By the above comments, given that \( \mathbb{C}[G] \cong V_1^{d_1} + \cdots + V_h^{d_h} \), the projection \( m_{e_i} \) is projection onto the summand \( V_i^{d_i} \), which is in fact intrinsically defined.

2. (More on dual groups of finite abelian groups) For a finite abelian group \( G \), let \( \hat{G} \) be the group of homomorphisms \( \chi: G \rightarrow \mathbb{C}^* \). We have seen that \( \hat{G} \) is also a finite abelian group and that \( #(\hat{G}) = #(G) \).

(a) Suppose that \( G_1 \) and \( G_2 \) are two finite abelian groups and that \( f: G_1 \rightarrow G_2 \) is a homomorphism. Show that, for each \( \chi \in \hat{G}_2 \), \( f^*(\chi) = \chi \circ f: G_1 \rightarrow \mathbb{C}^* \) is a homomorphism, i.e. that \( f^*(\chi) = \chi \circ f \in \hat{G}_1 \), and that the resulting function \( f^*: \hat{G}_2 \rightarrow \hat{G}_1 \) is a homomorphism of abelian groups.

(b) Suppose that \( G_1, G_2, \) and \( G_3 \) are three finite abelian groups and that \( f_1: G_1 \rightarrow G_2 \) and \( f_2: G_2 \rightarrow G_3 \) are homomorphisms. Show that \( (f_2 \circ f_1)^* = f_1^* \circ f_2^* \).

(c) Let \( G \) be a finite abelian group and let \( H \) be a subgroup of \( G \). If \( \pi: G \rightarrow G/H \) is the quotient homomorphism, show that \( \pi^*: \hat{G}/H \rightarrow \hat{G} \) is injective and that the image of \( \pi \) is the set of all homomorphisms \( \chi: G \rightarrow \mathbb{C}^* \) such that \( \chi(h) = 1 \) for all \( h \in H \).

(d) With \( G, H \) as in (c), let \( i: H \rightarrow G \) be the inclusion of the subgroup \( H \) in the group \( G \), so that, with \( i^*: \hat{G} \rightarrow \hat{H} \) as in (a), \( i^*(\chi) \) is the restriction of \( \chi \) to \( H \). Show that the kernel of \( i^* \) is equal to
the set of all homomorphisms $\chi: G \to \mathbb{C}^*$ such that $\chi(h) = 1$ for all $h \in H$ and hence that the kernel of $i^*$ is the image of $\pi^*$. Thus, as $\pi^*$ is injective, show that $\#(\text{Ker } i^*) = \#(\hat{G}/\hat{H}) = \#(G/H) = \#(G)/\#(H)$.

(e) By counting and the Fundamental Theorem for Homomorphisms, show that $\#(\text{Im } i^*) = \#(H) = \#(\hat{H})$, and hence show that $i^*$ is surjective.

3. Let $G$ be a finite abelian group. Show that the evaluation map $ev: G \to \hat{G}$ defined by

$$ev(g)(\chi) = \chi(g)$$

is an isomorphism, by first showing that it is a homomorphism, then that it is injective. (If $g \in \text{Ker } ev$, then for all $\chi \in \widehat{G}$, $\chi(g) = 1$. This would say that every $\chi \in \widehat{G}$ comes from a character on $G/\langle g \rangle$, i.e. that the homomorphism $G/\langle g \rangle \to \widehat{G}$ is surjective. Since $\#(\hat{G}) = \#(G)$ and $\#(\hat{G}/\hat{g}) = \#(G/\langle g \rangle) = \#(G)/\#(\langle g \rangle)$, show that the only possibility is $\#(\langle g \rangle) = 1$, hence $g = 1$. Thus ev: $G \to \widehat{G}$ is an isomorphism, since both sides have the same number of elements.)